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Delay Independent Static Output Feedback Variable Structure Control for Affine Nonlinear Systems

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Abstract— **In this paper, a stabilisation problem for a class of nonlinear control systems with uncertainties involving time-varying delay, is considered. It does not require that the nominal system is linearisable or partial linearisable. The bounds on both matched and mismatched uncertainties are nonlinear and time delayed. A static output feedback variable structure control is synthesized to stabilise the system uniformly asymptotically. A control strategy to enforce exponential stability is also derived. The Lyapunov-Razuminkhin approach is used to deal with the time delay, and the bounds on the uncertainties are used in the control design to enhance the robustness. The designed control does not depend on the time delay and thus it is not required that the time delay is known. Finally, the obtained results are tested on a simple mechanical system through simulation.**

I. INTRODUCTION

It is well known that linear dynamical systems cannot adequately describe many commonly observed phenomena. In the real world, nearly all systems exhibit nonlinearity. The desire to articulate complex phenomena has attracted many engineers and scientists to the field of nonlinear control systems [6], [13] where much of the existing effort has focused on state feedback control.

In reality, not all state variables are accessible and usually only a subset is available for measurement. One way to circumvent this problem is to design an observer to measure/estimate the system states [11], [2], and then use the estimated states to replace the true states to form the feedback loop. However, the separation principle usually does not hold for nonlinear systems [14], which implies that the approaches using the true state to design the control law may produce completely different results when estimated states are used in the control law. Moreover, in order to estimate systems states, extra devices or hardware are necessary which may be too expensive or complicated to implement. This motivates the need to consider static output feedback control approaches.

In recent decades, static output feedback control has been widely used in control design (see, e.g. [3], [15], [8], [7]). However, the systems considered in most of the existing work are either largely linear or delay free. A class of linear time delay systems is considered in [3] and a class of linear time delay systems with nonlinear disturbance is considered in [15] where it is required that the time-varying delay is precisely known. Luo *et al* considered a class of time-delay systems where both static and dynamic output feedback strategies are studied [8] but it is required that all of the uncertainties are matched. In nearly all of the existing static output feedback strategies for time-delay systems, it is required that the bounds on the uncertainties satisfy the linear growth condition (i.e. are linear functions of $\|x(t)\|$ and/or $\|x(t-d)\|$). Since the bounds on the uncertainties may have nonlinear forms in reality, it is pertinent to consider the case when the bounds on the uncertainties are nonlinear. Although a class of uncertainties bounded by nonlinear functions has been considered in [15] but it is required that the time delay is exactly known. Some interesting results on stabilisation of nonlinear time delay systems have been developed in [10] but state feedback is employed. The problem of static output feedback stabilisation for nonlinear systems is full of challenge especially when the system experiences both uncertainties and time delay.

For nonlinear control systems, there have been numerous tools to design asymptotic stabilizing controllers (see, for example, [9], [6]). Much of the existing work studying systems with disturbances is based on the fact that the nominal system has desired performance or nominal stability is preset [5], [16], [15]. In this paper, a class of affine nonlinear control systems with nonlinear uncertainties which involve time varying delay, is considered. Similar to the work in [5], [16], [15], it is assumed that the nominal system is output feedback stabilisable with an output feedback control law having been well-designed. Then, a robust static output feedback variable structure control is synthesised such that the corresponding closed loop system is uniformly asymptotically stable in the presence of uncertainties. Both matched and mismatched uncertainties are considered, and the accessible bounds on the uncertainties are employed in the control design to reduce the effects of the uncertainties. A set of sufficient conditions is developed to guarantee that the closed-loop system is exponentially stable. A simulation example of a mass-spring system is provided to illustrate the proposed approaches.

II. PRELIMINARIES

Notation: The notation in this paper is quite standard. A real vector X with dimension n is denoted by $X \in \mathcal{R}^n$. A $n \times m$ matrix Y with real entries is denoted by $Y \in \mathcal{R}^{n\times m}$ and for $P \in \mathcal{R}^{n\times n}$, the symbol $P > 0$ means

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that P is symmetric positive definite. The symbol $\mathcal{R}^+$ represents the set of non-negative real numbers. A function $f(x_1,\ldots,x_{n_1},y_1,\ldots,y_{n_2})$ is also written as $f(x,y)$ where $x=[x_1\ \ldots\ x_{n_1}]^T\in\mathcal{R}^{n_1}$ and $y=[y_1\ \ldots\ y_{n_2}]^T\in\mathcal{R}^{n_2}$. The symbol $\mathcal{C}$ denotes a set of continuous functions and $\mathcal{C}_{[a,b]}$ denotes a set of continuous functions defined in the interval $[a,b]$. A ball $\{x\in\mathcal{R}^n \mid \|x\|\le r\}$ is denoted by B_r. The expression $A\Rightarrow B$ means that if A holds, then B holds, and $A\Leftrightarrow B$ means that A and B are equivalent. Finally, $\|\cdot\|$ denotes the Euclidean norm or its induced norm.

Definition 1. A continuous function $\alpha:\ [0,a)\mapsto\mathcal{R}^+$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0)=0$. If a class $\mathcal{K}$ function is a C^1 function, then it is said to belong to class $\mathcal{K}C^1$. A continuous function $\beta:\mathcal{R}^n\times R^+\mapsto\mathcal{R}^+$ is said to be a class $\mathcal{KI}$ function if for any $x\in\mathcal{R}^n$ the function $\beta(x,s)$ is increasing with respect to the variable s in $\mathcal{R}^+$, that is, $\beta(x,s_1)\le\beta(x,s_2)$ for any $0\le s_1\le s_2$ and $x\in\mathcal{R}^n$.

Lemma 1. The following results hold:

i) If $\beta(x,s):\mathcal{R}^n\times R^+\mapsto\mathcal{R}^+$ is a class $\mathcal{KI}$ function, then $\beta^2(x,s)$ is a class $\mathcal{KI}$ function.

ii) Suppose that a function $\phi_1:\ [0,a)\mapsto\mathcal{R}^+$ is a C^1 function with $\phi_1(0)=0$. Then there exists a continuous function $\phi_2(\cdot)$ in $[0,a)$ such that

$$\phi_1(s)=\phi_2(s)s,\qquad\qquad s\in[0,a)$$

Proof: i) Suppose that $\beta(x,s):\mathcal{R}^n\times R^+\mapsto\mathcal{R}^+$ is a class $\mathcal{KI}$ function. Then for any $0\le s_1\le s_2$ and $x\in\mathcal{R}^n$,

$$\beta(x,s_1)\le\beta(x,s_2)$$

From $\beta(x,s)\ge0$ for any $(x,s)\in\mathcal{R}^n\times\mathcal{R}^+$

$$\begin{aligned}&\beta^2(x,s_1)-\beta^2(x,s_2)\\ =\ &(\beta(x,s_1)+\beta(x,s_2))(\beta(x,s_1)-\beta(x,s_2))\\ \le\ &0\end{aligned}$$

This shows that $\beta^2(x,s)$ is a class $\mathcal{KI}$ function

ii) Since the functions $\phi_1(\cdot)$ is a C^1 function in $[0,a)$, its derivative $\frac{d\phi_1(s)}{ds}$ is continuous in $[0,a)$. For any $s\in[0,a)$, construct a function

$$\phi_2(s):=\begin{cases}\frac{\phi_1(s)}{s}, & s\ne0\\ \frac{d\phi_1(s)}{ds}\mid_{s=0}, & s=0\end{cases}$$

From the definition of $\phi_2(\cdot)$, it is clear to see that

1) if $s\ne0$, then $\phi_1(s)=\phi_2(s)s$;
2) if $s=0$, then from $\phi_1(0)=0$, $\phi_1(s)=\phi_2(s)s$.

Therefore, the equality $\phi_1(s)=\phi_2(s)s$ holds. It remains to prove that $\phi_2(\cdot)$ is continuous.

It is clear that $\phi_2(s)$ is continuous in $(0,a)$. Since ϕ_1 is a C^1 function in $[0,a)$, from the continuity of $\frac{d\phi_1(s)}{ds}$ at $s=0$,

$$\lim_{s\to0}\phi_2(s)=\lim_{s\to0}\frac{\phi_1(s)}{s}=\frac{d\phi_1(s)}{ds}\mid_{s=0}=\phi_2(0)$$

which implies that $\phi_2(\cdot)$ is continuous at $s=0$. Therefore $\phi_2(\cdot)$ is continuous in $[0,a)$.

Hence the conclusion follows. ∇

The following Lemma 2 is used to identify a class of functions with certain properties which will be useful to the subsequent analysis.

Lemma 2. Let $\bar{d}>0$, $\xi:\mathcal{R}^+\mapsto\mathcal{R}^+$ be a class $\mathcal{K}$ function and $P\in\mathcal{R}^{n\times n}$ be symmetric positive definite. There exists $k>1$ such that $k\xi(r)\le\xi(c_0r)$ in $r\in\mathcal{R}^+$ for some constant $c_0>0$. Then,

i) the function $V(x):=\xi(x^TPx)$ $(x\in\mathcal{R}^n)$ is positive definite;

ii) there exists a nondecreasing continuous function $w:R^+\mapsto\mathcal{R}^+$ satisfying $w(r)>r$ for $r>0$ such that for any $\theta\in[-\bar{d},0]$,

$$\|x(t+\theta)\|\le\eta\|x\|\quad\text{if}\quad V(x(t+\theta))\le w\left(V(x(t))\right)$$

where η is a positive constant.

Proof: i) From the definition of the class $\mathcal{K}$ function, $\xi(\cdot)$ is strictly increasing in $\mathcal{R}^+$ with $\xi(0)=0$. Then from $P>0$ and thus $x^TPx\ge0$, it follows that if $x\ne0$

$$V(x)=\xi(x^TPx)>\xi(0)=0\quad\text{and}\quad V(0)=0\Longleftrightarrow x=0$$

This shows that the function $V(x)$ is positive definite.

ii) Let $w(r):=kr$ $(r\in\mathcal{R}^+)$. It is easy to see from $k>1$ that $w(\cdot)$ is nondecreasing and continuous satisfying $w(r)>r$ for $r>0$. From the fact that $\xi:\mathcal{R}^+\mapsto\mathcal{R}^+$ is strictly increasing in $\mathcal{R}^+$ and $k\xi(r)\le\xi(c_0r)$, it follows that when $V(x(t+\theta))\le w(V(x))=kV(x)$ for any $\theta\in[-\bar{d},0]$,

$$\begin{aligned}&\xi\left(x^T(t+\theta)Px(t+\theta)\right)\le k\xi\left(x^TPx\right)\le\xi(c_0x^TPx)\\ &\Longrightarrow\ x^T(t+\theta)Px(t+\theta)\le c_0x^TPx\\ &\Longrightarrow\ \begin{aligned}&\underline{\lambda}(P)\|x(t+\theta)\|^2\le x^T(t+\theta)Px(t+\theta)\\ &\le c_0x^TPx\le c_0\overline{\lambda}(P)\|x\|^2\end{aligned}\\ &\Longrightarrow\ \|x(t+\theta)\|\le\sqrt{c_0\overline{\lambda}(P)/\underline{\lambda}(P)}\|x\|.\end{aligned}$$

where $\overline{\lambda}(\cdot)$ and $\underline{\lambda}(\cdot)$ denote the maximum and minimum eigenvalues of the matrix P respectively. Hence the conclusion follows by choosing $\eta\ge\sqrt{c_0\overline{\lambda}(P)/\underline{\lambda}(P)}$. ∇

Remark 1. It is straightforward to check that the function $V(x)=(x^TPx)^\delta$ where the constant $\delta>0$ and the matrix $P>0$, satisfies the condition of Lemma 2. Thus $V(x)=(x^TPx)^\delta$ have the properties stated in both i) and ii) of Lemma 2. Furthermore, if $\delta\ge1$, then $V(\cdot)$ is differentiable.

The results given above will be used in the subsequent analysis.

III. System Description and Problem Formulation

Consider an affine nonlinear system described by

$$\begin{aligned}\dot{x}&=f(x)+g(x)(u+\Delta g(x,x_d))+\Delta f(x,x_d) &(1)\\ y&=h(x) &(2)\end{aligned}$$

where $x\in\mathcal{X}\subset\mathcal{R}^n$ ($\mathcal{X}$ is a neighbourhood of the origin), $u\in\mathcal{R}^m$, $y\in\mathcal{Y}\subset\mathcal{R}^p$ are system state variables, inputs and outputs respectively. The functions $f(x)$ and $g(x)$ with $f(0)=0$ are both known functions, and $\Delta g(x,x_d)$ and $\Delta f(x,x_d)$ are matched and mismatched uncertainties respectively which include all the disturbances and modeling error. The symbol $x_d := x(t - d(t))$ denotes the delayed states where $d(t)$ is a time-varying delay which is assumed to be continuous, nonnegative in R^+ and

$$\overline{d} := \sup_{t \in \mathcal{R}^+} \{d(t)\} < \infty$$

The initial condition relating to time delay is given by

$$x(t) = \phi(t), \qquad t \in [-\overline{d}, 0]$$

where $\phi(\cdot)$ is continuous in $[-\overline{d}, 0]$. It is assumed that all the nonlinear terms are smooth enough such that the existence and uniqueness of solutions of the unforced system is guaranteed.

For system (1)–(2), the following system

$$\dot{x} = f(x) + g(x)u \tag{3}$$

$$y = h(x) \tag{4}$$

is called the corresponding nominal system. The following assumptions are imposed on the system (1)–(2).

Assumption 1. There exist known continuous nonnegative functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\alpha_3(\cdot)$ and $\beta(\cdot)$ such that

$$\|\Delta g(x, x_d)\| \leq \alpha_1(y) + \alpha_2(y)\alpha_3(x, \|x_d\|) \tag{5}$$

$$\|\Delta f(x, x_d)\| \leq \beta(x, \|x_d\|) \tag{6}$$

where $\beta(x, r)$ is a class $\mathcal{KI}$ function.

Assumption 2. There exist a continuous function $u_1(y)$ in $\mathcal{Y}$, a C^1 function $V(x) : \mathcal{R}^n \mapsto \mathcal{R}$ and a continuous function $M(\cdot) \in \mathcal{R}^{1\times m}$ defined in $\mathcal{Y}$ such that

$$c_1(\|x\|) \leq V(x) \leq c_2(\|x\|) \tag{7}$$

$$\frac{\partial V}{\partial x}\left(f(x) + g(x)u_1(y)\right) \leq -c_3(\|x\|) \tag{8}$$

$$\left\|\frac{\partial V}{\partial x}\right\| \leq c_4(\|x\|) \tag{9}$$

$$\frac{\partial V}{\partial x} g(x) = M(y) \tag{10}$$

where $c_1(\cdot)$, $c_2(\cdot)$, $c_3(\cdot)$ and $c_4(\cdot)$ are class $\mathcal{K}$ functions in R^+, and $\frac{\partial V}{\partial x} =: \left(\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \cdots, \frac{\partial V}{\partial x_n}\right)$.

Remark 2. Assumption 2 limits the nominal system (3)–(4) where the conditions (7)–(9) guarantee that the nominal system is output feedback stabilised by $u = u_1(y)$. The condition (10) provides a constraint on the Lyapunov function $V(\cdot)$. The conditions (7)–(9) together with the limitation (10) is an extension of the well known constrained Lyapunov problem (CLP) ([4]). The CLP is for the linear case and has widely appeared in static output feedback control and observer design (see e.g. [12], [4]). It should be emphasised that the current consideration has extended the CLP to the non-square nonlinear case and renders all previous settings as special cases in this regard.

Assumption 3. There exists a continuous nondecreasing function $w(\cdot)$ defined in R^+ satisfying that $w(r) > r$ for $r > 0$ such that if $V(x(t+\theta)) \leq w\left(V(x(t))\right)$ for any $\theta \in [-\bar{d}, 0]$, then

$$\|x(t+\theta)\| \leq \gamma(\|x\|)$$

for some continuous nonnegative function $\gamma(\cdot)$.

In this paper, the objective is to design a variable structure control such that the corresponding closed-loop system is uniform asymptotically stable in the presence of time delayed uncertainties. The local case will be considered. For ease of exposition, the domain considered may not be specifically stated in the subsequent analysis unless it is necessary.

IV. Output Feedback Control Synthesis

In this section, a static output feedback control will be synthesised such that the corresponding closed-loop system is uniformly asymptotically stable. Then conditions for exponentially stabilisation will follow.

For system (1)–(2), consider an output feedback control law

$$u(y) = u_1(y) + u_2(y) \tag{11}$$

where the function $u_1(\cdot)$ is given in Assumption 2, and $u_2(\cdot)$ is defined by

$$u_2(y) = \begin{cases} -M^T(y)\Big(\frac{1}{\|M(y)\|}\alpha_1(y) + \frac{1}{2\varepsilon}\alpha_2^2(y)\Big), & M(y) \neq 0 \\ 0, & M(y) = 0 \end{cases} \tag{12}$$

where $M(\cdot) \in \mathcal{R}^{1\times m}$ is given in (10), ε is an adjustable positive constant, and $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are determined by (5).

Theorem 1 Under Assumptions 1-3, the closed-loop system formed by applying control (11)–(12) to the system (1)–(2) is uniformly asymptotically stable if there exists a class $\mathcal{K}$ function $\alpha(\cdot)$ such that in the considered domain

$$c_3(\|x\|) - \frac{\varepsilon}{2}\alpha_3^2(x, \gamma(\|x\|)) - c_4(\|x\|)\beta(x, \gamma(\|x\|)) \leq \alpha(\|x\|) \tag{13}$$

for some positive constant ε, where $c_3(\cdot)$ and $c_4(\cdot)$ satisfy Assumption 2 and $\alpha_3(\cdot)$ and $\beta(\cdot)$ satisfy Assumption 1.

Proof: The closed-loop system obtained by applying the control (11)–(12) to system (1)–(2) is described by

$$\dot{x} = f(x) + g(x)\left(u_1(y) + u_2(y) + \Delta g(x, x_d)\right) + \Delta f(x, x_d) \tag{14}$$

$$y = h(x) \tag{15}$$

For system (14)–(15), consider the Lyapunov function candidate $V(x)$ in Assumption 2. The time derivative of $V(x)$ along the trajectories of system (14)–(15) is given by

$$\begin{aligned} &\dot{V}\,|_{(14)-(15)} \\ =\;& \frac{\partial V}{\partial x}\left(f(x) + g(x)u_1(y)\right) + \frac{\partial V}{\partial x}g(x)\big(u_2(y) \\ &+\Delta g(x, x_d)\big) + \frac{\partial V}{\partial x}\Delta f(x, x_d) \\ \leq\;& -c_3(\|x\|) + \frac{\partial V}{\partial x}g(x)\big(u_2(y) + \Delta g(x, x_d)\big) \\ &+\frac{\partial V}{\partial x}\Delta f(x, x_d) \end{aligned} \tag{16}$$

where (8) is used to obtain the last inequality. From the definition of $u_2(\cdot)$ in (12), inequality (5) and equation (10), it follows that

i). if $M(y)=0$, then

$$\begin{aligned}&\frac{\partial V}{\partial x}g(x)\left(u_2(y)+\Delta g(x,x_d)\right)\\ =\ & M(y)\left(u_2(y)+\Delta g(x,x_d)\right)=0;\end{aligned}$$

ii). if $M(y)\neq 0$, then

$$\begin{aligned}&\frac{\partial V}{\partial x}g(x)\left(u_2(y)+\Delta g(x,x_d)\right)\\ &\le \frac{\partial V}{\partial x}g(x)u_2(y)+\left\|\frac{\partial V}{\partial x}g(x)\right\|\|\Delta g(x,x_d)\|\\ &\le -\|M(y)\|\alpha_1(y)-\frac{1}{2\varepsilon}\|M(y)\|^2\alpha_2^2(y)\\ &\quad+\|M(y)\|\alpha_1(y)+\|M(y)\|\alpha_2(y)\alpha_3(x,\|x_d\|)\\ &=-\frac{1}{2\varepsilon}\|M(y)\|^2\alpha_2^2(y)+\|M(y)\|\alpha_2(y)\alpha_3(x,\|x_d\|)\end{aligned} \quad (17)$$

Then from the special case of Youngs inequality $ab\le\frac{\varepsilon}{2}a^2+\frac{1}{2\varepsilon}b^2$, it follows that for any $\varepsilon>0$

$$\begin{aligned}&\|M(y)\|\alpha_2(y)\,\alpha_3(x,\|x_d\|)\\ &\quad\le \frac{1}{2\varepsilon}\alpha_2^2(y)\|M(y)\|^2+\frac{\varepsilon}{2}\alpha_3^2(x,\|x_d\|)\end{aligned} \quad (18)$$

Further from (18), (17) and the analysis in i) and ii) above, it is straightforward to see that

$$\frac{\partial V}{\partial x}g(x)\left(u_2(y)+\Delta g(x,x_d)\right)\le\frac{\varepsilon}{2}\alpha_3^2(x,\|x_d\|). \quad (19)$$

By (6) and (9),

$$\frac{\partial V}{\partial x}\Delta f(x,x_d)\le\left\|\frac{\partial V}{\partial x}\right\|\|\Delta f(x,x_d)\|\le c_4(\|x\|)\beta(x,\|x_d\|) \quad (20)$$

Substituting (19) and (20) into (16) yields

$$\dot V\le -c_3(\|x\|)+\frac{\varepsilon}{2}\alpha_3^2(x,\|x_d\|)+c_4(\|x\|)\beta(x,\|x_d\|) \quad (21)$$

Now, it is assumed that there exists a continuous nondecreasing function $w(\cdot)$ defined in R^+ satisfying $w(r)>r$ for $r>0$ and $V(x(t+\theta))\le w\left(V(x(t))\right)$. Then from Assumption 3, for any $d(t)\in[0,\bar d]$,

$$\|x_d\|\le\gamma(\|x\|) \quad (22)$$

From the result i) of Lemma 1 and the condition that $\alpha_3(\cdot)$ is a class $\mathcal{KI}$ function, the function $\alpha_3^2(\cdot)$ is class $\mathcal{KI}$. Since $\beta(\cdot)$ is also a class $\mathcal{KI}$ function, if $V(x(t+\theta))\le w\left(V(x(t))\right)$, then (22) holds and thus

$$\alpha_3(x,\|x_d\|)\ \le\ \alpha_3(x,\gamma(\|x\|)) \quad (23)$$

$$\beta(x,\|x_d\|)\ \le\ \beta(x,\gamma(\|x\|)) \quad (24)$$

Applying (23) and (24) to (21), it follows from (13) that

$$\begin{aligned}\dot V&\le -c_3(\|x\|)+\frac{\varepsilon}{2}\alpha_3^2(x,\gamma(\|x\|))+c_4(\|x\|)\beta(x,\gamma(\|x\|))\\ &\le -\alpha(\|x\|)\end{aligned}$$

Hence the conclusion follows from the well known Razumikhin Theorem. ∇

Next, an output feedback control law will be synthesised for system (1)–(2) such that the corresponding closed-loop system is exponentially stable. The Assumptions 1 and 2 are strengthened to the following Assumptions A1 and A2.

Assumption A1. The uncertainties $\Delta g(x,x_d)$ and $\Delta f(x,x_d)$ satisfy

$$\|\Delta g(x,x_d)\|\ \le\ \alpha_1(y)+\alpha_2(y)\|x_d\| \quad (25)$$

$$\|\Delta f(x,x_d)\|\ \le\ \beta_1(\|x\|)+\beta_2(\|x_d\|) \quad (26)$$

where $\alpha_1:\mathcal{Y}\mapsto\mathcal{R}^+$ and $\alpha_2:\mathcal{Y}\mapsto\mathcal{R}^+$ are known continuous and nonnegative functions, and $\beta_1:\mathcal{R}^+\mapsto\mathcal{R}^+$ and $\beta_2:\mathcal{R}^+\mapsto\mathcal{R}^+$ are known nonnegative and $\mathcal{K}C^1$ functions.

From Assumption A1, $\beta_1(\cdot)$ and $\beta_2(\cdot)$ are $\mathcal{K}C^1$ functions, and it follows from the result ii) of Lemma 1 that there exist continuous functions $\beta_3:\mathcal{R}^+\mapsto\mathcal{R}^+$ and $\beta_4:\mathcal{R}^+\mapsto\mathcal{R}^+$ such that

$$\beta_1(r)\ =\ \beta_3(r)r \quad (27)$$

$$\beta_2(r)\ =\ \beta_4(r)r \quad (28)$$

It should be noted that one of the choices to obtain $\beta_3(\cdot)$ and $\beta_4(\cdot)$ has been given in the proof of Lemma 1.

Assumption A2. There exist a continuous function $u_a:\mathcal{Y}\mapsto\mathcal{R}^m$, a function matrix $M(\cdot)\in\mathcal{R}^{1\times n}$ defined in $\mathcal{Y}$ and a C^1 function $U(x):\mathcal{R}^n\mapsto\mathcal{R}$ such that

$$\kappa_1\|x\|^2\ \le\ U(x)\le\kappa_2\|x\|^2 \quad (29)$$

$$\frac{\partial U}{\partial x}\left(f(x)+g(x)u_a(y)\right)\ \le\ -\kappa_3\|x\|^2 \quad (30)$$

$$\left\|\frac{\partial U}{\partial x}\right\|\ \le\ \kappa_4\|x\| \quad (31)$$

$$\frac{\partial U}{\partial x}g(x)\ =\ M(y) \quad (32)$$

for some positive constants κ_i for $i=1,2,3,4$, where $\frac{\partial U}{\partial x}=:\left(\frac{\partial U}{\partial x_1},\frac{\partial U}{\partial x_2},\cdots,\frac{\partial U}{\partial x_n}\right)$.

Remark 3. The conditions (29)–(31) in Assumption A2 imply that the nominal system (3)–(4) is stabilisable by $u=u_a(y)$. Suppose that $f(x)$ and $g(x)$ are C^1 functions. Then it follows from [6] that the Assumption A2 is satisfied if the nominal system (3)–(4) is exponentially stabilisable by a C^1 function $u=u_a(y)$, that is

$$\dot x=f(x)+g(x)u_a(y) \quad (33)$$

is exponentially stable.

Theorem 2. Under Assumptions A1 and A2, system (1)–(2) is exponentially stablised by the control

$$u(\cdot)=u_a(y)+u_2(y) \quad (34)$$

where $u_a(\cdot)$ satisfies Assumption A2 and $u_2(\cdot)$ is defined in (12), if there exist constants $\varepsilon>0$ and $q>1$ such that

$$\mu:=\kappa_3-\frac{\varepsilon}{2}\frac{q\kappa_2}{\kappa_1}-\nu>0 \quad (35)$$

where

$$\nu:=\sup_{x\in\mathcal{X}}\left\{\kappa_4\beta_3(\|x\|)+\kappa_4\sqrt{\frac{q\kappa_2}{\kappa_1}}\beta_4\left(\sqrt{\frac{q\kappa_2}{\kappa_1}}\|x\|\right)\right\} \quad (36)$$

where the constants κ_1 and κ_2 satisfy (29), k_3 and κ_4 satisfy (30) and (31) respectively, and $\beta_3(\cdot)$ and $\beta_4(\cdot)$ are defined by (27) and (28) respectively.

Proof: For the closed-loop system formed by applying the control (34) into (1)–(2), consider the Lyapunov function candidate $U(x)$ defined in Assumption A2. The time derivative of $U(x)$ along the trajectories of the closed loop system is described by

$$\begin{aligned}\dot{U} &= \frac{\partial U}{\partial x}\left(f(x)+g(x)u_a(y)\right)+\frac{\partial U}{\partial x}g(x)\big(u_2(y)\\&\quad +\Delta g(x,x_d)\big)+\frac{\partial U}{\partial x}\Delta f(x,x_d)\\&\leq -\kappa_3\|x\|^2+\frac{\partial U}{\partial x}g(x)\big(u_2(y)+\Delta g(x,x_d)\big)\\&\quad +\frac{\partial U}{\partial x}\Delta f(x,x_d)\end{aligned} \tag{37}$$

Following the analysis on the inequality (19) in the proof of Theorem 1, it is straightforward to see that

$$\frac{\partial U}{\partial x}g(x)\left(u_2(y)+\Delta g(x,x_d)\right)\leq\frac{\varepsilon}{2}\|x_d\|^2 \tag{38}$$

holds for any constant $\varepsilon>0$

From (26) and (31),

$$\frac{\partial U}{\partial x}\Delta f(x,x_d)\leq\kappa_4\beta_1(\|x\|)\|x\|+\kappa_4\beta_2(\|x_d\|)\|x\| \tag{39}$$

Substituting (38) and (39) into (37),

$$\dot{U}\leq-\kappa_3\|x\|^2+\frac{\varepsilon}{2}\|x_d\|^2+\kappa_4\beta_1(\|x\|)\|x\|+\kappa_4\beta_2(\|x_d\|)\|x\| \tag{40}$$

It is assumed that $U(x(t+\theta))\leq qU(x(t)$ for $q>1$ and $\theta\in[-\bar{d},0]$. Then from (29), for any $d(t)\in[0,\bar{d}]$,

$$\kappa_1\|x_d\|^2\leq U(x_d)\leq qU(x(t))\leq q\kappa_2\|x\|^2$$

and thus

$$\|x_d\|\leq\sqrt{\frac{q\kappa_2}{\kappa_1}}\|x\| \tag{41}$$

Since $\beta_2(\cdot)$ is class $\mathcal{K}C^1$, it follows from the nondecreasing property of $\beta_2(\cdot)$ and equation (28) that

$$\beta_2(\|x_d\|)\leq\beta_2\Big(\sqrt{\frac{q\kappa_2}{\kappa_1}}\|x\|\Big)=\sqrt{\frac{q\kappa_2}{\kappa_1}}\beta_4\Big(\sqrt{\frac{q\kappa_2}{\kappa_1}}\|x\|\Big)\|x\|$$

Therefore, when $U(x(t+\theta))\leq qU(x(t)$ for $q>1$ and $\theta\in[-\bar{d},0]$,

$$\begin{aligned}\dot{U} &\leq -\kappa_3\|x\|^2+\frac{\varepsilon}{2}\frac{q\kappa_2}{\kappa_1}\|x\|^2+\kappa_4\beta_3(\|x\|)\|x\|^2\\&\quad+\kappa_4\sqrt{\frac{q\kappa_2}{\kappa_1}}\beta_4\Big(\sqrt{\frac{q\kappa_2}{\kappa_1}}\|x\|\Big)\|x\|^2\\&= -\Big(\kappa_3-\frac{\varepsilon}{2}\frac{q\kappa_2}{\kappa_1}-\nu\Big)\|x\|^2\\&= -\mu\|x\|^2\\&\leq -\frac{\mu}{\kappa_2}U(x)\end{aligned} \tag{42}$$

where (29) is used to obtain the last inequality, and μ and ν are defined by (35) and (36) respectively. From (42),

$$\|x(t)\| \leq \sqrt{\frac{U(x(0))}{\kappa_1}}\exp\left\{-\frac{\mu}{2\kappa_2}t\right\}$$

which implies that system (1)–(2) is exponentially stable. Hence the conclusion follows. ∇

V. Case Study – A Mass-Spring System Control

Consider a mass-spring system which experiences a hardening spring, linear viscous friction and an external force described by (see [6])

$$m\ddot{s}+c\dot{s}+ks+ka^2s^3=u \tag{43}$$

where s denotes the displacement from the reference position, m is the mass of the object sliding on a horizontal surface, k is the spring constant and u is an external force which is considered as the control input. Let $x=\mathrm{col}(x_1,x_2)=(s,\dot{s})$. The parameters are chosen the same as that in [6] (see, pages 172-173). Then, the system is described by

$$\begin{aligned}\dot{x} &= \underbrace{\begin{bmatrix}x_2\\-(1+x_1^2)x_1-x_2\end{bmatrix}}_{f(\cdot)}+\underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{g(\cdot)}(u+\Delta g(x,x_d))\\&\quad+\Delta f(x,x_d)\end{aligned} \tag{44}$$

$$y = x_1+x_2 \tag{45}$$

where y is the system output. It should be pointed out that the uncertainties $\Delta f(x,x_d)$ and $\Delta g(x,x_d)$ are not an inherent property of the system. Here they are specifically added to illustrate the results obtained in this paper. It is assumed that

$$\|\Delta g(\cdot)\| \leq \underbrace{1+y\sin^2 y}_{\alpha_1(\cdot)}+\underbrace{y^2}_{\alpha_2(\cdot)}\underbrace{\|x_d\|}_{\alpha_3(\cdot)} \tag{46}$$

$$\|\Delta f(\cdot)\| \leq \underbrace{0.01\|x\|\,\|x_d\|^2}_{\beta(\cdot)} \tag{47}$$

Then, consider an output feedback control $u_1(y)=-y$ and a Lyapunov function candidate

$$V=x^\tau\begin{bmatrix}4&1\\1&1\end{bmatrix}x+\frac{1}{2}x_1^4$$

It is straightforward to see that $V(\cdot)$ is a continuous positive definite function. By direct computation,

$$0.697\|x\|^2\leq V(x)\leq 4.303\|x\|^2+\frac{1}{2}\|x\|^4 \tag{48}$$

$$\frac{\partial V}{\partial x}\left(f(x)+g(x)u_1(y)\right)=-2x_1^2-2x_2^2-2x_1^4\leq-2\|x\|^2$$

$$\left\|\frac{\partial V}{\partial x}\right\|\leq 8.606\|x\|+2\|x\|^3$$

$$\frac{\partial V}{\partial x}g(x)=2x_1+2x_2=2y=M(y)$$

Let

$$\begin{aligned}c_1(r)&=0.697r^2, & c_2(r)&=4.303r^2+\tfrac{1}{2}r^4,\\ c_3(r)&=2r^2, & c_4(r)&=8.6056r+2r^3\end{aligned}$$

It is clear to see that both Assumptions 1 and 2 are satisfied. Further, assume that

$$V(x(t+\theta))\leq qV(x(t))$$

for any $\theta \in [-\bar{d}, 0]$ and $q > 1$. Then from (48), for any $d \in [0, \bar{d}]$,

$$\begin{aligned} 0.697\|x_d\|^2 &\leq V(x(t-d)) \leq qV(x(t)) \\ &\leq 4.303q\|x\|^2 + \frac{1}{2}q\|x\|^4 \end{aligned}$$

Then,

$$\|x_d\|^2 \leq 6.2q\|x\|^2 + 0.72q\|x\|^4.$$

This implies that Assumption 3 is satisfied with

$$\gamma(r) = \sqrt{6.2qr^2 + 0.72qr^4}$$

Choose $\varepsilon = 0.1$ and $q = 1.05$. By direct computation, it is observed that the conditions of Theorem 1 are satisfied in the domain $\mathcal{X} := \{x \mid \|x\| \leq 1\}$ with $\alpha(r) = 0.01r^2$. Thus, the mass-spring system (44)–(45) is stabilised by the control

$$u = u_1 + u_2$$

where

$$u_1 = -y \tag{49}$$

$$u_2 = \begin{cases} -\left(\frac{y}{|y|}\left(1 + y\sin^2 y\right) + \frac{1}{\varepsilon}y^5\right), & y \neq 0 \\ 0, & y = 0 \end{cases} \tag{50}$$

For simulation purposes, the time delay is chosen as

$$d(t) = 2 - 1.5\cos(t)$$

and the initial condition relating to delay is chosen as $\phi(t) = \sin t$. The simulation results in Figure 1 are as expected.

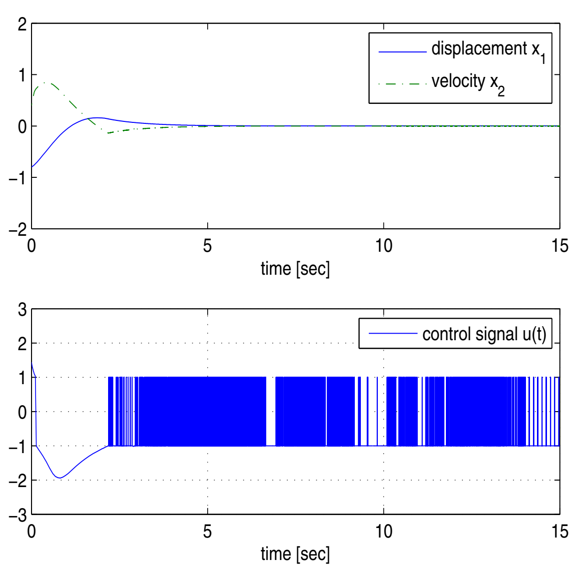

Fig. 1. The time response of state variables of mass-spring system (44)–(45) (Upper) and the control signal (Bottom)

Remark 4 Figure 1 shows that chattering occurs in the control signal. This comes from the discontinuous nature of the control $u_2(\cdot)$ in (50). One way of overcoming this drawback is to introduce a boundary layer about the discontinuity surface (see, [1]).

Remark 5 As in [12], the system output is chosen as a linear combination of the position and velocity. This situation may occur in some real systems such as certain remote control applications where the number of transmission and receive lines/frequencies is limited [12].

VI. Conclusion

In this paper, stabilistion using static output feedback control is achieved for a class of nonlinear systems with time delay disturbances. The sufficient conditions for both uniformly asymptotic stabilisation and exponential stabilisation are developed under the condition that the nominal system is output feedback stabilisable with pre-defined controllers. The bounds on the uncertainties are nonlinear and time delayed. It is not required that either the time delay is known or the system is square. This makes the work applicable to a wide class of nonlinear systems. The case study of a mass spring system shows the effectiveness and feasibility of the proposed control approach.

References

[1] J. A. Burton and A. S. I. Zinober. Continuous approximation of variable structure control. *Int. J. Systems Sci.*, 17:876–885, 1986.

[2] J. Davila, H. Rios, and L. Fridman. State observation for nonlinear switched systems using nonhomogeneous high-order sliding mode observers. *Asian Journal of Control*, 14(4):911–923, 2012.

[3] B. Du, J. Lam, and Z. Shu. Stabilization for state/input delay systems via static and integral output feedback. *Automatica*, 46(12):2000–2007, 2010.

[4] C. Edwards, X. G. Yan, and S. K. Spurgeon. On the solvability of the constrained Lyapunov problem. *IEEE Trans. on Automat. Control*, 52(10):1982–87, 2007.

[5] C.C. Hua, S.X. Ding, and X.P. Guan. Robust controller design for uncertain multiple-delay systems with unknown actuator parameters. *Automatica*, 48(1):211–218, 2012.

[6] H. K. Khalil. *Nonlinear Systems* (Third Edition). New Jersey: Prentice Hall, Inc., 2002.

[7] V.L. Kharitonov, S.-I. Niculescu, J. Moreno, and W. Michiels. Static output feedback stabilization: necessary conditions for multiple delay controllers. *IEEE Trans. on Automat. Control*, 50(1):82–86, 2005.

[8] N. Luo, M. De La Sen, and J. Rodellar. Robust stabilization of a class of uncertain time delay systems in sliding mode. *Int. J. Robust Nonlinear Control*, 7(1):59–74, 1997.

[9] R. Marino and P. Tomei. *Nonlinear Control Design: geometric, adaptive and robust*. Englewood Cliff: Prentice Hall International, 1995.

[10] P. Pepe. Input-to-state stabilization of stabilizable, time-delay, control-affine, nonlinear systems. *IEEE Trans. on Automat. Control*, 54(7):1688–1693, 2009.

[11] S. K. Spurgeon. Sliding mode observers: a survey. *Int. J. Systems Sci.*, 39(8):751–764, 2008.

[12] B. L. Walcott and S. H. Żak. Combined observer– controller synthesis for uncertain dynamical systems with application. *IEEE Trans. on Systems, Man and Cybernetics*, 18(1):88–104, 1988.

[13] X. G. Yan, J. Lam, and G. Z. Dai. Decentralized robust control for nonlinear similar large-scale systems. *Computer & Electrical Engineering*, 25(3):169–179, 1999.

[14] X. G. Yan, J. Lam, H. S. Li, and I. M. Chen. Decentralized control of nonlinear large-scale systems using dynamic output feedback. *J. Optim. Theory Appl.*, 104(2):459–475, 2000.

[15] X. G. Yan, S. K. Spurgeon, and C. Edwards. Static output feedback sliding mode control for time-varying delay systems with time-delayed nonlinear disturbances. *Int. J. Robust Nonlinear Control*, 20(7):777–788, 2010.

[16] A.I. Zecevic and D.D. Siljak. Design of robust static output feedback for large-scale systems. *IEEE Trans. on Automat. Control*, 49(11):2040–2044, 2004.