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Receding Horizon Observer and Control for linear 2x2 hyperbolic systems of conservation laws

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***Abstract*— This paper presents an *infinite-dimensional* Receding Horizon Observer for linear 2x2 hyperbolic systems with boundary measurements. The initial state is estimated as the optimal solution of an optimization problem which minimizes the distance between the measurements and the observer output. A constructive method is used to derive the existence and uniqueness of the solution. A composite strategy combining Receding Horizon Optimal Control and Receding Horizon Observer is also presented. Its effectiveness in guaranteeing closed-loop stability is also demonstrated. For the implementation, the calculus of variation approach is used to derive the adjoint state which will be discretized and solved with the observer state to obtain the optimal solution. Finally, a simulation with a linearized model of an open-channel system is carried out to validate the here-proposed approach.**

I. INTRODUCTION

The Receding Horizon Optimal Control (RHOC) scheme was quite widely studied for finite-dimensional systems (described by ordinary differential equations (ODEs)), even in the nonlinear case, as a promising approach to obtain a guaranteed stability and to deal with constrains (see e.g. [9], [10], [18]). These advantages motivated many extensions for infinite-dimensional systems (described by partial differential equations (PDEs)). In [13], the RHOC with terminal cost was proposed and applied to Navier-Stokes equations, semilinear wave equations and reaction diffusion systems. This control scheme was also considered for parabolic system with state and control constraints ([6], [7], [8]) or for quasilinear hyperbolic system [26]. The RHOC with zero terminal constraints for nonlinear hyperbolic systems was studied in [12] and validated in simulation. Our recent works provided a complete stability proof of RHOC for linear 2×2 hyperbolic systems of conservation laws ([21], [22], [24]). The usefulness of RHOC for these systems, however, is limited by the fact that the full current state needs to be accessible, which can not be obtained in practice for infinite-dimensional system. This obstacle raises the need of some observers in order to estimate the system state.

While the design of observers for linear ODEs has reached a certain maturity, this problem remains a difficult task for general nonlinear systems. However several families of observers can be found in the literature such as: Lyapunov-based observer [28], Luenberger-like observers [15], high gain observers ([3], [11]) or optimization-based observers ([2], [19]). Observer design for infinite-dimensional systems, on the other hand, has received less attention, except the backstepping observers (for parabolic equations [27] and for hyperbolic equations [29]), Luenberger-like observer for dissipative bilinear systems [30] or the Lyapunov-based observer for hyperbolic equations [1].

Our work aims at extending the optimization-based or Receding Horizon Observer (RHO) (see [2], [19]), thanks to its intuitive formulation and its possible extension to nonlinear cases, for the class of infinite-dimensional systems described by 2x2 hyperbolic equations of the form:

$$\begin{matrix} \partial_t y_1 = c_1 \partial_x y_1 + \gamma y_1 + \delta y_2 \\ \partial_t y_2 = c_2 \partial_x y_2 + \gamma y_1 + \delta y_2 \end{matrix}, x \in [0, L],\ t \geq 0. \qquad (1)$$

Here t and x classically stand for time and space coordinates, and ∂_t, ∂_x denotes the partial derivative w.r.t. t, x respectively. The system is completed with the initial and boundary conditions

$$\begin{cases} y_1(x,0) = y_{10}(x) \\ y_2(x,0) = y_{20}(x) \end{cases} \text{and} \begin{cases} y_1(L,t) = v_1(t) \\ y_2(0,t) = v_2(t) \end{cases}. \qquad (2)$$

The boundary measurements used by observer are given by:

$$\begin{cases} m_1(t) = y_1(0,t) \\ m_2(t) = y_2(L,t) \end{cases}, \quad t \geq 0. \qquad (3)$$

Coefficient c_i, λ and δ are supposed to be constant, $c_1 > 0 > c_2$ and λ, $\delta < 0$. As presented later, these equations can be used to describe the linearized dynamic of an open channel around a uniform equilibrium.

The paper also addresses a Receding Horizon Optimal Control - Observer (RHOC-O) scheme for this class of systems, and by using the stability results of the RHOC in [23], the stability of the composite strategy can be established.

The manuscript is structured as follows. First, the Receding Horizon Observer is presented and the proof of the existence and uniqueness of the optimal solution is given. Section III is dedicated to the RHOC-O scheme. Section IV discusses the method to find a numerical solution of the optimization problem using the calculus of variation approach. The proposed strategy is then validated in section V with linearized Saint-Venant equations describing an open-channel system. Finally, some conclusions end the paper in section VI.

II. PROBLEM FORMULATION

A. Receding Horizon Observer strategy

With a certain observer horizon T_o, the objective of the observer is to solve the following problem at each sample

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time t:

$$\min_{\hat{y}_{10}(\cdot),\hat{y}_{20}(\cdot)} J_o(\hat{y}_{10}(\cdot),\hat{y}_{20}(\cdot)) = \int_{t-T_o}^{t} [q_1(\hat{y}_1(0,s)-m_1(s))^2 + q_2(\hat{y}_2(L,s)-m_2(s))^2]ds \tag{4}$$

where q_1 and q_2 are some weighting parameters,

$$\text{s.t.} \left\{ \begin{array}{l} \partial_s \hat{y}_1 = c_1 \partial_x \hat{y}_1 + \gamma \hat{y}_1 + \delta \hat{y}_2 \\ \partial_s \hat{y}_2 = c_2 \partial_x \hat{y}_2 + \gamma \hat{y}_1 + \delta \hat{y}_2 \end{array} \right. , \; s \in [t-T_o, t]. \tag{5}$$

and

$$\left\{ \begin{array}{l} \hat{y}_1(\cdot,0) = \hat{y}_{10}(\cdot) \\ \hat{y}_2(\cdot,0) = \hat{y}_{20}(\cdot) \end{array} \right. , \left\{ \begin{array}{l} \hat{y}_1(L,s) = v_1(s) \\ \hat{y}_2(0,s) = v_2(s) \end{array} \right. , s \in [t-T_o, t].$$

The current state at time t is then estimated by:

$$y_1(\cdot,t) = \hat{y}_1(\cdot,t), \quad y_2(\cdot,t) = \hat{y}_2(\cdot,t). \tag{6}$$

Remark 1: The weighting parameters q_1 and q_2 can also be time-variant, in order, for instance, to put more importance on the newest data.

B. Existence and uniqueness of the optimal solution

Since our system is time invariant, we can consider only the problem on $[0, T_o]$. Given $m_1(s)$, $m_2(s)$ and $v_1(s)$, $v_2(s)$ ($s \in [0, T_o]$) the boundary measurements and the boundary controls as in (2)-(3), the following optimization problem needs to be solved:

$$\min_{\hat{y}_{10}(\cdot),\hat{y}_{20}(\cdot)} J_o(\hat{y}_{10}(\cdot),\hat{y}_{20}(\cdot)) = \int_{0}^{T_o} [q_1(\hat{y}_1(0,s)-m_1(s))^2 + q_2(\hat{y}_2(L,s)-m_2(s))^2]ds \tag{7}$$

$$\text{s.t.} \left\{ \begin{array}{l} \partial_s \hat{y}_1 = c_1 \partial_x \hat{y}_1 + \gamma \hat{y}_1 + \delta \hat{y}_2 \\ \partial_s \hat{y}_2 = c_2 \partial_x \hat{y}_2 + \gamma \hat{y}_1 + \delta \hat{y}_2 \end{array} \right. , \; s \in [0, T_o]. \tag{8}$$

and

$$\left\{ \begin{array}{l} \hat{y}_1(x,0) = \hat{y}_{10}(x) \\ \hat{y}_2(x,0) = \hat{y}_{20}(x) \end{array} \right. \text{and} \left\{ \begin{array}{l} \hat{y}_1(L,s) = v_1(s) \\ \hat{y}_2(0,s) = v_2(s) \end{array} \right. , s \in [0, T_o], \tag{9}$$

Proposition 2.1: Suppose that there is no difference between the model and the process and that the observer horizon satisfies

$$T_o > \max\left\{ \frac{L}{c_1}, \frac{-L}{c_2} \right\}, \tag{10}$$

then the optimization problem (7)-(9) has a unique solution.

Proof: The proof adopts the constructive method introduced in [16]. Denote by $T_1 = \frac{L}{c_1}$ (resp. $T_2 = \frac{-L}{c_2}$) the propagation time of the wave issued from $x = L$ (resp. $x = 0$). Without loss of generality, suppose that $T_1 > T_2$ (see Fig. 1). Consider the following problem:

$$\left\{ \begin{array}{l} \partial_x \hat{y}_1 = \frac{1}{c_1} \partial_s \hat{y}_1 - \frac{\gamma}{c_1} \hat{y}_1 - \frac{\delta}{c_1} \hat{y}_2 \\ \partial_x \hat{y}_2 = \frac{1}{c_2} \partial_s \hat{y}_2 - \frac{\gamma}{c_2} \hat{y}_1 - \frac{\delta}{c_2} \hat{y}_2 \end{array} \right. , x \in [0, L], \; s \in [0, T_o], \tag{11}$$

which has the same form as (8) where the role of x and s are interchanged. The initial condition is now specified at $x = 0$:

$$\begin{array}{l} \hat{y}_1(0,s) = m_1(s) \\ \hat{y}_2(0,s) = v_2(s) \end{array} , \quad s \in [0, T_o]. \tag{12}$$

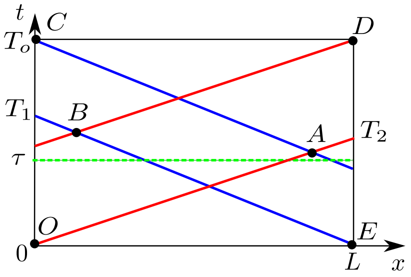

Fig. 1. Notation used in the proof of proposition 2.1

Since this problem is well-posed (see e.g. [25]), it has a unique solution $(\hat{y}_1^l, \hat{y}_2^l)$ on the maximum determinate domain (the triangle OAC). This solution is the restriction of the solution (y_1, y_2) of (1) on OAC. Similarly, the problem (11) with the initial condition specified at $x = L$:

$$\begin{array}{l} \hat{y}_1(L,s) = v_1(s) \\ \hat{y}_2(L,s) = m_2(s) \end{array} , \quad s \in [0, T_o], \tag{13}$$

has a unique solution $(\hat{y}_1^r, \hat{y}_2^r)$ on its maximum determinate domain (the triangle EDB). This solution is also the restriction of the solution (y_1, y_2) of (1) on EDB. Since $T_o > T_1$, the two triangles must have a common region. Hence, there exist some τ such that we can determine $\hat{y}_1(\cdot,\tau) = y_1(\cdot,\tau)$ and $\hat{y}_2(\cdot,\tau) = y_2(\cdot,\tau)$ (which is the combination of $(\hat{y}_1^l, \hat{y}_2^l)$ and $(\hat{y}_1^r, \hat{y}_2^r)$). We solve now problem (8) backward in time, with the initial condition at $t = \tau$ and the boundary condition

$$\begin{array}{l} \hat{y}_1(0,s) = m_1(s) \\ \hat{y}_2(L,s) = m_2(s) \end{array} , s \in [0, \tau]. \tag{14}$$

to obtain the solution $(\hat{y}_1^d, \hat{y}_2^d)$ which coincides with the solution (y_1, y_2) of (1) on $[0, L] \times [0, \tau]$. Particularly, $(\hat{y}_1^d(\cdot,0), \hat{y}_2^d(\cdot,0)) = (y_1(\cdot,0), y_2(\cdot,0))$ which implies that $(\hat{y}_1^d(\cdot,0), \hat{y}_2^d(\cdot,0))$ is the optimal solution of (7) since $J(\hat{y}_1^d(\cdot,0), \hat{y}_2^d(\cdot,0)) = 0$. Finally, by the construction, this is also the unique solution. □

Remark 2: Although the above constructive method gives us the exact initial state, a minimization formulation as (7) is necessary because of the presence of measurement noise and possible model error (see the simulation in section V).

III. COMBINATION OF RECEDING HORIZON OPTIMAL CONTROL AND RECEDING HORIZON OBSERVER

We propose here a control scheme to stabilize system (1)-(2) using boundary measurements, given that the observer horizon T_o is sufficiently large. In order to collect sufficient data for the observer, we need to fix an activation time $T_a > T_o$ at which the observer and the control are activated. The algorithm consists of the following steps:

1) For $t \leq T_a$, do nothing.
2) At time $t > T_a$, solve the optimization problem (7)-(9) using the measurements $m_1(s)$ and $m_2(s)$, $s \in [t - T_o, t]$.
3) With appropriate control horizon T_c, use the estimation $\hat{y}_1(\cdot,t)$ and $\hat{y}_2(\cdot,t)$ as initial condition to solve the following optimization problem

$$\min_{\bar{u}_1,\bar{u}_2} J_c = \int_{t}^{t+T} F(\bar{z}(\tau), \bar{u}(\tau))d\tau + E(\bar{z}(t+T_c))$$

with $\bar{z}(\tau) = (\bar{y}_1(\cdot,\tau) \quad \bar{y}_2(\cdot,\tau) \quad \bar{v}_1(\tau) \quad \bar{v}_2(\tau))^T$ and $\bar{u}(\tau) = (\bar{u}_1(\tau) \quad \bar{u}_2(\tau))^T$.

$$\text{s.t.} \left\{ \begin{array}{l} \partial_\tau \bar{y}_1 = c_1 \partial_x \bar{y}_1 + \gamma \bar{y}_1 + \delta \bar{y}_2 \\ \partial_\tau \bar{y}_2 = c_2 \partial_x \bar{y}_2 + \gamma \bar{y}_1 + \delta \bar{y}_2 \end{array} \right. , \tau \in (t, t+T_c]. \tag{15}$$

with the initial conditions

$$\left\{ \begin{array}{l} \bar{y}_1(\cdot,t) = \hat{y}_1(\cdot,t) \\ \bar{y}_2(\cdot,t) = \hat{y}_2(\cdot,t) \end{array} \right. . \tag{16}$$

and boundary conditions

$$\left\{ \begin{array}{l} \bar{y}_1(L,t) = \bar{v}_1(t) \\ \bar{y}_2(0,t) = \bar{v}_2(t) \end{array} \right. , \left\{ \begin{array}{l} \dot{\bar{v}}_1 = \bar{u}_1 \\ \dot{\bar{v}}_2 = \bar{u}_2 \end{array} \right. \tag{17}$$

where the notation $\bar{\cdot}$ stands for the predicted variables.

4) Apply the first part of the optimal control in period $[t, t+\sigma]$ with a small σ, and repeat step 2 at $t = t + \sigma$.

We would like to make some remarks on the formulation of the RHOC in step 3. Firstly, we extended the boundary state as in (17) in order to reformulate the optimal control problem in the abstract form in which the proof of the existence and uniqueness of the optimal solution can be obtained (see [23]). Secondly, the stage cost F and the final cost E must be well chosen to guarantee the stability of the closed-loop. In [23], we showed that if E and F take the following form (inspired from the Lyapunov function proposed in [5]):

$$\begin{aligned} E(z) &= \int_0^L \gamma y_1^2(x) + \delta y_2^2(x) dx + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}^T P_f \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ F(z,u) &= \int_0^L \epsilon (y_1^2(x) + y_1^2(x)) dx \\ &\qquad + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}^T P \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}^T R \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{aligned}$$

then there exist parameters P_f, P, ϵ, μ and R such that for all initial condition $(y_{10}(\cdot), y_{20}(\cdot))$, one can find a control u_K satisfying

$$\dot{E}(z(\tau)) + F(z(\tau), u_K(\tau)) \leq 0 \tag{18}$$

along the trajectory of (1) (with extended state $\dot{v}_i = u_i$, $i = 1, 2$). As a consequence, the RHOC using the real state guarantees an exponential stability of the closed-loop system.

We can notice that for $t \leq T_a$, since (1)-(2) is well-posed, the trajectory remains bounded (see e.g. [25]). At each sample time $t > T_a$, by Proposition 2.1 and without noise, the observer gives the exact state $(y_1(\cdot,t), y_2(\cdot,t))$ which is used in the control scheme. Combining with the stability result of [23], we can conclude that the proposed Receding Horizon Optimal Control - Observer scheme stabilizes system (1) in nominal condition.

Remark 3: The receding observer proposed in section II can also be applied to a more general class of 2×2 hyperbolic systems having the following form

$$\begin{array}{l} \partial_t y_1 = c_1(x) \partial_x y_1 + \gamma_1(x) y_1 + \delta_2(x) y_2 \\ \partial_t y_2 = c_2(x) \partial_x y_2 + \gamma_2(x) y_1 + \delta_2(x) y_2 \end{array}, x \in [0, L], \, t \geq 0. \tag{19}$$

where the coefficients $c_i(x)$, $\gamma_i(x)$ and $\delta_i(x)$ are smooth function of x. In this case, in the RHOC-O scheme, we need to use the RHOC with zero terminal constraint as in [24].

Remark 4: Unlike [29], the measurements here are supposed to be made on both boundaries. This assumption can however be relaxed to one measurement without changing the existence proof of the optimal state estimation (see [16]). This extension will be investigated deeper in our future work.

IV. NUMERICAL SOLUTION

We will use the calculus of variations approach to formulate the optimal control as a function of the adjoint state in order to be used for the numerical scheme.

A. Calculus of variations approach

Consider the problem (7)-(9) and introduce for it the Lagrangian as follows:

$$\begin{aligned} L \; = \; & \int_0^{T_o} \left[q_1(\hat{y}_1(0,s) - m_1(s))^2 \right. \\ & \qquad \left. + \, q_2(\hat{y}_2(L,s) - m_2(s))^2 \right] ds \\ & + \int_0^{T_o} \int_0^L [\lambda_1 (c_1 \partial_x \hat{y}_1 + \gamma \hat{y}_1 + \delta \hat{y}_2 - \partial_s \hat{y}_1) \\ & \qquad + \lambda_2 (c_2 \partial_x \hat{y}_2 + \gamma \hat{y}_1 + \delta \hat{y}_2 - \partial_s \hat{y}_2)] dx ds \end{aligned}$$

where λ_1 and λ_2 are the adjoint states corresponding to $\hat{y}_1$ and $\hat{y}_2$.

The first variation of this function must be zero at the optimal solution (since the initial conditions are not bounded).

Integrating by part leads to:

$$\begin{aligned} L = \; & \int_0^{T_o} \left[q_1(\hat{y}_1(0,s) - m_1(s))^2 \right. \\ & \qquad \left. + \, q_2(\hat{y}_2(L,s) - m_2(s))^2 \right] ds \\ & \int_0^{T_o} \int_0^L [-c_1 y_1 \partial_x \lambda_1 + \gamma y_1 \lambda_1 + \delta y_2 \lambda_1 + y_1 \partial_t \lambda_1 \\ & \qquad - c_2 y_2 \partial_x \lambda_2 + \gamma y_1 \lambda + \delta y_2 \lambda_1 + y_2 \partial_t \lambda_2] dx ds \\ & + \int_0^{T_o} [c_1 y_1 \lambda_1]_0^L ds - \int_0^L [y_1 \lambda_1]_0^T dx \\ & + \int_0^{T_o} [c_2 y_2 \lambda_2]_0^L ds - \int_0^L [y_2 \lambda_2]_0^T dx \end{aligned}$$

Then:

- First variations of $\hat{y}_1(x,t)$ and $\hat{y}_2(x,t)$ give the dynamics of the adjoint states:

$$\left\{ \begin{array}{l} \partial_t \lambda_1 - c_1 \partial_x \lambda_1 + \gamma \lambda_1 + \gamma \lambda_2 = 0 \\ \partial_t \lambda_2 - c_2 \partial_x \lambda_2 + \delta \lambda_1 + \delta \lambda_2 = 0 \end{array} \right. . \tag{20}$$

- First variations of $\hat{y}_1(x,T)$, $\hat{y}_2(x,T)$ impose the condition at $t = T$ for the adjoint states:

$$\lambda_1(x,T) = 0, \quad \lambda_2(x,T) = 0. \tag{21}$$

- First variations of $a(0,t)$, $a(1,t)$ and $b(0,t)$, $b(1,t)$ give the boundary conditions:

$$\begin{array}{c} 2q_1(\hat{y}_1(0,s) - m_1(s)) - \lambda_1(0,s) c_1 = 0 \\ 2q_2(\hat{y}_2(0,s) - m_2(s)) + \lambda_2(L,s) c_2 = 0 \end{array} . \tag{22}$$

- First variations of $\hat{y}_1(x,0)$ and $\hat{y}_2(x,0)$ give the optimal condition:

$$\lambda_1(x,0) = 0, \quad \lambda_2(x,0) = 0 \; . \tag{23}$$

The gradient of the cost function which is used later in the steepest descend method is given by:

$$L_{\hat{y}_0} = \left[\begin{array}{c} \int_0^L \lambda_1(x,0)dx \\ \int_0^L \lambda_2(x,0)dx \end{array} \right]. \tag{24}$$

where $\hat{y}_0 = (\hat{y}_{10} \quad \hat{y}_{20})^T$.

B. Computation of the numerical solution

We use here the steepest descend method to solve the optimal problem (7)-(9):

1) Choose an initial estimation of the solution $\hat{y}_0^{(0)}$.
2) Solve system (8) from $t = 0$ to $t = T_o$ by using a discretization method.
3) Solve adjoint state (20) from $t = T_o$ to $t = 0$.
4) Update the estimation by the steepest descend method
$$\hat{y}_0^{(k+1)} = \hat{y}_0^{(k)} - K_{\hat{y}} L_{\hat{y}_0}, \tag{25}$$
with the step $K_{\hat{y}} > 0$ which must be tuned to satisfy a trade-off between convergence speed and numerical stability.
5) Repeat step 2 while the norm of the gradient $L_{\hat{y}_0}$ is larger than a given tolerance.

C. Lattice Boltzmann Method

We will use the so-called Lattice Boltzmann Method (LBM) to solve (8) and (20). This method starts to attract the researchers as an alternative for traditional ones such as finite volume method thanks to its powerful capacities to simulate the fluid flows and other physical phenomena (see [4]).

In this method, we consider the movement of the particles on a regular grid. The particles at point (x,t) are split in several quantities $f_i(x,t)$ which has the velocities v_i. The underlying algorithm consists of the alternative of two phases: collision phase and streaming phase. In the collision phase, the particles enter the site x at instant t $f^{in}(x,t)$ collide and result a new distribution $f_i^{out}(x,t)$. Then during the period $[t, t + \Delta t]$ of the streaming phase, the new distribution moves to the lattice site in the direction of the velocity v_i. These two phases can be formulated as:

$$\begin{array}{ll} \text{Collision :} & f_i^{out}(x,t) = f_i^{in}(x,t) + \frac{1}{\tau}(f_i^{eq} - f_i^{in}) + F_i, \\ \text{Streaming :} & f_i^{in}(x + v_i\Delta t, t + \Delta t) = f_i^{out}(x,t), \end{array} \tag{26}$$

where F_i represents the contribution of the force term to the collision, Δx is the lattice spacing and Δt the time step. The collision phase above is based on the *Bhatnagar-Gross-Krook* (BGK) approximation with a single relaxation time τ and the so-called equilibrium distribution functions f_i^{eq} whose expression depends on the physical process to be described. To guarantee the stability of this explicit scheme, τ must be larger than 0.5.

Equations (26) may be combined to obtain the evolution equation:

$$f_i(x + v_i\Delta t, t + \Delta t) = f_i(x,t) + \frac{1}{\tau}(f_i^{eq} - f_i) + F_i, \tag{27}$$

where f stands for f^{in}.

Equations in (8) and (20) have the following form:

$$\partial_t \lambda + c\partial_x \lambda = F, \tag{28}$$

where F stands for the source term. Consider firstly the case $c > 0$. Hence, only the boundary condition at $x = 0$ needs to be defined as $\lambda(0,t) = u(t)$. We propose a lattice with 2 velocities as in Fig. 2 where $v_0 = 0$ and $v_1 = v = \Delta x/\Delta t$.

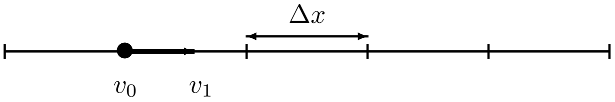

Fig. 2. Two velocities lattice ($c > 0$)

The equilibrium distribution function must satisfy:

$$\left\{ \begin{array}{rcl} f_0^{eq} + f_1^{eq} & = & \lambda \\ v f_1^{eq} & = & c\lambda \end{array} \right. \Leftrightarrow \left\{ \begin{array}{rcl} f_0^{eq} & = & \left(1 - \frac{c}{v}\right)\lambda \\ f_1^{eq} & = & \frac{c}{v}\lambda \end{array} \right. , \tag{29}$$

and λ is computed by $\lambda = \sum_i f_i = f_0 + f_1$. With this equilibrium distribution function, the LBM can be proved to solve (28) in the case without the source term F up to the precision $(\Delta x)^2$ and $(\Delta t)^2$ (see [14]).

In this simple case, the evolution equation (27) has the form:

$$\begin{array}{rcl} f_0(x, t + \Delta t) & = & f_0(x,t) + \frac{1}{\tau}(f_0^{eq} - f_0) + \Delta t F, \\ f_1(x + v\Delta t, t + \Delta t) & = & f_1(x,t) + \frac{1}{\tau}(f_1^{eq} - f_1). \end{array}$$

The boundary condition at point $x = 0$ is guaranteed by imposing $f_1(0,t) = u(t) - f_0(0,t)$. In the case of $c < 0$ we use the lattice where $v_0 = 0$ and $v_1 = -v = -\Delta x/\Delta t$.

V. APPLICATION TO LINEARIZED SAINT-VENANT EQUATIONS

In order to illustrate and validate the proposed control technique and numerical implementation scheme, we apply the above approach to the linearized Saint-Venant equations.

A. Linearized Saint-Venant equations around a non-uniform profile

An open-channel system is usually described by a set of two partial differential equations (PDEs) named Saint-Venant equations which represent the mass and the momentum conservation as follows:

$$\begin{array}{rcl} B\partial_t h + \partial_x Q & = & 0, \\ \partial_t Q + \partial_x \left(\frac{Q^2}{Bh} + \frac{1}{2}Bgh^2 \right) & = & gBh(I - J(Q,h)), \end{array}$$

where h denotes the water depth, Q the discharge, g the gravitational acceleration, B the channel width, I the slope and J the friction term. The friction is modeled by the classical Manning formula [17]:

$$J(h,Q) = \frac{Q^2}{k^2 B^2 h^2 \left(\frac{Bh}{B+2h}\right)^{4/3}}, \tag{30}$$

where k is the Manning-Strickler coefficient. The boundary conditions are specified by the gate equation:

$$\begin{array}{l} Q^2(0,t) = K_2^2 \theta_2^2(t) 2g(h_{us} - h(0,t)), \\ Q^2(L,t) = K_1^2 \theta_1^2(t) 2g(h(L,t) - h_{ds}), \end{array} \tag{31}$$

where h_{us} is the water height at the upstream of the gate at $x = 0$, h_{ds} is the water height at the downstream of the gate at $x = L$. $\theta_1(t)$ and $\theta_2(t)$, the opening of the gates, are control variables. K_1 and K_2 are constants which depend on the structure of the gates. We consider now a uniform steady state of the system $(\bar{h}, \bar{Q})$ which has to satisfy $\bar{Q} = \text{constant}$ and $J(\bar{h}, \bar{Q}) = I$.

We consider next the deviation of the state h and Q around the steady state $(\bar{h}, \bar{Q})$: $\tilde{h} = h - \bar{h}$, $\tilde{Q} = Q - \bar{Q}$. We obtain then:

$$\begin{aligned} \tilde{h}_t &= -B^{-1}\tilde{Q}_x, \\ \tilde{Q}_t &= \alpha(x)\tilde{h}_x + \beta(x)\tilde{Q}_x + \gamma(x)\tilde{h} + \delta(x)\tilde{Q}, \end{aligned} \tag{32}$$

with

$$\begin{aligned} &\alpha = -\left(Bg\bar{h} - \tfrac{\bar{Q}^2}{B\bar{h}^2}\right), \quad \beta = -\tfrac{2\bar{Q}}{B\bar{h}}, \\ &\rho = -gB\bar{h}\bar{J}_h, \quad \phi = -gB\bar{h}\bar{J}_Q. \end{aligned} \tag{33}$$

For a reason of readability, we adopt the notations that f_α represents the derivative of a function f in α, and $\bar{f}$ is the value of f at the steady state $(\bar{h}, \bar{Q})$. Let us additionally define:

$$G(x) = \begin{pmatrix} 0 & -B^{-1} \\ \alpha(x) & \beta(x) \end{pmatrix}, \quad H(x) = \begin{pmatrix} 0 & 0 \\ \gamma(x) & \delta(x) \end{pmatrix}.$$

In the sub-critical regime (low flow speed), G has two eigenvalues satisfying

$$c_1 = -\frac{\bar{Q}}{B\bar{h}} + \sqrt{g\bar{h}} > 0 \text{ and } c_2 = -\frac{\bar{Q}}{B\bar{h}} - \sqrt{g\bar{h}} < 0.$$

By applying the transformation

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = P^{-1}\begin{pmatrix} \tilde{h} \\ \tilde{Q} \end{pmatrix} \text{ with } P = \begin{pmatrix} 1 & 1 \\ -B\lambda_1 & -B\lambda_2 \end{pmatrix},$$

we have a new system:

$$\partial_t \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \partial_x \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \gamma & \delta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{34}$$

with $\gamma = -\frac{\rho - B\phi\lambda_1}{2B\sqrt{g\bar{h}}} < 0$ and $\delta = -\frac{-\rho + B\phi\lambda_1}{2B\sqrt{g\bar{h}}} < 0$. The gate opening θ_1 and θ_2 can be calculated in order to implement the boundary conditions in the form

$$y_1(L, t) = v_1(t), \quad y_2(0, t) = v_2(t). \tag{35}$$

In fact, these gate openings are given by:

$$\begin{aligned} \theta_1(t) &= \frac{-2B\sqrt{g\bar{h}}v_2(t) - B\lambda_2\tilde{h}(L,t) - \bar{Q}}{K_1\sqrt{2g(\tilde{h}(L,t) + \bar{h} - h_{ds})}}, \\ \theta_2(t) &= \frac{-2B\sqrt{g\bar{h}}v_2(t) - B\lambda_1\tilde{h}(0,t) - \bar{Q}}{K_2\sqrt{2g(h_{us} - \tilde{h}(0,t) - \bar{h})}}. \end{aligned} \tag{36}$$

System (34)-(35) has the form of (1) and can be stabilized by the RHOC-O scheme.

B. Simulation results

In this section a simulation has been carried out with the real parameters of a section of the *canal de la Bourne* in France ($L = 3000(m)$, $B = 4.36(m)$, $I = 2.4.10^{-4}$ and $k = 30$). The linearized model is around an equilibrium ($\bar{h} = 1.97(m)$ and $\bar{Q} = 4.48(m^3/s)$). The observer and the control activated at $T_a = 3000(s)$ are both solved by the steepest descent method where the PDEs are discretized by LBM. The control-observer scheme is tested with a nonlinear model made by the LBM with three speeds (see [20]). In order to validate the robustness of the proposed algorithm, we added a zero-mean noise in the measurements. The observer takes $T_o = 918(s)$ and $q_{1,2} = 1$. The parameters of the control scheme are presented in Table I. The numerical simulation is carried out with space step $\Delta x = 300(m)$ and time step $\Delta t = 1(s)$.

Parameters	Values	Parameters	Values
T_c	$30(s)$	σ	$10(s)$
R	$\text{diag}(10, 10)$	ϵ	4×10^{-6}
P_f	$\text{diag}(0, 01; 0, 01)$	P	$\text{diag}(0, 55; 0, 55)$

TABLE I

CONTROL PARAMETERS

The results are presented in Fig. 3-Fig. 5. We can observe from Fig. 3 and Fig. 4 that the estimated water level and discharge at the boundaries fit well with the real values, especially in the stationary regime. Fig. 5 presents the normalized error Err of the observer calculated at each sample time by:

$$\begin{aligned} Err(t) = &\frac{1}{L\bar{h}^2}\int_0^L (\tilde{h}(x,t) - \hat{\tilde{h}}(x,t))^2 dx \\ &+ \frac{1}{L\bar{Q}^2}\int_0^L (\tilde{Q}(x,t) - \hat{\tilde{Q}}(x,t))^2 dx. \end{aligned} \tag{37}$$

where $\hat{\tilde{h}}$ and $\hat{\tilde{Q}}$ are the water level and discharge given by the observer. We realize that this error is less than 0.05%. These results coincide well with the theories developed in the previous sections.

One final remark is that the computation time was around $30(s)$ for the observer and $1(s)$ for the optimal control on an Intel Core 2 Duo 2.4Ghz, 1.95Go RAM Laptop. As mentioned in [20], a C++ implementation of the LBM is typically 100 times faster, without any code optimization, which means that real-time control is possible.

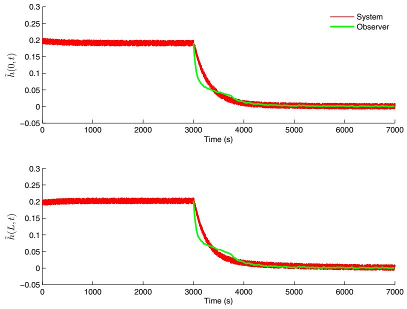

Fig. 3. Water level of the closed-loop system at the boundaries

VI. CONCLUSIONS

In this paper, a Receding Horizon Observer for a class of 2×2 hyperbolic systems of conservation laws was proposed and validated by simulation. The existence of the optimal state estimation is established by using a direct constructive method. For the purpose of practical implementation, the calculus of variations was employed to derive the adjoint state. The state and the adjoint state were then discretized, and the optimal solution was obtained by the steepest descend method. A Receding Horizon Optimal Control -

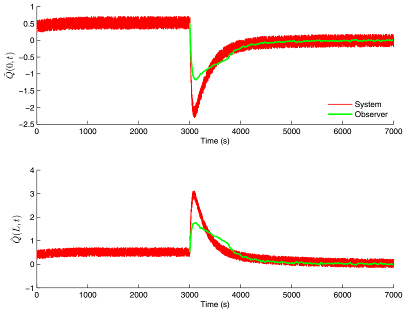

Fig. 4. Discharge of the closed-loop system at the boundaries

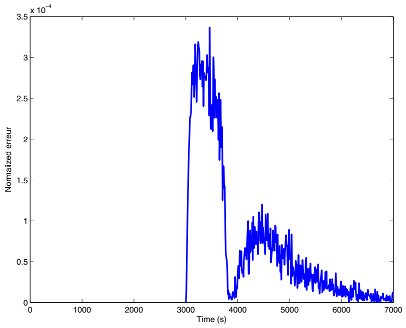

Fig. 5. Normalized error of the estimated state

Observer (RHOC-O) was also presented. Finally, a numerical simulation of a real channel demonstrated the effectiveness of the here-proposed approach.

These results open some directions for future works. Among them, we can mention the possibility to extend the approach for a network of conservation laws, such as a channel network. Another goal is to investigate a nonlinear observer to increase accuracy, knowing that the constructive method can be applied to nonlinear hyperbolic systems as well (see [16]). Other discretization methods are under investigation to improve computation time. The theory developed here encourages the implementation of the RHOC-O for a real-time application up to when the linearized model is still valid.

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