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Balancing time-varying demand-supply in distribution networks: an internal model approach

Claudio De Persis

***Abstract*— The problem of load balancing in a distribution network under unknown time-varying demand and supply is studied. A set of distributed controllers which regulate the amount of flow through the edges is designed to guarantee convergence of the solution to the steady state solution. The results are then extended to a class of nonlinear systems and compared with existing results. Incremental passivity and internal model principle are the main analytical tools.**

I. Introduction

Cooperative control systems have been widely investigated in a variety of different contexts [18], [14], [17], [3]. Less attention has been devoted to cooperative control in the framework of dynamical flow networks, with some interesting exceptions [10], [6], [8], [7], [19], [5]. The aim of this paper is to study a class of cooperative control algorithms in the context of distribution networks under exogenous inputs.

Main contribution. We analyze and design distributed controllers at the edge which achieve load balancing in the presence of time-varying demand and supply (exogenous signals). The role of internal model and incremental passivity ([16]) is investigated for the problem at hand. Similar tools have been used for controlled synchronization and leader-follower formation control in e.g. [20], [3], [17], [12] and references therein. We address a different problem and we tackle it in a novel way. The load distribution problem is then considered for a more general class of systems and this allows us to make a comparison with the results of [2] and [10].

Literature review. The literature on the control of flow or distribution networks is wide and multi-disciplinary. Here we focus on a model which takes into account the amount of stored material at the nodes and mass balance. This class of systems has been used to model data networks [15] and supply chains [1] for instance. Our paper focuses on the problem of stabilizing the flow network to a steady state solution in the presence of exogenous time-varying demand and supply under the scenario in which the controllers aim at a uniform distribution of the material among the nodes. This is a problem which has attracted considerable attention in the area of parallel and distributed computation [18] and has been recently reconsidered for instance in [5] where input and state constraints have been taken into account and a connection with [14] has been established. The work [5] did not consider the presence of external inputs. A large amount of work on the topic of flow control in the presence of disturbances has been carried out in works such as [6]-[8] where the problem is cast in the robust control framework. The approach in our paper is based on the theory of output regulation and to the best of our knowledge this has not been considered before. A similar problem has been tackled in [19] but the authors restrict themselves to the class of constant disturbances.

Organization of the paper. The class of systems under study is introduced in Section II, the design of the edge regulators is carried out in Section III and the extension to a class of nonlinear system in Section IV. The conclusions are discussed in the last section.

The proofs are omitted for lack of space and can be found in [11].

C. De Persis is with ITM, Faculty of Mathematics and Natural Sciences, University of Groningen, 9747 AG Groningen, The Netherlands, c.de.persis@rug.nl and Department of Computer, Control and Management Engineering, Sapienza Università di Roma, Italy.
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II. Distribution networks and demand supply balancing

Consider the system

$$\dot{x} \quad = \quad B\lambda + Qd \tag{1}$$

with $x \in \mathbb{R}^n$ the state, $\lambda \in \mathbb{R}^m$ the control vector and $d \in \mathbb{R}^q$, $q \leq n$, a disturbance vector. The $(n \times m)$ matrix B is the incidence matrix of an *undirected* graph $G = (V, E)$ where $|V| = n$, $|E| = m$. The ends of the edges of G are labeled with a '+' and a '-'. Then

$$b_{ik} = \begin{cases} +1 & i \text{ is the positive end of } k \\ -1 & i \text{ is the negative end of } k \\ 0 & \text{otherwise} \end{cases}$$

The system above is a simple model of a flow network [6] and it has been used also to model data networks [15] and supply chains [1]. The state $x_i \in \mathbb{R}$, $i \in \mathcal{I} := 1, 2, \ldots, n$ represents the quantity of material stored at the node i, $\lambda_k \in \mathbb{R}$, $k = 1, 2, \ldots, m$ the flow through the edge k. The disturbance $d_j \in \mathbb{R}$ represents the inflow or the outflow at some node.

The available measurements are the differences among the quantities stored at the nodes namely, $z = B^T x$.

We assume that each disturbance d_j is generated by the exosystem $\dot{w}_j = S_j^d w_j,\ d_j = \Gamma_j^d w_j, \quad j = 1, \ldots, q,$ where $w_j \in \mathbb{R}^{p_j}$ is the state of the exosystem which describes the evolution of the inflow/outflow j and Γ_j^d, S_j^d are suitable matrices. The class of exosystems that are considered in this paper generate periodic and constant disturbances. Considering more general classes of exosystems that could take into account more realistic demand-supply profiles is left for future research.

We give the exosystems above the compact form

$$\dot{w} = S^d w, \quad d = \Gamma^d w, \tag{2}$$

where $w = (w_1^T \ldots w_q^T)^T$, $d = (d_1^T \ldots d_q^T)^T$, $S^d = \text{block.diag}(S_1^d, \ldots, S_q^d)$, $\Gamma^d = \text{block.diag}(\Gamma_1^d, \ldots, \Gamma_q^d)$. Setting $P = Q\Gamma^d$, the model (1) and the overall exosystem (2) return the closed-loop system

$$\begin{array}{rcl} \dot{w} & = & S^d w \\ \dot{x} & = & B\lambda + Pw \\ z & = & B^T x\ . \end{array} \tag{3}$$

We are interested in the problem of distributing the cumulative imbalance of the network due to the in- and out-flow among the nodes. More formally the problem at hand is as follows:

Load balancing at the nodes Find distributed dynamic feedback control laws

$$\begin{array}{rcl} \dot{\eta}_k & = & \Phi_k \eta_k + \Lambda_k z_k \\ \lambda_k & = & \Psi_k \eta_k + \Gamma_k z_k, \quad k = 1, \ldots, m \end{array} \tag{4}$$

such that, for each initial condition (w_0, x_0, η_0), the solution of the closed-loop system (3), (4) satisfies $\lim_{t \to +\infty} z(t) = 0$.

III. Design of regulators at the edges

We focus on flow networks whose underlying graph satisfies the following standing assumption:

Assumption 1 *The graph G is connected.*

The first result concerns the characterization of a "steady state" solution to the problem:

Lemma 1 *Let Assumption 1 hold. For each w solution to $\dot{w} = S^d w$, if there exist a function $\lambda^w : \mathbb{R}_+ \to \mathbb{R}^m$ and a continuously differentiable function $x^w : \mathbb{R}_+ \to \mathbb{R}^n$ solution to*

$$\dot{x}^w = B\lambda^w + Pw, \quad 0 = B^T x^w \tag{5}$$

then

$$x^w = \mathbf{1}_n x_*^w, \quad \dot{x}_*^w = \frac{\mathbf{1}_n^T P w}{n} \tag{6}$$

and $\lambda^w = Mw$, for some matrix M. If the graph is a tree, then the matrix M is unique.

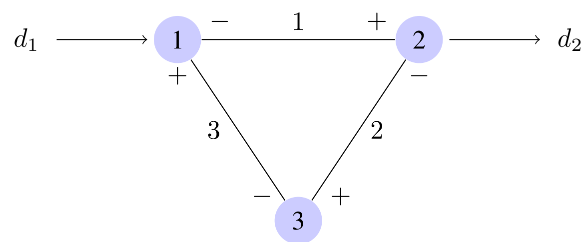

Fig. 1: The distribution network considered in the Example 1.

In what follows, we assume that a solution to (5) exists. Moreover, if $m > n - 1$, then without loss of generality we let the last $m - n + 1$ components of λ^w be identically zero. This choice in general is not optimal. See [9] for choices of λ^w that are optimal with respect to suitable cost functions.

Remark 1 From (6), by integration, one has

$$x^w(t) = \mathbf{1}_n \left(x_*^w(0) + \int_0^t \frac{\mathbf{1}_n^T P w(s)}{n} ds \right).$$

Observe that x^w depends on the initial condition and strictly speaking cannot be referred to as a steady state solution. Bearing in mind the interpretation of (1) as a flow network and of Pw as the vector of the inflows and outflows of the network, the integral $\int_0^t \frac{\mathbf{1}_n^T P w(s)}{n} ds$ can be seen as the *cumulative* imbalance of the network. In other words, if for any given w a solution to the load balancing problem exists, then the state at each node equals – up to a constant – the cumulative imbalance of the network. In the case of a network with no imbalance, i.e. $\mathbf{1}_n^T P w(t) = 0$ for all $t \geq 0$, x^w is a *constant* vector.

Example 1 Consider the graph depicted in Fig. 1. The graph corresponds to system (1) with

$$B = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

The solutions to (5) (with $w = d$) are as follows

$$\begin{array}{rcl} \dot{x}_*^w & = & \frac{d_1 - d_2}{3} \\ \lambda_1^w & = & \lambda_3^w + \frac{2d_1 + d_2}{3} \\ \lambda_2^w & = & \lambda_3^w + \frac{d_1 - d_2}{3}. \end{array}$$

A possible solution is obtained letting e.g. $\lambda_3^w = 0$.

We introduce now a system which generates the control signal λ^w in Lemma 1. Consider the input λ_k^w associated with the edge k, with $k = 1, 2, \ldots, m$. In general, such input may depend on all the components of the disturbance vector w. Hence, to generate λ_k^w, the following system is proposed:

$$\dot{\eta}_k = S^d \eta_k \quad u_k = H_k \eta_k \tag{7}$$

The statement below is immediate.

Lemma 2 *For any w solution to $\dot{w} = S^d w$, there exists a solution η_k^w to (7) such that $H_k \eta_k^w(t) = \lambda_k^w(t)$ for all $t \geq 0$, where λ_k^w is the kth entry of λ^w in Lemma 1.*

Remark 2 From the proof of Lemma 1 the $m-n+1$ components of λ^w can be chosen identically zero. The matrices H_k corresponding to these components are then identically zero as well. Hence, for $k = n, n+1, \ldots, m$, the system (7) reduces trivially to $u_k = 0$.

The system (7) is completed by adding control inputs v_{k1}, v_{k2} to be designed for guaranteeing that the response of the closed-loop system converges to the desired response for x. Hence, we set

$$\begin{array}{rcl} \dot{\eta}_k & = & S^d \eta_k + v_{k1} \\ u_k & = & H_k \eta_k + v_{k2}, \quad k = 1, 2, \ldots, n-1 \end{array} \tag{8}$$

with $\eta_k, v_{k1} \in \mathbb{R}^q$, $v_{k2} \in \mathbb{R}$, and $u_k = v_{k2}$ for $k = n, n+1, \ldots, m$. We write (8) in the form

$$\dot{\eta} = \overline{S}\eta + v_1 \quad \lambda = \overline{H}\eta + v_2 \tag{9}$$

where $\eta = (\eta_1^T \ \eta_2^T \ldots \eta_{n-1}^T)^T$, $\overline{S} = I_{n-1} \otimes S^d$, $\otimes$ denotes the Kronecker product, and

$$\overline{H} = \begin{pmatrix} \overline{H}_1 \\ \mathbf{0} \end{pmatrix}, \quad \overline{H}_1 = \text{block.diag}(H_1, \ldots, H_{n-1}).$$

Observe that by Lemma 2, for any w and provided that $v_1 = \mathbf{0}$, $v_2 = \mathbf{0}$, there exists a solution η^w to (9) which satisfies $\dot{\eta}^w = \overline{S}\eta^w$, $\lambda^w = \overline{H}\eta^w$.

The theorem below is the main result of this section and it solves the distributed dynamic load balancing problem formulated in the previous section.

Theorem 1 *Consider the system (1), where B is the incidence matrix of a graph G and d is a disturbance generated by the system (2).*
Under Assumption 1, provided that S_j^d is skew symmetric for each $j = 1, 2, \ldots, q$, the dynamic feedback controller (9) with $v_1 = -\overline{H}^T B^T x$ and $v_2 = -B^T x$, namely

$$\dot{\eta} = \overline{S}\eta - \overline{H}^T B^T x, \quad \lambda = \overline{H}\eta - B^T x \tag{10}$$

guarantees boundedness of the state of the closed-loop system and asymptotic convergence of $x(t)$ to $\mathbf{1}_n(\frac{\mathbf{1}^T x(0)}{n} + \int_0^t \frac{\mathbf{1}_n^T P w(s)}{n} ds)$.

Remark 3 The result states that under the effect of a time-varying demand/supply all the components of the state $x(t)$ asymptotically converge to the average of the initial distribution of material at the nodes plus the cumulative imbalance equally divided among the nodes.

Considering the block diagonal nature of the matrices $\overline{S}$, $\overline{H}$ and the definition $z = B^T x$, the dynamic feedback controller (10) can be decomposed as the following set of dynamic feedback controllers at the edges:

$$\begin{array}{rcl} \dot{\eta}_k & = & S^d \eta_k - H_k^T z_k \\ \lambda_k & = & H_k \eta_k - z_k, \quad k = 1, 2, \ldots, n-1 \end{array} \tag{11}$$

which only requires the knowledge of the difference between the quantities stored at the two nodes connected by the edge. As such the proposed controller (10) is fully distributed and solves the load balancing problem formulated in Section II, with $\Phi_k = S^d, \Lambda_k = -H_k^T, \Psi_k = H_k, \Gamma_k = -1$. By Remark 2, for $k = n, n+1, \ldots, m$ for which $H_k = \mathbf{0}$ the edge controller becomes a static one, i.e. $\lambda_k = -z_k$. The system (11) is an *internal-model-based* controller which embeds a copy of the exosystem. To compare with other results in coordinated control, consider the closed-loop system (1), (10)

$$\begin{array}{rcl} \dot{w} & = & Sw \\ \dot{x} & = & -BB^T x + B\overline{H}\eta + Pw \\ \dot{\eta} & = & \overline{S}\eta - \overline{H}^T B^T x. \end{array} \tag{12}$$

The equation of the x-subsystem includes a term that coincides with the standard consensus algorithm ($-BB^T x$), a perturbative term (Pw) and an additional control input $B\overline{H}\eta$. The latter is needed to compensate for the disturbance and is provided by the internal-model-based controller $\dot{\eta} = \overline{S}\eta - \overline{H}^T B^T x$.

Example 1 (Cont'd) Assume that $d_1 = \alpha + \beta \sin(\omega t + \varphi)$, with $\alpha > \beta > 0$ and $d_2 = \alpha$. The supply is a periodic fluctuation around a constant value while the demand is a constant. Then the matrices S^d and Γ^d in (2) write as

$$S^d = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & -\omega & 0 \end{pmatrix}, \quad \Gamma^d = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let $\lambda_3^w = 0$. Then, for $k = 1, 2$, the matrices H_k which allow to reproduce λ_k^w are

$$H_1 = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & 0 \end{pmatrix}.$$

Then the controllers at the edges 1 and 2 are given by (11) with S^d and H_k as above and $z_1 = -x_1 + x_2$, $z_2 = -x_2 + x_3$. The controller at edge 3 is the static control law $\lambda_3 = -z_3 = -(x_1 - x_3)$.

Remark 4 (Passivity-based reinterpretation) The proof of Theorem 1 can be reinterpreted as follows. In view of Lemma 1, the system $\dot{\tilde{x}} = B\tilde{\lambda}$, $z = B^T\tilde{x}$ is the incremental model associated with system (1). Similarly, by Lemma 2, system $\dot{\tilde{\eta}} = \overline{S}\tilde{\eta} + \overline{H}^T\tilde{v}$, $\tilde{u} = \overline{H}\tilde{\eta}$, where $\tilde{u} = u - u^w$ and $u^w := \overline{H}\eta^w$, is the incremental model associated with the internal model $\dot{\eta} = \overline{S}\eta + \overline{H}^T v$, $u = \overline{H}\eta$. The systems are passive with respect to the storage functions $V_1(\tilde{x}) = \frac{1}{2}\tilde{x}^T\tilde{x}$ and $V_2(\tilde{\eta}) = \frac{1}{2}\tilde{\eta}^T\tilde{\eta}$ provided that S^d is skew symmetric. The negative feedback interconnection of the two systems, namely $\tilde{\lambda} = \lambda_{ext} - \tilde{u}$, $\tilde{v} = u_{ext} + z$, is passive as well from the input (λ_{ext}, u_{ext}) to the output $(z, \tilde{u})$. The output feedback

$$\begin{pmatrix} \lambda_{ext} \\ u_{ext} \end{pmatrix} = -\begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} z \\ \tilde{u} \end{pmatrix}$$

gives asymptotic convergence of the closed-loop system to the largest invariant set where $z = 0$.

We discuss briefly the difficulties related to the presence of possible state constraints. A discussion on the presence of edge capacity constraints and its relation to the output regulation problem with input saturation ([13]) is given in [11].

State constraints. Consider a variation of the model (1) in which the positivity constraint on the amount of material stored at the nodes is enforced. The model becomes $\dot{x} = (B\lambda + Pw)_x^+$ where $(b_i\lambda + p_i w)_{x_i}^+$ is the ith component of the vector $(B\lambda + Pw)_x^+$ and

$$(\zeta_i)_{x_i}^+ = \begin{cases} \zeta_i & \text{if } (x_i > 0) \text{ or } (i = 0 \text{ and } \zeta_i \geq 0) \\ 0 & \text{if } (x_i = 0 \text{ and } \zeta_i < 0) \end{cases}.$$

We consider the special case of balanced demand and supply, i.e. $\mathbf{1}_n^T P w = 0$. As a consequence, $Pw = -B\lambda^w$ and $\dot{x} = \dot{\tilde{x}} = (B\tilde{\lambda})_x^+$. The function $V_1(\tilde{x}) = \frac{1}{2}\tilde{x}^T\tilde{x}$, with $\tilde{x} = x - \mathbf{1}x_*^w$ and $x_*^w > 0$, satisfies $\dot{V}_1(\tilde{x}) = \tilde{x}^T(B\tilde{\lambda})_x^+$.

Observe that $\tilde{x}^T(B\tilde{\lambda})_x^+ = \sum_{i=1}^n \tilde{x}_i(b_i\tilde{\lambda})_{x_i}^+ = \tilde{x}^T(B\tilde{\lambda})$. This shows that the system $\dot{\tilde{x}} = (B\tilde{\lambda})_x^+$, $z = B^T\tilde{x}$is passive and the arguments of the previous remark can be used. A formal analysis requires to take into account the discontinuity of the system. This is not pursued here for lack of space.

IV. Flow networks with nonlinear dynamics at the nodes

In section III, the dynamics describing the evolution of the storage variable at each node was given by

$$\dot{x}_i = b_i\lambda + p_i w, \quad i = 1, 2, \ldots, n \tag{13}$$

where b_i and p_i are the ith row of the incidence matrix B and P respectively. Consider now a different case of a flow network in which the way material accumulates at the node is described by a non-trivial dynamics, namely

$$\dot{x}_i = f_i(x_i) + b_i\lambda + p_i w, \quad i = 1, 2, \ldots, n \tag{14}$$

with vector of measurements given by $y_i = b_i^T x_i \in \mathbb{R}^m$. The nonlinear system (14) allows us to put the results of the paper in a broader context and compare them with those in [2], [10] (see the end of the section). Observe that for $k = 1, 2, \ldots, m$, y_{ik} is either x_i, $-x_i$ or 0. The sum of the outputs y_i over all the nodes returns the vector of relative measurements $z = B^T x = \sum_{i=1}^n y_i$. Each system

$$\begin{array}{rcl} \dot{x}_i & = & f_i(x_i) + b_i\lambda + p_i w \\ y_i & = & b_i^T x_i, \quad i = 1, 2, \ldots, n \end{array} \tag{15}$$

is assumed to be incrementally passive.

Assumption 2 *There exists a regular*[1] *storage function* $V_i : \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\begin{array}{l} \dfrac{\partial V_i}{\partial t} + \dfrac{\partial V_i}{\partial x_i}(f_i(x_i) + b_i\lambda + p_i w) + \\ \dfrac{\partial V_i}{\partial x_i'}(f_i(x_i') + b_i\lambda' + p_i w) \leq (y_i - y_i')^T(\lambda - \lambda'). \end{array}$$

Remark 5 (A class of incrementally passive systems) Consider the linear dynamics at the node (13) and the function $V_i = \frac{1}{2}(x_i - x_i')^2$. Then the right-hand side of the inequality above becomes

$$\begin{array}{l} (x_i - x_i')(b_i\lambda + p_i w) - (x_i - x_i')(b_i\lambda' + p_i w) \\ = (x_i - x_i')b_i(\lambda - \lambda') = (b_i^T(x_i - x_i'))^T(\lambda - \lambda') \\ = (y_i - y_i')^T(\lambda - \lambda') \end{array}$$

and satisfies the dissipation inequality in Assumption 2. Suppose that the dynamics f_i are equal to ∇F_i, with F_i a twice continuously differentiable and concave function. Then the static nonlinearity $-f_i(x_i)$ is incrementally passive, that is $(x_i - x_i')(f_i(x_i) - f_i(x_i')) \leq 0$. As a matter of fact $f_i(x_i) - f_i(x_i') = \nabla F_i(x_i) - \nabla F_i(x_i') = \nabla^2 F_i(\xi_i)(x_i - x_i')$ for some ξ_i lying in the segment connecting x_i, x_i'. By concavity, $\nabla^2 F_i(\xi_i) \leq 0$ and therefore $(x_i - x_i')(f_i(x_i) - f_i(x_i')) \leq 0$. Hence any system (15) with $f_i(x_i) = \nabla F_i(x_i)$ and F_i defined as before satisfies Assumption 2.

Lemma 1 is replaced by the following:

Lemma 3 *For each $i = 1, 2, \ldots, n$, for each w solution to $\dot{w} = S^d w$, there exist a function $\lambda^w : \mathbb{R}_+ \to \mathbb{R}^m$ and continuously differentiable bounded functions $x_i^w : \mathbb{R}_+ \to \mathbb{R}$ that satisfy*

$$\begin{array}{l} \dot{x}_i^w = f_i(x_i^w) + b_i\lambda^w + p_i w, \quad i = 1, 2, \ldots, n \\ 0 = \displaystyle\sum_{i=1}^n b_i^T x_i^w \end{array} \tag{16}$$

only if there exists a solution $x_*^w : \mathbb{R}_+ \to \mathbb{R}$ defined for all $t \geq 0$ to

$$\dot{x}_*^w = \frac{\mathbf{1}_n^T f(x_*^w)}{n} + \frac{\mathbf{1}_n^T P w}{n}, \tag{17}$$

[1] See [16] for a definition.

where $f(x) = (f_1(x) \dots f_n(x))^T$. If this is the case, then $x_i^w = x_*^w$, $i = 1, 2, \dots, n$,

$$\lambda^w = \begin{pmatrix} \lambda_a^w \\ \lambda_b^w \end{pmatrix} = \begin{pmatrix} M_1 f(x_*^w) + M_2 w \\ \mathbf{0} \end{pmatrix}$$

with $\lambda_a^w \in \mathbb{R}^{n-1}, \lambda_b^w \in \mathbb{R}^{m-n+1}$, and M_1, M_2 suitable matrices.

Remark 6 If the inflow and outflow are balanced, i.e. $\mathbf{1}_n^T P w = 0$, then the solution x_*^w to (17) exists for all t and is bounded. In fact, consider the system $\dot{y} = \frac{\mathbf{1}_n^T f(y)}{n}$ and the radially unbounded function $V(y) = \frac{1}{2}y^2$. Then $\dot{V}(y) = y\frac{\mathbf{1}_n^T f(y)}{n} = \sum_{i=1}^n \frac{y f_i(y)}{n}$.
By the incremental passivity property of $-f_i$, $y f_i(y) \leq 0$ for all i and this implies $\dot{V}(y) \leq 0$. Hence every solution to the system above is bounded and so is x_*^w.

Remark 7 The proof of the result, that is omitted and that can be found in [11], shows that in the case the dynamics at the nodes are all the same, i.e. $f_i = f_j$ for all i, j, then the expression of λ^w simplifies as

$$\lambda^w = \begin{pmatrix} \lambda_a^w \\ \lambda_b^w \end{pmatrix} = \begin{pmatrix} M_2 \\ \mathbf{0} \end{pmatrix} w.$$

In the remaining of the section we assume that a solution to (16) exists.
The parallel interconnection of the n subsystems (15) with input λ and output $z = \sum_{i=1}^n y_i$ returns an incrementally passive systems. Formally

Lemma 4 *The parallel interconnection*

$$\begin{array}{rcl} \dot{x}_1 & = & f_1(x_1) + b_1\lambda + p_1 w \\ & \dots & \\ \dot{x}_n & = & f_n(x_n) + b_n\lambda + p_n w \\ z & = & \displaystyle\sum_{i=1}^n b_i^T x_i, \end{array}$$

denoted as

$$\begin{array}{rcl} \dot{x} & = & f(x) + B\lambda + Pw \\ z & = & B^T x \end{array} \tag{18}$$

is such that the storage function $V(x, x') = \sum_{i=1}^n V_i(x_i, x_i')$ satisfies

$$\frac{\partial V}{\partial x}(f(x) + B\lambda + Pw) + \frac{\partial V}{\partial x'}(f(x') + B\lambda' + Pw) \leq (z - z')^T(\lambda - \lambda').$$

The proof is straightforward and is omitted. Consider now systems of the form

$$\begin{array}{rcl} \dot{\eta}_k & = & \phi_k(\eta_k, v_k) \\ u_k & = & \psi_k(\eta_k), \quad k = 1, 2, \dots, n-1, \end{array} \tag{19}$$

with the following two additional properties:

Assumption 3 *For each $k = 1, 2, \dots, n-1$, there exists regular functions $W_k(\eta_k, \eta_k')$ such that*

$$\frac{\partial W_k}{\partial \eta_k}\phi(\eta_k, v_k) + \frac{\partial W_k}{\partial \eta_k'}\phi(\eta_k', v_k') \leq (u_k - u_k')(v_k - v_k').$$

Assumption 4 *For each $k = 1, 2, \dots, n-1$, for each w solution to $\dot{w} = S^d w$, there exists a bounded solution η_k^w to $\dot{\eta}_k = \phi_k(\eta_k, 0)$ such that $\lambda_k^w = \psi_k(\eta_k^w)$.*

Assume that the system

$$\dot{\eta}_{ka}^w = \frac{\mathbf{1}_n^T f(\eta_{ka}^w)}{n} + \frac{\mathbf{1}_n^T P \eta_{kb}^w}{n}, \quad \dot{\eta}_{kb}^w = S^d \eta_{kb}^w$$

is forward complete. Initialize the system as $\eta_{ka}^w(0) = x_*^w(0)$ and $\eta_{kb}^w(0) = w(0)$. Then $\eta_{ka}^w(t) = x_*^w(t)$ and $\eta_{kb}^w(t) = w(t)$ for all $t \geq 0$. Hence $\lambda_k^w = M_{1k} f(\eta_{ka}) + M_{2k}\eta_{kb}$, $k = 1, 2, \dots, n-1$, where M_{1k} and M_{2k} are the kth rows of M_1 and M_2 respectively. On the other hand, $\lambda_k^w = 0$, $k = n, n+1, \dots, m$. An expression for ϕ_k, ψ_k, $k = 1, 2, \dots, n-1$ is

$$\phi_k(\eta_k, 0) = \begin{pmatrix} \dfrac{\mathbf{1}_n^T f(\eta_{ka})}{n} + \dfrac{\mathbf{1}_n^T P \eta_{kb}}{n} \\ S^d \eta_{kb} \end{pmatrix},$$

$\psi_k(\eta_k) = M_{1k} f(\eta_{ka}) + M_{2k}\eta_{kb}$.
In the special case of nodes with the same dynamics ($f_i = f_j = \bar{f}$ for all i, j) $\psi_k(\eta_k)$ simplifies as $M_{2k}\eta_{kb}$ and a system that satisfies Assumptions 3 and 4 is

$$\begin{array}{rcl} \dot{\eta}_k & = & S^d \eta_k + M_{2k}^T v_k \\ u_k & = & M_{2k}\eta_k, \end{array}$$

with storage function $W_k(\eta_k) = \frac{1}{2}\eta_k^T \eta_k$. Collect the systems (19) into a system with state variable $\eta = (\eta_1^T \dots \eta_{n-1}^T)^T$, input $v = (v_1 \dots v_m)^T$ and output $u = (u_1 \dots u_m)^T$, namely

$$\begin{array}{rcl} \dot{\eta} & = & \Phi(\eta, v) \\ u & = & \Psi(\eta) \end{array} \tag{20}$$

with $\Phi(\eta, v) = (\phi_1^T \dots \phi_{n-1}^T)^T$, $\Psi(\eta) = (\psi_1 \dots \psi_{n-1}\ \mathbf{0}^T)^T$. The system is incrementally passive from v to u with storage function $W(\eta, \eta') = \sum_{k=1}^{n-1} W_k(\eta_k, \eta_k')$.
The following holds:

Theorem 2 *Let Assumptions 1-4 hold. Suppose that a solution to (16) exists and x_*^w is bounded. Consider the systems (18), with input λ and output z, and (20), with input v and output u, interconnected via the relations $v = -z + v_{ext}, \quad \lambda = u + \lambda_{ext}$.*
The interconnected system is incrementally passive from the input $(\lambda_{ext}^T\ v_{ext}^T)^T$ to the output $(z^T\ u^T)^T$. Moreover, the feedback $(\lambda_{ext}^T\ v_{ext}^T)^T = (-Kz^T\ \mathbf{0}^T)^T$, with K a positive definite diagonal matrix, guarantees $\lim_{t\to+\infty} z(t) = \mathbf{0}$.

Analogously to the case of linear systems, the result shows that distributed internal-model-based controllers exist that achieve load balancing in the presence of time-varying inflow-outflow. The following consequence further discusses the result for a special class of systems.

Corollary 1 *If (i)* $f_i = \bar{f}$ *for all* $i = 1, 2, \ldots, n$*, (ii) there exists a twice continuously differentiable convex function* $F(x)$ *such that* $\nabla F(x) = \bar{f}(x)$ *and (iii)* $\mathbf{1}_n^T P w = 0$ *for all* $t \geq 0$*, then the controllers*

$$\begin{array}{rcl} \dot{\eta}_k & = & S^d \eta_k - M_{2k}^T z_k \\ \lambda_k & = & M_{2k}\eta_k - z_k, \quad k = 1, 2, \ldots, n-1, \end{array}$$

and $\lambda_k = -z_k$*,* $k = n, n+1, \ldots, m$*, guarantee* $\lim_{t \to +\infty} z(t) = \mathbf{0}$.

The closed-loop system given in the corollary above takes the form

$$\begin{array}{rclrcl} \dot{x} & = & \nabla F(x) + B\lambda + Pw, & z & = & B^T x \\ \dot{\eta} & = & \overline{S}\eta - M_2^T z, & \lambda & = & M_2 \eta - z\ , \end{array}$$

where we are assuming that $m = n - 1$ for the sake of simplicity. This system can be compared with similar ones appeared in the recent literature ([2], [10]), where models of the form

$$\begin{array}{rclrcl} \dot{x} & = & \nabla F(x) + B\lambda, & z & = & B^T x \\ \dot{\eta} & = & z, & \lambda & = & -\psi(\eta) \end{array}$$

with ψ a non-decreasing monotonic non-linearity (such as a saturation function), were studied. The presence of the non-trivial dynamics $\overline{S}$ in our controller is due to the time varying-nature of the external input. In [2], $\nabla F(x)$ has a unique equilibrium at the origin and the system $\dot{x} = \nabla F(x) + B\lambda$ is strictly passive. In [10] it is shown that if the components of the vector field $\nabla F(x)$ have different equilibria, $\nabla F(x)$ is strongly concave and ψ introduces saturation constraints, then the system's response exhibits state clustering.

V. Conclusions

We have presented an internal model approach to the problem of balancing demand and supply in a class of distribution networks. Extensions to nonlinear systems have also been discussed. Compared with other papers where the robustness to time-varying inputs is studied using a frequency domain approach ([4]), our state space approach allows us to consider more general classes of cooperative control systems.

Further research will focus on a detailed investigation of state and input constraints and more complex models of demand and supply. The fulfillment of the internal model principle has to be understood for more general classes of nonlinear systems than those in Corollary 1. This will shed light on the relation between the results in this paper and the saddle-point perspective of [10]. Other results that connect our approach to optimal network flow control problems are discussed in [9].

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