

# Differential flatness of a class of $n$ -DOF planar manipulators driven by an arbitrary number of actuators

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**Abstract**—This paper deals with the problem of feedback linearization, either by dynamic or static feedback, of a planar robot manipulator with  $n$  links and  $m$  inputs. The robot is provided with a particular inertia distribution in order to reduce nonlinearities, and with  $n - m$  springs to keep controllability. Under these design conditions, the paper answers the question of where to place the actuators such that the system is static feedback linearizable. For those systems that there are not static feedback linearizable, the paper shows that are linearizable by prolongations and, therefore, dynamic feedback linearizable.

## I. INTRODUCTION

In references ([1], [2], [5], [9]) the authors have studied the same structure of a planar robot manipulator with only one or two inputs. This paper generalizes the results to the general multi-input case.

We keep the same design conditions as the previous papers: the center of mass of the last link  $n$  lies on the preceding joint axis  $n$ ; the center of mass of the last two links  $n$  and  $n - 1$  lies on the joint axis  $n - 1$ ; this procedure is iterated until the center of mass of the last  $n - 2$  links lies on the second joint axis. In order to keep the controllability property, torsional springs are added at each joint where there is no any actuator. This design eliminates many of the nonlinearities arising out of inertia function. The locations of the center of masses are obtained by adding counter masses to each of the last  $n - 2$  links.

Even if the manipulator is designed in such a way, it was already proven in [5] that the placement of the actuators is important in order to guarantee feedback linearization of such a system with two inputs. For the case studied here, the location of the inputs is also relevant for the robot to have the property of feedback linearization.

A system which is static feedback linearizable or linearizable by prolongations is, also, differentially flat [3]. In any under-actuated system that possesses the property of differential flatness, the problem of point to point trajectory planning or trajectory tracking can be solved, i.e. the nonlinear system is transformed into a linear one by means of a diffeomorphism and a feedback law. Moreover, the resulting linear system is in Brunovsky canonical form and, therefore, the problem

of trajectory planning reduces to find interpolating polynomials (or any other basis functions) for the initial and final conditions.

The paper is structured as follows: In Section II, a description to the problem leading to the state space equations is given. Section III recalls some technical lemmas that will be used through the paper. Section IV introduces a matricial notation to study static feedback linearization and Section V studies the rank of these matrices. In Section VI and Section VII, we study static feedback linearization and linearization by prolongations using the matricial notation. Finally, Section VIII, there is a simulation of a specific system and Section IX summarizes the mains conclusions of the work.

## II. DESCRIPTION OF THE PROBLEM

The dynamic equations of motion for open-chain robots are given by the following:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = u, \quad (1)$$

where  $M(q)$  is the inertia matrix,  $C(q, \dot{q})$  are the Coriolis and Centripetal terms,  $g(q)$  are the gravity terms, and  $u$  is the vector of joint inputs. It is also well known from Lagrangian structure of the dynamic equations, if the inertia matrix  $M(q)$  is independent of  $q$ , the Coriolis term  $C(q, \dot{q}) = 0$ .

According to the special inertia distribution explained in ([5]), the  $(n \times n)$  inertia mass matrix  $M(q)$  is a constant and has a specific structure given by

$$M = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ a_2 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n-1} & a_{n-1} & a_{n-1} & \dots & a_{n-1} & a_n \\ a_n & a_n & a_n & \dots & a_n & a_n \end{pmatrix}, \quad (2)$$

where  $a_i$  are constants for all  $i = 1, \dots, n$ .

On the other hand,  $C(q, \dot{q})$  vanishes since  $M$  is a constant,  $g(q)$  is still non-linear since the center of mass is on joint 2. In order to preserve controllability of the system, we add a spring in every un-actuated joint except when it occurs on joint 1. The addition of springs provides dynamic coupling between successive joints, which helps preserve controllability of the robot. Summarizing, robots have the following dynamic equations of motion:

$$M\ddot{q} = \begin{pmatrix} -mg \sin q_1 + u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, \quad (3)$$

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where  $u_i$  is substituted by  $k_i q_i$  if  $u_i = 0$ , for all  $i \geq 2$ . On multiplying by the inverse of the matrix  $M$ , and considering a  $2n$ -dimensional system of first order differential equations (positions and velocities), the state space equations are obtained:

$$\dot{x} = \begin{pmatrix} x_{n+1} \\ \vdots \\ x_{2n} \\ M^{-1}b \end{pmatrix} + \sum_{i=1}^n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ M^{-1}e_i \end{pmatrix} u_i, \quad (4)$$

where  $x = (q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n)$ ,  $b = (-mg \sin x_1, k_2 x_2, \dots, k_n x_n)^T \in R^n$ , and  $e_i$  stands for the  $i$ th vector of the canonic base of  $R^n$ . Let us remark that only some of the  $u_i$  will be nonzero and as stated before,  $k_i = 0$  if  $u_i \neq 0$ . A simple computation shows that

$$M^{-1} = \begin{pmatrix} \lambda_1 & -\lambda_1 & 0 & \dots & 0 \\ -\lambda_1 & \lambda_1 + \lambda_2 & -\lambda_2 & \dots & 0 \\ 0 & -\lambda_2 & \lambda_2 + \lambda_3 & \dots & 0 \\ 0 & 0 & -\lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\lambda_{n-1} \\ 0 & 0 & 0 & \dots & -\lambda_{n-1} \lambda_n \end{pmatrix}$$

where  $\lambda_i = \frac{1}{a_i - a_{i+1}}$ ,  $\forall i < n$ ,  $\lambda_n = -\frac{a_{n-1}}{a_n}$ . The drift vector field is  $f(x) =$

$$\begin{pmatrix} x_{n+1} \\ \vdots \\ x_{2n} \\ \lambda_1 G_1(x_1) - \lambda_1 k_2 x_2 \\ -\lambda_1 G_1(x_1) + (\lambda_1 + \lambda_2) k_2 x_2 - \lambda_2 k_3 x_3 \\ -\lambda_2 k_2 x_2 + (\lambda_2 + \lambda_3) k_3 x_3 - \lambda_3 k_4 x_4 \\ \vdots \\ -\lambda_{n-2} k_{n-2} x_{n-2} + \dots - \lambda_{n-1} k_n x_n \\ -\lambda_{n-1} k_{n-1} x_{n-1} - \lambda_{n-1} \lambda_n k_n x_n \end{pmatrix}$$

with  $G_1(x_1) = -mg \sin x_1$ . The input vector fields are

$$\begin{aligned} g_i &= -\lambda_{i-1} \frac{\partial}{\partial x_{n+i-1}} + (\lambda_{i-1} + \lambda_i) \frac{\partial}{\partial x_{n+i}} - \lambda_i \frac{\partial}{\partial x_{n+i+1}} \\ \forall 1 < i < n \\ g_1 &= \lambda_1 \left( \frac{\partial}{\partial x_{n+1}} - \frac{\partial}{\partial x_{n+2}} \right) \\ g_n &= -\lambda_{n-1} \left( \frac{\partial}{\partial x_{2n-1}} + \lambda_n \frac{\partial}{\partial x_{2n}} \right) \end{aligned}$$

### III. MATHEMATICAL BACKGROUND

This section is devoted to some technical lemmas that will be useful in proving the main results of the following sections.

**Lemma 1:** 1)  $ad_f^2 g_i = -\lambda_{i-1} k_{i-1} g_{i-1} + (\lambda_{i-1} + \lambda_i) k_i g_i - \lambda_i k_{i+1} g_{i+1}$ ,  $\forall 2 < i < n$ .  
2)  $ad_f^2 g_1 = \lambda_1 \frac{\partial G_1(x_1)}{\partial x_1} g_1 - \lambda_1 k_2 g_2$ .  
3)  $ad_f^2 g_2 = -\lambda_1 \frac{\partial G_1(x_1)}{\partial x_1} g_1 + (\lambda_1 + \lambda_2) k_2 g_2 - \lambda_2 k_3 g_3$ .  
4)  $ad_f^2 g_n = k_{n-1} g_{n-1} - \lambda_{n-1} \lambda_n k_n g_n$ .

The proof can be found in [5].

Since the main objective of this paper is to study under what choices of inputs is the designed robot static feedback linearizable or linearizable by simple prolongations, we recall here the conditions for such properties to be fulfilled.

**Theorem 2:** A nonlinear system  $\dot{x} = f(x) + \sum_{i=1}^m g_i(x) u_i$ , is static feedback linearizable (SFL) if and only if ([7], [6], [8]) the following distributions have constant rank and are involutive:

$$\begin{aligned} D_0 &= \langle g_1, \dots, g_m \rangle \\ D_i &= \langle D_{i-1}, ad_f^i g_1, \dots, ad_f^i g_m \rangle \quad i = 1, \dots, n-2 \end{aligned}$$

and the rank of  $D_{n-1}$  is  $n$ .

Given a nonlinear system  $\dot{x} = f(x) + \sum_{i=1}^m g_i(x) u_i$ , a simple prolongation of this system is

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^m g_i(x) u_i^0 \\ \dot{u}_i^0 &= u_i^1 \quad \forall i = 1 \dots m \\ &\vdots \\ \dot{u}_i^{k_i-1} &= v_i, \end{aligned}$$

where  $u_i^j$ , that correspond to  $u_i^{(j)}$ , are new state variables  $\forall i = 1 \dots m$   $j = 0 \dots k_i - 1$  and the new inputs are  $v_i$ . The system is said to be linearizable by prolongations ([4]) if there exists a simple prolongation of the original system which is static feedback linearizable.

### IV. RESULTS AND NOTATION

We begin the section with a result that will be key for all the work:

**Theorem 3:** For a system with  $n$  joints and  $m$  actuators  $u_{i_1}, \dots, u_{i_m}$ , the system is static feedback linearizable, if only if,  $g_1 \in D_{2(i_1-1)}$ .

*Proof:* The conditions for a system to be static feedback linearizable are given in Theorem 2, and reduces to check the involutivity and constant rank of the distributions associated with the system. On studying the successive distributions:

$$\begin{aligned} D_0 &= \langle g_{i_1}, g_{i_2}, \dots, g_{i_m} \rangle \\ D_{2(1)} &= \langle D_1, \sum_{l_1=i_1-1}^{i_1+1} \mu_{l_1}^1 g_{l_1}, \sum_{l_2=i_2-1}^{i_2+1} \mu_{l_2}^1 g_{l_2}, \dots, \\ &\quad \sum_{l_m=i_m-1}^{i_m+1} \mu_{l_m}^1 g_{l_m} \rangle \\ D_{2(2)} &= \langle D_3, \sum_{l_1=i_1-2}^{i_1+2} \mu_{l_1}^2 g_{l_1}, \sum_{l_2=i_2-2}^{i_2+2} \mu_{l_2}^2 g_{l_2}, \dots, \\ &\quad \sum_{l_m=i_m-2}^{i_m+2} \mu_{l_m}^2 g_{l_m} \rangle \\ &\vdots \\ D_{2(i_1-1)} &= \langle D_{2i_1-3}, \sum_{l_1=1}^{2i_1-1} \mu_{l_1}^{i_1-1} g_{l_1}, \sum_{l_2=i_2-i_1+1}^{i_2+i_1-1} \mu_{l_2}^{i_1-1} g_{l_2}, \\ &\quad \dots, \sum_{l_m=i_m-i_1+1}^{i_m+i_1-1} \mu_{l_m}^{i_1-1} g_{l_m} \rangle \end{aligned} \quad (5)$$

where  $\mu_j^i$  are nonzero constants resulting from the product of the constants  $\lambda$ 's and constants  $k$ 's. All distributions, except  $D_{2(i_1-1)}$ , contain only constant vector fields, and therefore,

they are involutive. Consequently, it is only necessary to study the involutivity of  $D_{2(i_1-1)}$ . Hence, we need to show

$$D_{2(i_1-1)} \text{ involutive} \iff g_1 \in D_{2(i_1-1)}$$

$\implies$

Assume the system is SFL. We want to see that  $g_1 \in D_{2(i_1-1)}$ . If the system is static feedback linearizable,  $D_{2(i_1-1)}$  is involutive by [6]. So, it is enough to find two elements of the distribution whose Lie bracket will be a multiple of  $g_1$ .

Note that all Lie brackets between elements of  $D_{2(i_1-1)}$  are 0, because all of them are constant, with the following exception: The Lie bracket between  $ad_f^{2(i_1-1)}g_{i_1} \in D_{2(i_1-1)}$  and  $ad_f^{2(i_1-1)-1}g_{i_1} \in D_{2i_1-3}$  then

$$[ad_f^{2(i_1-1)}g_{i_1}, ad_f^{2(i_1-1)-1}g_{i_1}] = k \frac{\partial^2 G_1(x_1)}{\partial x_{i_1}^2} g_1$$

where  $k$  is a constant. Hence, since the system is involutive,  $g_1 \in D_{2(i_1-1)}$ .

$\longleftarrow$

Conversely, assume  $g_1 \in D_{2(i_1-1)}$ . As shown before, all Lie brackets between elements of  $D_{2(i_1-1)}$  are 0, except  $[ad_f^{2(i_1-1)}g_{i_1}, ad_f^{2(i_1-1)-1}g_{i_1}]$ , which is a multiple of  $g_1$ .

Since, by hypotheses,  $g_1 \in D_{2(i_1-1)}$ , the distributions are involutive and, therefore, the system is static feedback linearizable. ■

In theorem 3, we have seen that a system is static feedback linearizable if only if  $g_1 \in D_{2(i_1-1)}$ . Consequently, we are going to study these distributions in order to know whether or not  $g_1 \in D_{2(i_1-1)}$ . To achieve it, we are going to write down all distributions associated with the system as matrices, where the matrix rows contain combinations of vector fields  $g$ 's. First, it is important to observe the distributions until  $D_{2(i_1-1)}$  shown in Theorem 3.

Note that  $g_1$  appears for the very first time in  $D_{2(i_1-1)}$  as a combination with other vector fields. On the other hand, some  $\mu$  constants are 0 because  $k_j = 0$  when there is an actuator in the  $j$ -joint. For this reason, the following notation is introduced:

$$\begin{aligned} t_1 &= 1 \\ t_j &= \max(i_j - i_1 + 1, i_{j-1} + 1) \quad \forall j \in [2, m] \\ t^j &= \min(i_j + i_1 - 1, i_{j+1} - 1) \quad \forall j \in [1, m-1] \\ t^m &= \min(i_m + i_1 - 1, n) \end{aligned}$$

Using this notation, (5) can be rewritten as:

$$D_{2(i_1-1)} = \langle D_{2i_1-3}, \sum_{l_1=t_1}^{t^1} \mu_{l_1}^{i_1-1} g_{l_1}, \sum_{l_2=t_2}^{t^2} \mu_{l_2}^{i_1-1} g_{l_2}, \dots, \sum_{l_m=t_m}^{t^m} \mu_{l_m}^{i_1-1} g_{l_m} \rangle$$

**Definition 1:** Given a system with  $n$  joints and  $m$  actuators  $u_{i_1}, \dots, u_{i_m}$ , let us define  $M_0$  as the associated matrix with the  $D_0$  distribution.  $M_0 =$

$$\begin{pmatrix} 0 & \dots & 0 & 1 & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 & 1 & 0 & \dots & \dots & \dots & 0 \\ \vdots & & & & & & & & & & \vdots \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & 1 & 0 & \dots & 0 \end{pmatrix}_{m \times n}$$

and define  $G$  as the matrix.

$$G = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_{n-1} \\ g_n \end{pmatrix}_{n \times 1}$$

**Remark 1:** Using definition 1,  $D_0$  can be built as:

$$D_0 = \langle g_{i_1}, g_{i_2}, \dots, g_{i_m} \rangle = \langle \{\text{rows of } M_0 G\} \rangle$$

**Definition 2:** Extending the definition to the  $D_{2(1)}$  distribution, let us define  $M_1 =$

$$\begin{pmatrix} 0 & \dots & \dots & \mu_{i_1-1}^1 & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & 0 & \mu_{i_1-1}^1 & 0 & \mu_{i_1+1}^1 & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 & \mu_{i_2+1}^1 & 0 & \mu_{i_2+1}^1 & 0 & \dots & 0 \\ \vdots & & & & & & & & & & \vdots \\ 0 & \dots & \dots & \dots & \dots & \dots & \mu_{i_m-1}^1 & 0 & \mu_{i_m+1}^1 & \dots & 0 \end{pmatrix}$$

as a  $(2m \times n)$  matrix and consequently,

$$D_2 = \langle D_1, \{\text{rows of } M_1 G\} \rangle$$

**Definition 3:** Analogously,  $M_{i_1-1}$  is defined as the associated matrix with  $D_{2(i_1-1)}$  distribution (where  $g_1$  appears for the first time in combination with other vector fields).

$M_{i_1-1} =$

$$\begin{pmatrix} \boxed{C_{i_1}} \\ \boxed{C_{i_2}} \\ \dots \\ \boxed{C_{i_m}} \end{pmatrix}_{mi_1 \times n}$$

where the  $C_{i_j}$  boxes are obtained using lemma 1.

**Remark 2:** The  $M_{i_1-1}$  matrix and the  $C_{i_j}$  boxes satisfy the following conditions:

- (I) The first column of  $M_{i_1-1}$  is not empty, since  $g_1$  appears in  $D_{2(i_1-1)}$  in combination with other vector fields.
- (II)  $\text{Dim}(M_{i_1-1}) = (mi_1 \times n)$ , because it is made of  $m$  boxes (one for each actuator) of  $i_1$  rows. The number of columns equals the number of joints.
- (III) If  $i_m + i_1 - 1 < n$ , the matrix  $M_{i_1-1}$  has zero columns. Hence, we consider

$$\text{Dim}(M_{i_1-1}) = (mi_1 \times t^m - t_1 + 1) = (mi_1 \times t^m)$$

- (IV) As a consequence, we redefine  $G$  as:

$$G = \begin{pmatrix} g_1 \\ \vdots \\ g_{t^m} \end{pmatrix}_{t^m \times 1}$$

- (V)  $D_{2(i_1-1)} = \langle D_{2i_1-3}, \{\text{rows of } M_{i_1-1} G\} \rangle$

- (VI)  $\text{Dim}(C_{i_j}) = (i_1 \times t^j - t_j + 1)$ .

Using this notation, we can state all distributions as a product of matrices. The importance of the rank of  $M_{i_1-1}$  to study static feedback linearizability of the systems will be highlighted in the next section.

## V. RANK OF $M_{i_1-1}$

**Definition 4:** We say there is a gap between 2 actuators  $u_{i_j}$  and  $u_{i_{j+1}}$  if  $i_{j+1} - i_j > 2i_1 - 2$ .

**Definition 5:** Let have a system with a gap between  $u_{i_j}$  and  $u_{i_{j+1}}$ , then we define  $M_{i_1-1}^j$  as a submatrix of  $M_{i_1-1}$ , where  $M_{i_1-1}^j$  is made of  $C_{i_l} \forall l = 1, \dots, j$  boxes.

$$\dim(M_{i_1-1}^j) = (j i_1 \times t^j - t_1 + 1)$$

**Definition 6:** In this case, a new matrix  $G^j$  is defined as the matrix with all vectors fields that appear in  $M_{i_1-1}^j$ , i.e. vector fields  $g_1$  to  $g_{t^j}$ , and  $\dim(G^j) = (t^j - t_1 + 1 \times 1)$ .

**Remark 3:** If there is a gap between  $u_{i_j}$  and  $u_{i_{j+1}}$ , then  $C_{i_j}$  and  $C_{i_{j+1}}$  have no common vector fields.

**Proposition 1:** Consider a system with a gap between  $u_{i_j}$  and  $u_{i_{j+1}}$ .

The system is SFL  $\iff g_1 \in \langle \{\text{rows of } M_{i_1-1}^j G^j\} \rangle$

**Theorem 4:** Given a system with  $n$  joints and  $m$  actuators  $u_{i_1}, \dots, u_{i_m}$ .

- **Without gaps**, if  $\text{Rank}(M_{i_1-1}) \geq t^m$ , then the system is static feedback linearizable.
- **With a gap between  $u_{i_j}$  and  $u_{i_{j+1}}$** , if  $\text{Rank}(M_{i_1-1}^j) \geq t^j$ , then the system is static feedback linearizable.

However, this result neither establishes the rank of  $M_{i_1-1}$  or  $M_{i_1-1}^j$ , nor says what happens when these conditions are not met.

### A. Rank of the $C_{i_j}$ boxes

**Definition 7:** We define  $R_j$  as the rank of  $C_{i_j}$ .

$$R_j = \text{Rank}(C_{i_j})$$

**Proposition 2:**

$$R_j = \begin{cases} \min(i_1, i_{j+1} - i_{j-1} - 1) & \forall j \in [1, m-1] \\ \min(i_1, n - i_{m-1}) & j = m \end{cases}$$

Due to the form of  $C_{i_j}$  boxes and using lemma 1.

### B. Rank of the $C_{i_j}$ boxes with an actuator in the last joint

**Remark 4:** It is interesting to consider systems with an actuator in the last joint ( $i_m = n$ ). When this occurs,  $C_{i_m}$  is an antidiagonal matrix with 1 in the antidiagonal. Hence, all vector fields  $g_j \in D_{2(i_1-1)} \forall j \in [d, n]$  where

$$d = \max(i_{m-1} + 1, i_m - i_1 + 1)$$

**Theorem 5:** Given a system with an actuator in the last joint and let  $k$  be the highest index satisfying  $i_{k+1} - i_k > i_1$ , then:

- If  $k > 1$ , then  $R_k = \min(i_1, i_{k+1} - i_1 - i_{k-1})$  and  $t^k = \min(i_k + i_1 - 1, i_{k+1} - i_1)$ .
- However, if  $k = 1$ , then the system is static feedback linearizable.

### C. Rank of the $M_{i_1-1}^j$ matrices

**Definition 8:** Let  $Ra_j = \sum_{l=1}^j R_l$ , the accumulated sum of ranks of the  $C_{i_j}$  boxes of  $M_{i_1-1}^j$ .

**Definition 9:** Let  $\#G^j$  the number of rows of  $G^j$ , i.e.  $\#G^j = t^j$ .

**Theorem 6:**  $\text{Rank}(M_{i_1-1}^j) = \min(\#G^j, Ra_j)$ .

*Proof:* As  $C_{i_k}$  has rank  $R_k$ , then  $C_{i_k}$  has  $R_k$  rows linearly independent. As each row of  $M_{i_1-1}^j$  has only elements of one  $C_{i_k}$  box, therefore  $M_{i_1-1}^j$  has  $Ra_j$  independent rows. Consequently  $M_{i_1-1}^j = \min(\#G^j, Ra_j)$  ■

With this result, we finish the study of the  $M_{i_1-1}^j$  matrix ranks and we can state necessary and sufficient conditions to study static feedback linearizability.

## VI. STATIC FEEDBACK LINEARIZABILITY FOR M-INPUT SYSTEMS

**Theorem 7:** Consider a system with  $m$  inputs and  $n$  joints. Define  $k$  as follows:

- $k$  is the lowest index such that there is a gap between  $u_{i_k}$  and  $u_{i_{k+1}}$ .
- If there is no gap, and if  $i_m = n$ ,  $k$  is the highest index satisfying  $i_{k+1} - i_k > i_1$ .
- Otherwise  $k = m$ .

Then, for this index  $k$ :

**The system is SFL**  $\iff Ra_j \geq \#G^j$  for some  $j \in [1, k]$

**Remark 5:** Theorem 7 says that a system is SFL, if only if, some submatrix of  $M_{i_1-1}^j$  has maximum rank.

### A. Algorithm of theorem 7

- 1) Detect witch actuators are needed to study static feedback linearizability.
- 2) Write the actuators position in the  $i_j$  row.
- 3) Write the rank of the  $C_{i_j}$  box in the  $R_j$  row.
- 4) Write the accumulated rank of the  $C_{i_j}$  boxes in the  $Ra_j$  row.
- 5) Write the number of vector fields of  $G^j$  in the  $\#G^j = t^j$  row.
- 6) If  $\exists j \in \{1, \dots, k\} \mid Ra_j \geq \#G^j$ , where  $k$  is the index of theorem 7, then the system is static feedback linearizable.

**Example 1:** Consider a system with 24 joints and 5 actuators at joints  $i_1 = 5, i_2 = 7, i_3 = 10, i_4 = 14$  and  $i_5 = 23$ .

Note that the system has a gap between  $u_{i_4}$  and  $u_{i_5}$ . Hence, we do not need to use  $u_{i_5}$  to study static feedback linearizability. To know whether or not the system is static feedback linearizable, we need to compute  $R_j, Ra_j$  and  $\#G^j$ .

	$i_1$	$i_2$	$i_3$	$i_4$
$i_j$	5	7	10	14
$R_j$	5	4	5	5
$Ra_j$	5	9	14	19
$\#G^j$	6	9	13	18

We can see that  $M_{i_1-1}^2$  has maximum rank, hence the system is static feedback linearizable.  $M_{i_1-1}^3$  and  $M_{i_1-1}^4$  have maximum rank as well. But to check static feedback linearizability is enough satisfying  $Ra_j \geq \#G^j$  one time.

## VII. DYNAMIC FEEDBACK LINEARIZABILITY FOR M-INPUT SYSTEMS BY PROLONGATIONS

**Theorem 8:** Any not static feedback linearizable system with  $n$  joints and  $m$  actuators  $u_{i_1}, u_{i_2}, \dots, u_{i_m}$  with  $m > 1$  is linearizable by prolongations. The following prolongations make any system, not static feedback linearizable, linearizable by prolongations:

If the system has 3 or more actuators,

$$2(\max(1, i_3 - 2i_1 - 1), 0, \dots, 0)$$

While if the system has 2 actuators,

$$2(\max(1, n - 2i_1), 0, \dots, 0)$$

*Example 2:* Let the system be with 7 joints and 2 actuators on joints  $i_1 = 2$  and  $i_2 = 4$ . Using Theorem 8, consider the prolongation:

$$2(\max(1, n - 2i_1), 0, \dots, 0) = 2(3, 0)$$

for a system with 2 actuators. With this prolongation, the prolonged drift vector field is :

$$\bar{f}(x) = \begin{pmatrix} x_8 \\ \vdots \\ x_{14} \\ u_{i_1}^1 \\ u_{i_1}^3 \\ u_{i_1}^5 \\ u_{i_1}^7 \\ \lambda_1 G_1(x_1) - \lambda_1 u_{i_1} \\ -\lambda_1 G_1(x_1) - \lambda_2 k_3 x_3 + (\lambda_1 + \lambda_2) u_{i_1} \\ + (\lambda_2 + \lambda_3) k_3 x_3 - \lambda_2 u_{i_2} \\ -\lambda_3 k_3 x_3 + (\lambda_{n-2} - \lambda_4 k_5 x_5 \\ + (\lambda_4 + \lambda_5) k_5 x_5 - \lambda_5 k_6 x_6 \\ -\lambda_5 k_5 x_5 + (\lambda_5 + \lambda_6) k_6 x_6 - \lambda_6 k_7 x_7 \\ -\lambda_6 k_6 x_6 - \lambda_6 \lambda_7 k_7 x_7 \\ u_{i_1}^2 \\ u_{i_1}^4 \\ 0 \end{pmatrix}$$

where  $u_{i_1}^k = \frac{d^k}{dt^k} u_{i_1}$ ,  $k = 0, \dots, 5$ , are the 6 new state variables. On the other hand, the input vector fields  $\bar{g}$  are:

$$\begin{aligned} \bar{g}_4 &= -\lambda_3 \frac{\partial}{\partial x_{13}} + (\lambda_3 + \lambda_4) \frac{\partial}{\partial x_{14}} - \lambda_4 \frac{\partial}{\partial x_{15}} \\ \bar{g}_{10} &= \frac{\partial}{\partial u_{i_1}^5} \end{aligned}$$

while the remaining vector fields are:

$$\bar{g}_i = -\lambda_{i-1} \frac{\partial}{\partial x_{9+i}} + (\lambda_{i-1} + \lambda_i) \frac{\partial}{\partial x_{10+i}} - \lambda_i \frac{\partial}{\partial x_{11+i}}$$

for  $i = 2, \dots, 6$ . On the other hand,

$$\bar{g}_1 = \lambda_1 \left( \frac{\partial}{\partial x_{11}} - \frac{\partial}{\partial x_{12}} \right), \quad \bar{g}_7 = -\lambda_6 \left( \frac{\partial}{\partial x_{16}} - \lambda_7 \frac{\partial}{\partial x_{17}} \right)$$

On computing the distributions of the prolonged system,

$$\begin{aligned} \bar{D}_0 &= \langle \bar{g}_4, \bar{g}_{10} \rangle \\ \bar{D}_2 &= \langle \bar{D}_1, -\lambda_3 k_3 \bar{g}_3 - \lambda_4 k_5 \bar{g}_5, \bar{g}_9 \rangle \\ \bar{D}_4 &= \langle \bar{D}_3, \mu_3^2 \bar{g}_3 + \mu_5^2 \bar{g}_5 + \mu_6^2 \bar{g}_6, \bar{g}_8 \rangle \\ \bar{D}_6 &= \langle \bar{D}_5, \bar{g}_2, \mu_3^3 \bar{g}_3 + \mu_5^3 \bar{g}_5 + \mu_6^3 \bar{g}_6 + \mu_7^3 \bar{g}_7 \rangle \\ \bar{D}_8 &= \langle \bar{D}_7, -\lambda_1 \frac{\partial G_1(x_1)}{\partial x_1} \bar{g}_1 - \lambda_2 k_3 \bar{g}_3, \\ &\quad \mu_3^4 \bar{g}_3 + \mu_5^4 \bar{g}_5 + \mu_6^4 \bar{g}_6 + \mu_7^4 \bar{g}_7 \rangle \end{aligned}$$

$\bar{D}_8$  contains 10 independent combinations of the 10 vector fields  $\bar{g}_i$ ; then the prolonged system is static feedback linearizable, and the original system is linearizable by prolongations. Using theorem 7 we can see :

	$i_1$	$i_2$
$i_j$	2	4
$R_j$	2	5
$Ra_j$	2	7
$\#G^j$	3	7

As  $\bar{R}a_2 = \#G^2$  the system is linearizable by prolongations..

## VIII. SIMULATIONS

In this section, we are going to perform a simulation for a given system. For a system that is static feedback linearizable, we can find a diffeomorphism and feedback law, so that the original system becomes a linear and controllable one.

Let us consider a system of 7 joints and 3 actuators located at joints 3, 5 and 7. Following the algorithm given in Theorem 7, it is a straightforward computation to check that this system is static feedback linearizable. For this system, the equation of motion is:

$$M\ddot{q} = \begin{pmatrix} -mg \sin(q_1) \\ k_2 q_2 \\ u_3 \\ k_4 q_4 \\ u_5 \\ k_6 q_6 \\ u_7 \end{pmatrix}$$

where  $q = (q_1, q_2, q_3, q_4, q_5, q_6, q_7)$ ,  $k_2, k_4$  and  $k_6$  are negative constants;  $u_3, u_5$  and  $u_7$  are inputs, and  $M$  is a constant matrix with the following form:

$$M = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_6 & a_7 \\ a_2 & a_2 & a_3 & \cdots & a_6 & a_7 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_6 & a_6 & a_6 & \cdots & a_6 & a_7 \\ a_7 & a_7 & a_7 & \cdots & a_7 & a_7 \end{pmatrix}$$

We have chosen the following values for the parameters:  $a_i = i/100 \quad \forall i = 1 \dots 7$ ,  $m = 0.15kg$ ,  $g = 9.81m/s^2$   $k_2 = -0.6$ ,  $k_4 = -0.7$  and  $k_6 = -0.8$ .

In order to linearize the system, a diffeomorphism  $z = \phi(x)$  and a feedback law  $u = \alpha(z) + \beta(z)u$  must be obtained, so that in the new variables  $(z, v)$  the system reads:  $\dot{z} = Az + Bv$  which is linear and controllable. The diffeomorphism is given by  $y_1, y_2$  and  $y_3$  and the following derivatives:

$$\begin{aligned}
y_1 &= \sum_{i=1}^7 a_i x_i(t) \\
\dot{y}_1 &= \sum_{i=1}^7 a_i \dot{x}_i(t) = \sum_{i=1}^7 a_{i+7} x_{i+7}(t) \\
\ddot{y}_1 &= -mg \sin(x_1(t)) \\
y_1^{(3)} &= -mg \cos(x_1(t)) x_8(t) \\
y_1^{(4)} &= mg \sin(x_1(t)) (x_8(t))^2 \\
&\quad - mg \cos(x_1(t)) (\lambda_1 mg \sin(x_1(t)) + \lambda_2 k_2 x_2) \\
y_1^{(5)} &= mg \cos(x_1(t)) (x_8(t))^3 \\
&\quad + 2mg \sin(x_1(t)) x_8(t) (-\lambda_1 mg \sin(x_1(t)) - \lambda_2 k_2 x_2(t)) \\
&\quad - mg \sin(x_1(t)) x_8(t) (\lambda_1 mg \sin(x_1(t)) + \lambda_2 k_2 x_2(t)) \\
&\quad + mg \cos(x_1(t)) (\lambda_1 mg \cos(x_1(t)) x_8(t) + \lambda_2 k_2 x_9(t)) \\
y_2 &= a_4 \left( \sum_{i=1}^4 x_i(t) \right) + \sum_{i=5}^7 a_i x_i(t) \\
\dot{y}_2 &= a_4 \left( \sum_{i=1}^4 \dot{x}_i(t) \right) + \sum_{i=5}^7 a_i \dot{x}_i(t) \\
\ddot{y}_2 &= k_4 x_4(t) \\
y_2^{(3)} &= k_4 \dot{x}_4(t) = k_4 x_{11}(t) \\
y_3 &= a_6 \left( \sum_{i=1}^6 x_i(t) \right) + a_7 x_7(t) \\
\dot{y}_3 &= a_6 \left( \sum_{i=1}^6 \dot{x}_i(t) \right) + a_{14} x_{14}(t) \\
\ddot{y}_3 &= k_6 x_6(t) \\
y_3^{(3)} &= k_6 \dot{x}_6(t) = k_6 x_{13}(t)
\end{aligned}$$

We can consider  $z = (y_1, \dots, y_1^{(5)}, y_2, \dots, y_2^{(3)}, y_3, \dots, y_3^{(3)})^T$ . In the simulation, the initial and final values are:  $x_i(0) = 0$  and  $x_i(1) = 0.05 \forall i = 1, \dots, 14$ . These initial and final conditions are mapped into the  $z$  variables through the diffeomorphism. Therefore, six initial and final conditions are found for  $y_1$  (similarly, four initial and final conditions are found for  $y_2$  and  $y_3$ ). For these variables, an interpolating polynomial satisfying the boundary conditions can be computed. Finally, the states  $x$  and inputs  $u$  are obtained again through the diffeomorphism and the feedback law. Plugging the inputs  $u(t)$  and integrating the system, the trajectories for the  $x$  variables are found. Below, a figure representing the trajectories of  $x_1(t)$  and  $\dot{x}_1(t)$  is plotted:

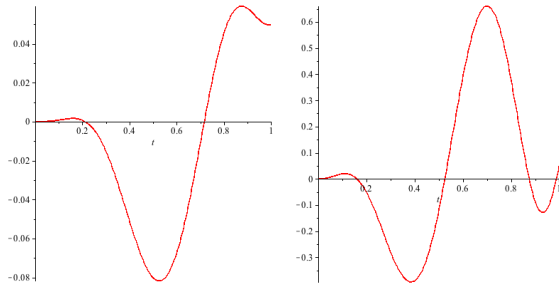


Fig. 1: Trajectories de  $x_1(t)$  y  $\dot{x}_1(t)$  respectively.

## IX. CONCLUSIONS

In this work, we have studied in detail the feedback linearization, both static and dynamic, of a robotic manipulator with a specific inertia distribution. Necessary and sufficient

conditions are given for feedback linearization of any system with an arbitrary number of actuators. More precisely,

- 1) A system is SFL iff  $g_1$  belongs to  $D_{2(i_1-1)}$  distribution. This result allows us to determine whether or not the system is static feedback linearizable, just by checking if a vector field belongs to a certain distribution.
- 2) The matrix notation was introduced as static feedback linearizability can be given interns of the rank of matrices. So, if the rank fulfills some constraints, the given system is static feedback linearizable.
- 3) A theorem and an algorithm has been introduced. This algorithm shows which systems are static feedback linearizable.
- 4) The consistency of the results with those of [5], has been established.
- 5) It has been shown that all systems, which are not static feedback linearizable, are linearizable by prolongations. The prolongation which makes any system linearizable has been given.
- 6) The simulation tests some of the results proven in this work.

**Acknowledgments:** The authors acknowledge support of Ministerio de ciencia e innovación, through project MTM 2011-22585.

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