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Embedding the generalized Acrobot into the n-link with an unactuated cyclic variable and its application to walking design*

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Abstract— **The Acrobot is the well-known and widely studied underactuated mechanical system having two links and one actuated joint between them. It may be also viewed as the simplest possible walking like mechanism without knees and the ankle-joint actuation, alternatively also referred to as the underactuated Compass gait walker. To extend techniques used to control the Acrobot to a more general underactuated n-link having an unactuated cyclic variable, this paper defines the so-called generalized Acrobot. Further, it is shown that for every set of virtual constraints there exists a generalized Acrobot that is linearly embedded into this n-link. Based on this property and results valid for the Acrobot, walking strategies for the n-link are provided. Important achievement here is that the exponentially stable tracking during the swing phase only is possible, i.e. the stabilizing effect of the impact map is not needed. Computer simulations of the 4-link case are provided.**

I. INTRODUCTION

Underactuated mechanical systems have been widely and intensively studied during the recent decades with special stress to underactuated walking [25], [17], [27], [11], [16]. The simplest model in this respect is the so-called Acrobot, or also underactuated Compass gait walker (i.e. the one without anle-joint), [8], [9], [10], [21], [14], [28], [13], [2], [1], [4], [6], [3] which has two links and one actuator between them, see Fig. 1. When supported on the planar surface, it may serve as the simplest, though hardly implementable, biped walking-like mechanism without knees and ankles. Numerous results on its walking-like movement suggest the idea to try to extend them to the general n-link case.

In this paper, the general planar n-link, which is underactuated at the pivot point on the walking surface will be considered. To use efficient approaches developed for the Acrobot, the technique known in robotics as the virtual constraint, see *e.g.* [7], will be used. This technique was further developed and interpreted via control theoretic terminology *e.g.* in [27], [16], [11]. Nevertheless, virtual constraints were typically used there to restrict the n-degrees of freedom system with $n-1$ actuators into a certain one degree of freedom system without actuation. This lead to the key and original notion of [27], [16], [11]: the so-called **hybrid zero dynamics** enabling to design exponentially stable multi-step walking trajectories. The strength of the hybrid zero dynamics approach is that it achieves the exponential stability of the overall hybrid system combining the swing and impact phases despite the fact that swing phase itself is unstable.

The current paper offers an alternative approach. The constrained dynamics being the two degrees of freedom system with a single actuator will be imposed, so-called later on as the **generalized Acrobot**. Results developed previously for the Acrobot will be further improved and used straightforwardly to impose the exponentially stable tracking of the n-link target trajectory even during a single swing phase only. This opens new ways how to design hybrid multi-step stable walking later on.

The decomposition of the n-link was first studied in [17]. For the Acrobot there is always an integrable (up to a suitable factor) generalized momentum, which leads to special canonical partially linear forms [26], [12], [22], [24], [23], [13]. The natural approach in [17] was therefore to find the largest integrable part of the generalized momentum, this approach, nevertheless leads to a decomposition which is independent of any selected particular virtual constraints, therefore it can not provide embedding reflecting these constraints. The approach presented in the current paper might be in that context also viewed as selecting an integrable part of the generalized momentum depending on the particular selection of virtual constraints to be imposed.

Summarizing, the current paper contribution is as follows. First, the so-called generalized Acrobot is defined and a special state and input coordinate transformations of the general n-link unactuated at the pivot point is developed. Such a transformation depends on arbitrarily selected $n-2$ smooth functions, the so-called virtual constraints. For every selection of these $n-2$ smooth virtual constraints the unique generalized Acrobot is defined and embedded into the original n-link. Furthermore, the design of the Acrobot target walking-like trajectory and its exponential tracking [13], [2], [1], [4], [6], [3], [5] is extended to the case of the generalized Acrobot and further improved. These results open new ways to n-link walking design which is illustrated by the 4-link case. The current paper concentrates on the swing phase of the single step only and using the full state feedback controllers. The interesting feature here is that walking-like trajectory can be exponentially tracked during the swing phase only. Some outlooks for the future multi-step walking and the measurement feedback controllers are given, relying on the results of [4], [6], [3].

The paper is organized as follows. The next section intro-

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duces embedding of nonlinear systems, while Section 3 repeats briefly some facts on the mechanical systems modelling and defines the notion of the generalized Acrobot. Main results are collected in Section 4. Section 5 presents control design techniques for the generalized Acrobot, including a new results, Theorem 3. The illustrative design for the 4-link is presented in the same section. Conclusions and outlooks for future research are briefly discussed in the final section.

II. NONLINEAR SYSTEMS EMBEDDING

Consider smooth manifolds M_1, M_2 and let τ_1, τ_2 be their topologies, cf. [18], [19]. Let $\mathcal{T} : M_1 \mapsto M_2$ be a smooth mapping and denote $\mathrm{d}\mathcal{T}$ its tangent map. This smooth mapping $\mathcal{T}$ is called as the embedding of M_1 into M_2, if:

(i) $\mathcal{T}$ is injective;

(ii) $\mathrm{rank}[\mathrm{d}\mathcal{T}] = \dim M_1$ everywhere on M_1;

(iii) $\forall U_1 \in \tau_1 \; \exists U_2 \in \tau_2$, such that $\mathcal{T}(U_1) = U_2 \cap \mathcal{T}(M_1)$.

The last property means that arbitrarily close points of $\mathcal{T}(M_1)$ should correspond to arbitrarily close points of M_1.

Definition 1: Consider the following controlled systems having states x, ξ and inputs u, μ (in the sequel, set $\mathbb{R}^0 := \emptyset$)

$$\dot{x} = f^1(x,u), \; x \in \mathbb{R}^{n_1}, \; u \in \mathbb{R}^{m_1} \quad (1)$$

$$\dot{\xi} = f^2(\xi,\mu), \; \xi \in \mathbb{R}^{n_2}, \; \mu \in \mathbb{R}^{m_2}. \quad (2)$$

where $n_1 \geq n_2 \geq 1$, $m_1 \geq m_2 \geq 0$. System (2) is said to be globally embedded into the system (1), if there exists an embedding $\mathcal{T}(\xi,\mu)$ of $\mathbb{R}^{n_2} \times \mathbb{R}^{m_2}$ into $\mathbb{R}^{n_1} \times \mathbb{R}^{m_1}$ such that

$$\mathcal{T}(\xi,\mu) = \begin{bmatrix} T(\xi) \\ \alpha(\xi,\mu) \end{bmatrix}, \quad (3)$$

$$T : \mathbb{R}^{n_2} \mapsto \mathbb{R}^{n_1}, \qquad \alpha : \mathbb{R}^{n_2} \times \mathbb{R}^{m_2} \mapsto \mathbb{R}^{m_1}, \quad (4)$$

$$\left[\frac{\partial T(\xi)}{\partial \xi}\right] f^2(\xi,\mu) = f^1(T(\xi), \alpha(\xi,\mu)). \quad (5)$$

Remark 1: For $n_1 = n_2$ and $m_1 = m_2$ Definition 1 gives the well-known equivalence of systems via smooth global change of coordinates and static state feedback. Nevertheless, in general, this definition is not related with dynamic feedback equivalence. Notice also that if a system Σ_2 is embedded globally into an another system Σ_1, then Σ_2 is also obviously embedded into any globally state and feedback equivalent system to Σ_1.

In the sequel, yet another notion will be needed.

Definition 2: System (2) is said to be globally linearly embedded into the system (1), if there is a controllable pair (F, G), F, G having the appropriate dimensions, such that

(i) (1) is globally smooth state and feedback equivalent to the following system having the state $z = (z^1, z^2)^\top$ and the input $v = (v^1, v^2)^\top$

$$\dot{z}^1 = Fz^1 + Gv^1, \quad z^1 \in \mathbb{R}^{n_1-n_2}, \; v^1 \in \mathbb{R}^{m_1-m_2}, \quad (6)$$

$$\dot{z}^2 = \bar{f}^2(z^1, z^2, v^1, v^2), \quad z^2 \in \mathbb{R}^{n_2}, \; v^2 \in \mathbb{R}^{m_2}, \quad (7)$$

(ii) (2) is globally embedded into (6,7) via the embedding

$$[\xi, \mu]^\top \mapsto [0_{n_1-n_2}, T^2(\xi), 0_{m_1-m_2}, \alpha^2(\xi,\mu)]^\top, \quad (8)$$

where $[T^2(\xi), \alpha^2(\xi,\mu)]^\top : \mathbb{R}^{n_2} \times \mathbb{R}^{m_2} \mapsto \mathbb{R}^{n_2} \times \mathbb{R}^{m_2}$ is a global diffeomorphism.

Definition 2 may be viewed as a generalization of the well-known input-output linearization and zero dynamics notion as shown by the following proposition.

Proposition 1: Let $m_2 = 0$, *i.e.* (2) does not have any input. Then, (2) is globally linearly embedded into the system (1) in the sense of Definition 2 if and only if there exists output $h(x)$ for which (1) is globally input-output linearizable and (2) is globally state equivalent to its zero dynamics.

Proof: Note that Definition 2 requires that the system

$$\dot{z}^2 = \bar{f}^2(0, z^2, 0, v^2), \quad (9)$$

is globally equivalent to (2) via the smooth state and feedback transformation $[T^2(\xi), \alpha^2(\xi,\mu)]^\top$. Furthermore, let $m_2 = 0$ and let matrix H be such that (H, F) is an observable pair, then (6-7) having the output Hz^1 is globally input-output linear and (9) is its zero dynamics. Therefore, proofs follows by choosing $h(x)$ as Hz^1 transformed into x-coordinates. □

The following result will be used later on to reduce the exponentially stable tracking design of the walking like trajectory of the general n-link to the same problem for the so-called generalized Acrobot.

Theorem 1: Let (2) be globally linearly embedded into system (1) via embedding (3) and let $\xi^r(t), t \in [0,\infty)$ be the solution of (2) for some $\xi^r(0) = \xi_0$ and globally uniformly bounded $\mu(t) = \mu^r(t), \; t \in [0,\infty)$. If (2) with $\mu = F_{tr}(\xi^r(t), \xi, \mu^r(t))$ locally exponentially tracks $x^r(t)$, then there exists $u = u(x, \xi^r(t), \mu^r(t))$ making (1) to track locally exponentially $x^r(t) = T(\xi^r(t))$, T given by (3).

Proof: Let (2) be globally linearly embedded into (1) and $\mathcal{T} = [T_l(x), \alpha_l(x,u)]^\top, \mathcal{T}^{-1} = [T_l^{-1}(z), \alpha_l^I(z^1, z^2, v^1, v^2)]^\top$ be the global state and feedback transformations taking (1) to the form (6). Further, let K be a matrix of appropriate dimension such that $F + GK$ is Hurwitz. Then it is straightforward to prove that the control law (substitute $\xi = T(x)$ and $z = T_l(x)$)

$$u = \alpha^I\Big(0, T^2(\xi), Kz^1, \alpha^2\big(\xi, F_{tr}(\xi^r(t), \xi, \mu^r(t))\big)\Big) \quad (10)$$

provides the desired local exponential tracking. □

III. N-LINK AND GENERALIZED ACROBOT

The walking like mechanisms studied in this paper is the planar n-link having $n-1$ actuators between their joints and being supported on the surface at the end of of those links, called as the **pivot point**. This mechanical system can be described by n generalized coordinates being the angle q_1 at the pivot point and the angles $q_2, \ldots, q_n$ between the remaining links. The coordinate q_1 is therefore the unactuated one and the overall n-link is the so-called underactuated mechanical system [25]. Its model can be obtained using the well-known Euler-Lagrange approach, [15], using the following Lagrangian

$$\mathcal{L}(q,\dot{q}) = K(q,\dot{q}) - V(q) = \frac{1}{2}\dot{q}^T D(q)\dot{q} - V(q), \quad (11)$$

where $q = (q_1, \dots, q_n)^\top$ denotes the n-dimensional vector of the generalized coordinates, $\dot q = (\dot q_1, \dots, \dot q_n)^\top$ is that of the generalized velocities, $D(q)$ is the inertia matrix, K is the kinetic energy and V is the potential energy of the system. Dynamical equations can be obtained via the following Euler-Lagrange equations

$$\left[\begin{array}{c} \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q_1} - \frac{\partial \mathcal{L}}{\partial q_1} \\ \vdots \\ \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q_n} - \frac{\partial \mathcal{L}}{\partial q_n} \end{array}\right] = u = \left[\begin{array}{c} 0 \\ \tau_2 \\ \vdots \\ \tau_n \end{array}\right], \tag{12}$$

where u stands for the vector of generalized forces, being the torques at the actuated points used to control the system. This gives the standard model of the mechanical systems

$$D(q)\ddot q + C(q, \dot q)\dot q + G(q) = u = [0, \tau_2, \dots, \tau_n]^\top. \tag{13}$$

Here, $C(q, \dot q)$ contains Coriolis and centrifugal terms, $G(q) = -\nabla V(q)$ contains the gravity terms.

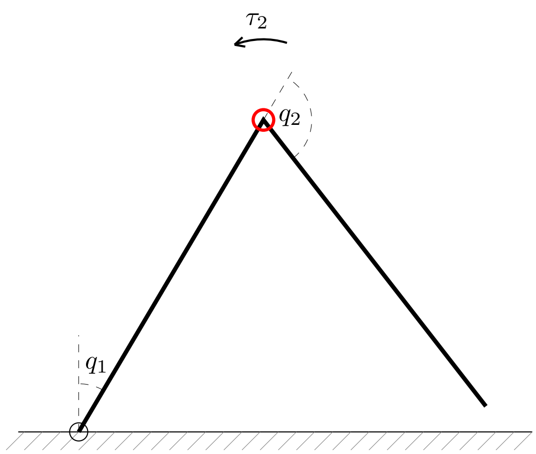

Fig. 1. The Acrobot.

The crucial property of the above walking like configuration of the underactuated planar n-link is the so-called **kinetic symmetry** with respect to the unactuated variable q_1

$$D(q) = D(q_2, \dots, q_n), \ \forall q \in \mathbb{R}^n. \tag{14}$$

Variable q_1 is also often called as the **cyclic** one.

The positive definite inertia matrix $D(q)$ in (13) is globally invertible and the dynamics of the system can be rewritten into the standard form defining the variables $x^1 = q, x^2 = \dot q$:

$$\begin{array}{rcl} \dot x^1 &=& x^2 \\ \dot x^2 &=& D(x^1)^{-1}(Bu - C(x^1, x^2)x^2 - G(x^1)). \end{array} \tag{15}$$

Example 1: The Acrobot depicted in Figure 1 is the particular case of the above described underactuated n-link where the corresponding model details have the following form

$$D = \left[\begin{array}{cc} 2(\theta_1 - \theta_2 + \theta_3 \cos q_2) & \theta_1 - 2\theta_2 + \theta_3 \cos q_2 \\ \theta_1 - 2\theta_2 + \theta_3 \cos q_2 & \theta_1 \end{array}\right] \tag{16}$$

$$C(q, \dot q) = \left[\begin{array}{cc} -2\theta_3 \sin q_2 \dot q_2 & -\theta_3 \sin q_2 \dot q_2 \\ \theta_3 \sin q_2 \dot q_1 & 0 \end{array}\right], \tag{17}$$

$$G(q) = \left[\begin{array}{c} -\theta_4 \sin q_1 - \theta_5 \sin(q_1 + q_2) \\ -\theta_5 \sin(q_1 + q_2) \end{array}\right], \tag{18}$$

$$\begin{array}{c} \theta_1 = m(l_c^2 + l^2) + I, \ \theta_2 = m l_c l, \ \theta_3 = ml(l - l_c), \\ \theta_4 = mg(l + l_c), \ \theta_5 = mg(l - l_c). \end{array} \tag{19}$$

As written in the Introduction in detail, numerous results and various techniques were obtained for the Acrobot walking like movement [1], [4], [6], [5], [3], [2], [13]. It turns out that all these techniques are merely relying on the above mentioned kinetic symmetry with respect to the unactuated variable (14). That suggests the idea to introduce the so-called **generalized Acrobot**.

Definition 3: **Generalized Acrobot (GA)** is the dynamical control system having the input $\bar u$ and the state $\bar q = (\bar q_1, \bar q_2)^\top, \dot{\bar q} = (\dot{\bar q}_1, \dot{\bar q}_2)^\top$ of the following form

$$\left[\begin{array}{c} \frac{d}{dt}\frac{\partial \overline{\mathcal{L}}}{\partial \dot{\bar q}_1} - \frac{\partial \overline{\mathcal{L}}}{\partial \bar q_1} \\ \frac{d}{dt}\frac{\partial \overline{\mathcal{L}}}{\partial \dot{\bar q}_2} - \frac{\partial \overline{\mathcal{L}}}{\partial \bar q_2} \end{array}\right] = \bar u = \left[\begin{array}{c} 0 \\ \bar\tau_2 \end{array}\right], \tag{20}$$

$$\overline{\mathcal{L}} = \overline{K}(\bar q_2, \dot{\bar q}) - \overline{V}(\bar q), \bar q = (\bar q_1, \bar q_2)^\top \in \mathbb{R}^2. \tag{21}$$

Here, the smooth function $\mathcal{L}$ is called as the GA Lagrangian, the smooth function $\overline{V}(\bar q) \geq 0$ is called as the GA potential energy, the smooth function $\overline{K}(\bar q_2, \dot{\bar q})$

$$K(\bar q_2, \dot{\bar q}) = \frac{1}{2}\dot{\bar q}^\top \overline{D}(\bar q_2)\dot{\bar q}, \ \overline{D}(\bar q_2) = \overline{D}(\bar q_2)^\top > 0, \tag{22}$$

is called as the GA kinetic energy and the (2×2) matrix $D(\bar q_2)$ as the GA inertia matrix.

Proposition 2: The generalized Acrobot is the dynamical control system having the input $\bar u$ and the state $\bar q = (\bar q_1, \bar q_2)^\top, \dot{\bar q} = (\dot{\bar q}_1, \dot{\bar q}_2)^\top$ of the following form

$$\overline{D}(\bar q_2)\ddot{\bar q} + \overline{C}(\bar q, \dot{\bar q})\dot{\bar q} + \overline{G}(\bar q) = \bar u = \left[\begin{array}{c} 0 \\ \bar\tau_2 \end{array}\right], \tag{23}$$

$$\overline{C}(\bar q, \dot{\bar q}) = \left[\frac{\partial \overline{D}(\bar q_2)}{\partial \bar q_2}\dot{\bar q}_2\right] - \frac{1}{2}\left[\begin{array}{c} 0 \\ \dot{\bar q}^\top \frac{\partial \overline{D}(\bar q_2)}{\partial \bar q_2} \end{array}\right]. \tag{24}$$

Proof: Straightforward computations. □

IV. EMBEDDING THE GENERALIZED ACROBOT

The generalized Acrobot introduced in the previous section has two degrees of freedom and kinetic symmetry with respect to the unactuated cyclic variable q_1. It will be shown in this section that for any n-link (12, 13) having unactuated cyclic variable q_1 and any collection of $n - 2$ smooth functions $\phi_3(q_2), ..., \phi_n(q_2)$ there exists a unique generalized Acrobot and its embedding into this n-link.

To proceed with, define the global smooth coordinate change of the state space of the n-link model (12) and smooth feedback as follows. Choose some set of smooth functions $\phi_3(q_2), ..., \phi_n(q_2)$ and denote new coordinates as $\bar q_1, \dots, \bar q_n, \dot{\bar q}_1, \dots \dot{\bar q}_n, \bar\tau_2, \dots, \bar\tau_n$. The coordinate change taking the "old" coordinates in (12, 13) into these new coordinates is defined as follows:

$$\begin{aligned}
\bar{q}_1 &= q_1,\ \bar{q}_2 = q_2,\\
\dot{\bar{q}}_1 &= \dot{q}_1,\ \dot{\bar{q}}_2 = \dot{q}_2,\ \bar{\tau}_2 = \tau_2,\\
\bar{q}_3 &= q_3 - \phi_3(q_2),\\
\dot{\bar{q}}_3 &= \dot{q}_3 - \tfrac{\partial \phi_3(q_2)}{\partial q_2}\dot{q}_2,\\
\bar{\tau}_3 &= \ddot{q}_3 - \tfrac{\partial \phi_3(q_2)}{\partial q_2}\ddot{q}_2 - \tfrac{\partial^2 \phi_3(q_2)}{\partial q_2^2}\dot{q}_2^2,\\
&\vdots\\
\bar{q}_n &= q_n - \phi_n(q_2),\\
\dot{\bar{q}}_n &= \dot{q}_n - \tfrac{\partial \phi_n(q_2)}{\partial q_2}\dot{q}_2,\\
\bar{\tau}_n &= \ddot{q}_n - \tfrac{\partial \phi_n(q_2)}{\partial q_2}\ddot{q}_2 - \tfrac{\partial^2 \phi_n(q_2)}{\partial q_2^2}\dot{q}_2^2,
\end{aligned} \tag{25}$$

where $\ddot{q}_2, \dots, \ddot{q}_n$ are substituted from

$$\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \\ \ddot{q}_n \end{bmatrix} = D^{-1}(q)\left(\begin{bmatrix} 0 \\ \tau_2 \\ \vdots \\ \tau_n \end{bmatrix} - C(q,\dot{q})\dot{q} - G(q)\right). \tag{26}$$

Note, that (26) follows easily from (13). To show that (25,26) defines a smooth coordinate change one has to prove that these transformations are smooth invertible. To do so, the corresponding smooth inverse transformations will be computed in an explicit and computationally convenient way that is easy to use in control implementations later on.

First, introduce the following notation (as a rule, lower indices denote scalar entries, upper indices denote a vector block or a matrix block of an appropriate dimension):

$$q = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}, D(q) = [d_{ij}] = \begin{bmatrix} d_{11} & D^{12} \\ (D^{12})^\top & D^2 \end{bmatrix}, \tag{27}$$

$$q = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}, C = \begin{bmatrix} C^1(q,\dot{q}) \\ C^2(q,\dot{q}) \\ \vdots \\ C^n(q,\dot{q}) \end{bmatrix}, G = \begin{bmatrix} G_1(q) \\ G_2(q) \\ \vdots \\ G_n(q) \end{bmatrix}. \tag{28}$$

To obtain the inverse transformation of (25), one has by (26)

$$\ddot{q}_1 = -\frac{1}{d_{11}}\left(D^{12}\begin{bmatrix} \ddot{q}_2 \\ \vdots \\ \ddot{q}_n \end{bmatrix} + C^1(q,\dot{q})\dot{q} + G_1(q)\right), \tag{29}$$

which enables to express actuating torques as follows

$$\begin{aligned}
\begin{bmatrix} \tau_2 \\ \tau_3 \\ \vdots \\ \tau_n \end{bmatrix} &= -\frac{(D^{12})^\top}{d_{11}}\left(D^{12}\begin{bmatrix} \ddot{q}_2 \\ \ddot{q}_3 \\ \vdots \\ \ddot{q}_n \end{bmatrix} + C^1\dot{q} + G_1\right)\\
&+D^2(q)\begin{bmatrix} \ddot{q}_2 \\ \ddot{q}_3 \\ \vdots \\ \ddot{q}_n \end{bmatrix} + \begin{bmatrix} C^2(q,\dot{q}) \\ C^3(q,\dot{q}) \\ \vdots \\ C^n(q,\dot{q}) \end{bmatrix}\dot{q} + \begin{bmatrix} G_2(q) \\ G_3(q) \\ \vdots \\ G_n(q) \end{bmatrix},
\end{aligned} \tag{30}$$

or, equivalently

$$\begin{bmatrix} \tau_2 \\ \tau_3 \\ \vdots \\ \tau_n \end{bmatrix} = \mathcal{D}(q)\begin{bmatrix} \ddot{q}_2 \\ \ddot{q}_3 \\ \vdots \\ \ddot{q}_n \end{bmatrix} + \mathcal{C}(q,\dot{q})\dot{q} + \mathcal{G}(q), \tag{31}$$

$$\mathcal{D} = D^2 - \frac{(D^{12})^\top D^{12}}{d_{11}},\ \mathcal{C} = \begin{bmatrix} C^2 \\ \vdots \\ C^n \end{bmatrix} - \frac{(D^{12})^\top C^1}{d_{11}}, \tag{32}$$

$$\mathcal{G} = \begin{bmatrix} G_2 \\ \vdots \\ G_n \end{bmatrix} - \frac{(D^{12})^\top G_1}{d_{11}}. \tag{33}$$

Previous idea goes back to M. Spong [25]. In fact, it extends the computed torque technique from the fully actuated systems to the underactuated case and (31,32,33) is sometimes referred to as the "Spong's linearization".

Furthermore, one has easily from (25, 26) and (30) that

$$\begin{bmatrix} -\frac{\partial \phi_3}{\partial q_2} & 1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ -\frac{\partial \phi_n}{\partial q_2} & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{q}_2 \\ \ddot{q}_3 \\ \vdots \\ \ddot{q}_n \end{bmatrix} = \begin{bmatrix} \bar{\tau}_3 + \frac{\partial^2 \phi_3}{\partial q_2^2}(\dot{q}_2)^2 \\ \vdots \\ \bar{\tau}_n + \frac{\partial^2 \phi_n}{\partial q_2^2}(\dot{q}_2)^2 \end{bmatrix}$$

$$\begin{aligned}
\bar{\tau}_2 = \tau_2 &= (d_{22} - \frac{d_{12}^2}{d_{11}})\ddot{q}_2 + (d_{23} - \frac{d_{12}d_{13}}{d_{11}})\ddot{q}_3 + \cdots +\\
&(d_{2n} - \frac{d_{12}d_{1n}}{d_{11}})\ddot{q}_n + (C^2 - \frac{d_{12}}{d_{11}}C^1)\dot{q} + (G_2 - \frac{d_{12}}{d_{11}}G_1).
\end{aligned}$$

These expressions can be compactly expressed as follows

$$\mathcal{B}(q)\begin{bmatrix} \ddot{q}_2 \\ \ddot{q}_3 \\ \vdots \\ \ddot{q}_n \end{bmatrix} = \begin{bmatrix} \bar{\tau}_2 \\ \bar{\tau}_3 \\ \vdots \\ \bar{\tau}_n \end{bmatrix} + \mathcal{A}(q,\dot{q}), \tag{34}$$

$$\mathcal{B}(q) = \begin{bmatrix} d_{22} - \frac{d_{12}^2}{d_{11}} & d_{23} - \frac{d_{12}d_{13}}{d_{11}} & \cdots & d_{2n} - \frac{d_{12}d_{1n}}{d_{11}} \\ -\frac{\partial \phi_3}{\partial q_2} & 1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ -\frac{\partial \phi_n}{\partial q_2} & 0 & 0 & 1 \end{bmatrix}, \tag{35}$$

$$\mathcal{A}(q,\dot{q}) = \begin{bmatrix} -(C^2 - \frac{d_{12}}{d_{11}}C^1)\dot{q} - (G_2 - \frac{d_{12}}{d_{11}}G_1) \\ \frac{\partial^2 \phi_3}{\partial q_2^2}(\dot{q}_2)^2 \\ \vdots \\ \frac{\partial^2 \phi_n}{\partial q_2^2}(\dot{q}_2)^2 \end{bmatrix}. \tag{36}$$

Finally, combining (31) and (34) gives the inverse of (25,26):

$$\begin{aligned}
\begin{bmatrix} \tau_2 \\ \tau_3 \\ \vdots \\ \tau_n \end{bmatrix} &= \mathcal{D}(q)\mathcal{B}(q)^{-1}\begin{bmatrix} \bar{\tau}_2 \\ \bar{\tau}_3 \\ \vdots \\ \bar{\tau}_n \end{bmatrix}\\
&+ \mathcal{D}(q)\mathcal{B}(q)^{-1}\mathcal{A}(q,\dot{q}) + \mathcal{C}(q,\dot{q})\dot{q} + \mathcal{G}(q),
\end{aligned} \tag{37}$$

$$
\begin{aligned}
q_1 &= \bar{q}_1,\ q_2 = \bar{q}_2,\ \dot{q}_1 = \dot{\bar{q}}_1,\ \dot{q}_2 = \dot{\bar{q}}_2,\ \tau_2 = \bar{\tau}_2,\\
q_3 &= \bar{q}_3 + \phi_3(\bar{q}_2),\\
\dot{q}_3 &= \dot{\bar{q}}_3 + \tfrac{\partial \phi_3(\bar{q}_2)}{\partial q_2}\dot{\bar{q}}_2,\\
&\vdots\\
q_n &= \bar{q}_n + \phi_n(\bar{q}_2),\\
\dot{q}_n &= \dot{\bar{q}}_n + \tfrac{\partial \phi_n(\bar{q}_2)}{\partial q_2}\dot{\bar{q}}_2.
\end{aligned} \tag{38}
$$

For the sake of shortness, the "old " coordinates $q, \dot{q}$ are kept on the right hand side of (37). To have the complete and explicit inverse transformation, one has just to substitute from (38) the right hand side of (37), which is straightforward.

Note, that in (37,38) the only numerical computation is the matrix $\mathcal{B}(q)$ inversion. Due to its structure (35), the corresponding computations are not demanding, even for higher degrees of freedom n. Moreover, for control applications only that inverse transformation (37,38) is needed as one should finally apply the control action in real physical control variables $\tau_2, \ldots, \tau_n$.

Finally, let us obtain the transformed dynamics in the new coordinates (25). The following notation will be used:

$$
q = \begin{bmatrix} q^a \\ q^r \end{bmatrix},\ \dot{q} = \begin{bmatrix} \dot{q}^a \\ \dot{q}^r \end{bmatrix},\ D(q) = \begin{bmatrix} D^a & D^{ar} \\ (D^{ar})^\top & D^r \end{bmatrix}, \tag{39}
$$

$$
C(q,\dot{q}) = \begin{bmatrix} C^a & C^{ar} \\ -C^{ar} & C^r \end{bmatrix},\quad G(q) = \begin{bmatrix} G^a \\ G^r \end{bmatrix}. \tag{40}
$$

Here, D^a, C^a are (2×2) , D^r, C^r are $((n-2)\times(n-2))$, D^{ar}, C^{ar} are $(2\times(n-2))$ matrices, $G^a, q^a, \dot{q}^a$ are 2-dimensional and $G^a, q^a, \dot{q}^a$ are $(n-2)$-dimensional column vectors.

Substituting from (37,38) into (13) and using (39,40) gives

$$
D^a \begin{bmatrix} \ddot{\bar{q}}_1 \\ \ddot{\bar{q}}_2 \end{bmatrix} + D^{ar} \begin{bmatrix} \bar{\tau}_3 + \frac{\partial \phi_3(q_2)}{\partial q_2}\ddot{\bar{q}}_2 + \frac{\partial^2 \phi_3(q_2)}{\partial q_2^2}\dot{\bar{q}}_2^2 \\ \vdots \\ \bar{\tau}_n + \frac{\partial \phi_n(q_2)}{\partial q_2}\ddot{\bar{q}}_2 + \frac{\partial^2 \phi_n(q_2)}{\partial q_2^2}\dot{\bar{q}}_2^2 \end{bmatrix} +
$$

$$
C^a \begin{bmatrix} \dot{\bar{q}}_1 \\ \dot{\bar{q}}_2 \end{bmatrix} + C^{ar} \begin{bmatrix} \dot{\bar{q}}_3 + \frac{\partial \phi_3}{\partial q_2}(q_2)\dot{\bar{q}}_2 \\ \vdots \\ \dot{\bar{q}}_n + \frac{\partial \phi_n}{\partial q_2}(q_2)\dot{\bar{q}}_2 \end{bmatrix} + G^a = \begin{bmatrix} 0 \\ \bar{\tau}_2 \end{bmatrix}.
$$

Denote (substitute for $q, \dot{q}$ from (38) where necessary):

$$
\overline{D}^a(\bar{q}) = D^a(q) + \left[\begin{array}{c|c} \begin{matrix} 0 \\ 0 \end{matrix} & D^{ar}(q) \begin{bmatrix} \frac{\partial \phi_3}{\partial q_2}(q_2) \\ \vdots \\ \frac{\partial \phi_n}{\partial q_2}(q_2) \end{bmatrix} \end{array}\right], \tag{41}
$$

$$
\overline{C}^a(\bar{q},\dot{\bar{q}}) = C^a(q,\dot{q}) + \left[\begin{array}{c|c} \begin{matrix} 0 \\ 0 \end{matrix} & C^{ar}(q,\dot{q}) \begin{bmatrix} \frac{\partial \phi_3}{\partial q_2}(q_2) \\ \vdots \\ \frac{\partial \phi_n}{\partial q_2}(q_2) \end{bmatrix} \end{array}\right] \tag{42}
$$

$$
+ \left[\begin{array}{c|c} \begin{matrix} 0 \\ 0 \end{matrix} & D^{ar} \begin{bmatrix} \frac{\partial^2 \phi_3}{\partial q_2^2}(q_2)\dot{\bar{q}}_2 \\ \vdots \\ \frac{\partial^2 \phi_n}{\partial q_2^2}(q_2)\dot{\bar{q}}_2 \end{bmatrix} \end{array}\right],\ \overline{G}^a(\bar{q}) = G^a(q). \tag{43}
$$

Finally, using (41,42,43) one concludes that the dynamics (13) in the coordinates (37,38) takes the following form

$$
\overline{D}^a(\bar{q}) \begin{bmatrix} \ddot{\bar{q}}_1 \\ \ddot{\bar{q}}_2 \end{bmatrix} + \overline{C}^a(\bar{q},\dot{\bar{q}}) \begin{bmatrix} \dot{\bar{q}}_1 \\ \dot{\bar{q}}_2 \end{bmatrix} + \overline{G}^a(\bar{q}) \tag{44}
$$

$$
+D^{ar}(q) \begin{bmatrix} \bar{\tau}_3 \\ \vdots \\ \bar{\tau}_n \end{bmatrix} + C^{ar}(q,\dot{q}) \begin{bmatrix} \dot{\bar{q}}_3 \\ \vdots \\ \dot{\bar{q}}_n \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{\tau}_2 \end{bmatrix}, \tag{45}
$$

$$
\begin{aligned}
\ddot{\bar{q}}_3 &= \bar{\tau}_3,\\
&\vdots\\
\ddot{\bar{q}}_n &= \bar{\tau}_n.
\end{aligned} \tag{46}
$$

The main result of the paper is the following theorem on the generalized Acrobot embedding.

Theorem 2: Consider any n-degrees of freedom mechanical system model (13) such that

$$
G(q) = \left[\frac{\partial V(q)}{\partial q}\right]^\top,\ D(q) = D(q_2,\ldots,q_n),\ C(q,\dot{q}) =
$$

$$
\frac{\partial D(q)}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial D(q)}{\partial q_n}\dot{q}_n - \frac{1}{2}\begin{bmatrix} 0 \\ \dot{q}^\top \frac{\partial D(q)}{\partial q_2} \\ \vdots \\ \dot{q}^\top \frac{\partial D(q)}{\partial q_n} \end{bmatrix}, \tag{47}
$$

where $V(q)$ is a given smooth potential energy and $D(q_2,\ldots,q_n)$ is a given smooth inertia matrix. Then, for any collection of smooth functions $\phi_3(q_2),\ldots,\phi_n(q_2)$ the generalized Acrobot (23) with

$$
\begin{aligned}
\overline{D}(\bar{q}_2) &= \overline{D}^a(\bar{q}_1,\bar{q}_2,0,\ldots,0),\\
\overline{C}(\bar{q}_1,\bar{q}_2,\dot{\bar{q}}_1,\dot{\bar{q}}_2) &= \overline{C}^a(\bar{q}_1,\bar{q}_2,0,\ldots,0),\\
\overline{G}(\bar{q}_1,\bar{q}_2) &= \overline{G}^a(\bar{q}_1,\bar{q}_2,0,\ldots,0),
\end{aligned} \tag{48}
$$

where $\overline{D}^a, \overline{C}^a, \overline{G}^a$ are obtained from given D, C, G using (27, 28, 41, 42, 43), is globally linearly embedded into (13).

Proof: To be able to check the definition of embedding, it is obviously sufficient to show embedding into the transformed system, which is almost immediately seen as

$$
\begin{bmatrix} \bar{q} \\ \dot{\bar{q}} \\ \bar{\tau}_2 \\ \vdots \\ \bar{\tau}_n \end{bmatrix} \mapsto [\bar{q}_1, \bar{q}_2, 0, \ldots, 0, \dot{\bar{q}}_1, \dot{\bar{q}}_2, 0, \ldots, 0, \bar{\tau}_2, 0, \ldots, 0]^\top.
$$

More precisely, reformulate first the above second order derivatives model into the standard form via

$$
z_1 = \bar{q}_1, z_3 = \bar{q}_2, \ldots, z_{2n-1} = \bar{q}_n,
$$

$$
z_2 = \dot{\bar{q}}_1, z_4 = \dot{\bar{q}}_2, \ldots, z_{2n} = \dot{\bar{q}}_n
$$

to obtain the system

$$
\dot{z}_1 = z_2,\quad \dot{z}_3 = z_4,
$$

$$
\begin{bmatrix} \dot{z}_2 \\ \dot{z}_4 \end{bmatrix} = [\overline{D}^a]^{-1}\left[\begin{bmatrix} 0 \\ \bar{\tau}_2 \end{bmatrix} - \overline{C}^a \begin{bmatrix} z_2 \\ z_4 \end{bmatrix} - \overline{G}^a \right. \tag{49}
$$

$$-D^{ar}\begin{bmatrix}\bar{\tau}_3\\ \vdots\\ \bar{\tau}_n\end{bmatrix} - C^{ar}\begin{bmatrix}z_6\\ \vdots\\ z_{2n}\end{bmatrix}\Bigg]\Bigg], \quad (50)$$

$$\begin{aligned}\dot{z}_3 &= z_4, \quad \dot{z}_4 = \bar{\tau}_3,\\ &\vdots \qquad\quad \vdots\\ \dot{z}_{2n-1} &= z_n, \quad \dot{z}_{2n} = \bar{\tau}_n.\end{aligned} \quad (51)$$

The rest of the proof is straightforward. □

Remark 2: Theorem 2 can be briefly stated as follows: Consider any mechanical system (12) having Lagrangian

$$\mathcal{L}(q,\dot{q}) = \frac{1}{2}\dot{q}^\top D(q_2,\ldots,q_n)\dot{q} - V(q).$$

Then for any smooth functions $\phi_3(q_2),\ldots,\phi_n(q_2)$ there is the unique generalized Acrobot embedded into (12) having, moreover, the dynamics

$$\begin{aligned}&\tfrac{\mathrm{d}}{\mathrm{d}t}\left[\tfrac{\partial\overline{\mathcal{L}}}{\partial\dot{\bar{q}}_1}\right] - \tfrac{\partial\overline{\mathcal{L}}}{\partial\bar{q}_1} = 0, \quad \tfrac{\mathrm{d}}{\mathrm{d}t}\left[\tfrac{\partial\overline{\mathcal{L}}}{\partial\dot{\bar{q}}_2}\right] - \tfrac{\partial\overline{\mathcal{L}}}{\partial\bar{q}_2} = \bar{\tau}_2,\\ &\overline{\mathcal{L}}(\bar{q}_1,\bar{q}_2,\dot{\bar{q}}_1,\dot{\bar{q}}_2) := \mathcal{L}(\bar{q}_1,\bar{q}_2,\phi_3(\bar{q}_2),\ldots,\\ &\ldots,\phi_n(\bar{q}_2),\dot{\bar{q}}_1,\dot{\bar{q}}_2,\phi_3'(\bar{q}_2)\dot{\bar{q}}_2,\ldots,\phi_n'(\bar{q}_2)\dot{\bar{q}}_2).\end{aligned} \quad (52)$$

Indeed, it is easy to see that the above GA dynamics coincides with (44, 45) after substituting $\bar{q}_3 = \ldots = \bar{q}_n = 0$, $\dot{\bar{q}}_3 = \dot{\bar{q}}_4 = \ldots = \dot{\bar{q}}_n = 0, \bar{\tau}_3 = \ldots = \bar{\tau}_n = 0$ there.

V. WALKING DESIGN VIA EMBEDDING

A. Generalized Acrobot walking design

In the series of work [13], [1], [2], [3], [4], [6], the Acrobot was controlled, including multi-step walking-like movement with impacts, based on the partial exact feedback linearization of order 3 possible thanks to its kinetic symmetry with respect to the unactuated angle. Here, the main ideas of this approach will be briefly repeated for the generalized Acrobot (23,24). To start with, define the following transformations

$$\begin{aligned}\xi_1 &= \bar{q}_1 + \textstyle\int_0^{\bar{q}_2}\bar{d}_{11}^{-1}(s)\bar{d}_{22}(s)\mathrm{d}s, \ \overline{D}(\bar{q}) = [\bar{d}_{ij}],\\ \xi_2 &= \bar{d}_{11}(\bar{q}_2)\dot{\bar{q}}_1 + \bar{d}_{12}(\bar{q}_2)\dot{\bar{q}}_2,\\ \xi_3 &= -\overline{G}_1(\bar{q}),\\ \xi_4 &= -\tfrac{\partial\overline{G}_1}{\partial\bar{q}_1}(\bar{q})\dot{\bar{q}}_1 - \tfrac{\partial\overline{G}_1}{\partial\bar{q}_2}(\bar{q})\dot{\bar{q}}_2\\ w &= -\dot{\bar{q}}^\top\tfrac{\partial^2\overline{G}_1}{\partial\bar{q}^2}(\bar{q})\dot{\bar{q}} - \left[\tfrac{\partial\overline{G}_1}{\partial\bar{q}_1}(\bar{q}),\tfrac{\partial\overline{G}_1}{\partial\bar{q}_2}(\bar{q})\right]\\ &\quad\times\overline{D}(\bar{q})^{-1}\left[\begin{bmatrix}0\\ \bar{\tau}\end{bmatrix} - \overline{C}(\bar{q},\dot{\bar{q}})\dot{\bar{q}} - \overline{G}(\bar{q})\right].\end{aligned} \quad (53)$$

Note, that $\dot{\xi}_1 = \bar{d}_{11}^{-1}(\bar{q}_2)\xi_2$, further, $\xi_2 = (\partial\overline{\mathcal{L}}/\partial\dot{\bar{q}}_1)$ and by (52) $\dot{\xi}_2 = -G_1 = \xi_3$. Finally, it is straightforward, that $\dot{\xi}_3 = \xi_4$, $\dot{\xi}_4 = w$. Summarizing, it holds that

$$\begin{aligned}\dot{\xi}_1 &= \bar{d}_{11}^{-1}(\varphi_2(\xi_1,\xi_3))\xi_2, \quad \dot{\xi}_2 = \xi_3,\\ \dot{\xi}_3 &= \xi_4, \quad \dot{\xi}_4 = w,\end{aligned} \quad (54)$$

where $\varphi_2(\xi_1,\xi_3) = \bar{q}_2$ is to be obtained by inverting (53). Dynamics (54) is the partial exact state feedback linearization of the generalized Acrobot using the transformations (53).

Such a favorable form (54) can be used in two ways.

First, it is possible to design a single step reference walking trajectory $\bar{q}^r(t), \dot{\bar{q}}^r(t)$ as follows [13]. Let the step begin at time 0 and end at time T. Fixing $\bar{q}_1^r(0), \bar{q}_2^r(0)$ and $\bar{q}_1^r(T), \bar{q}_2^r(T)$ determines $\xi_1^r(0)$, $\xi_1^r(T)$ and $\xi_3^r(0)$, $\xi_3^r(T)$. The corresponding reference virtual input is taken as $w^r = 0$, so that the constant reference $\xi_4^r(t) \equiv [\xi_3^r(T) - \xi_3^r(0)]/T$ guarantees conditions for $\xi_3^r(0)$, $\xi_3^r(T)$. Therefore, the only remaining condition to be fulfilled is that of $\xi_3^r(T)$ and the only yet undetermined reference step parameter is $\xi_2^r(0)$. This parameter can be determined by simple numerical tuning, *e.g.* using dichotomy. Resulting trajectory $\xi^r(t)$ is called in [13] as the **pseudo-passive** one. By construction it satisfies

$$\begin{aligned}\dot{\xi}_1^r &= \bar{d}_{11}^{-1}(\varphi_2(\xi_1^r,\xi_3^r))\xi_2^r, \quad \dot{\xi}_2^r = \xi_3^r,\\ \dot{\xi}_3^r &= \xi_4^r, \quad \dot{\xi}_4^r = 0,\end{aligned} \quad (55)$$

Secondly, (54) can also be used to design exponentially stable state feedback to track a given reference trajectory. To show that, subtract mutually (54) and (55) to obtain via Taylor expansion that

$$\begin{aligned}\dot{e}_1 &= \mu_1(t)e_1 + \mu_2(t)e_2 + \mu_3(t)e_3 + o(e)\\ \dot{e}_2 &= e_3, \quad \dot{e}_3 = e_4, \quad \dot{e}_4 = w,\end{aligned} \quad (56)$$

where $e := \xi - \xi^r$ and $\mu_1(t), \mu_2(t), \mu_3(t)$ are given as follows

$$\mu_1(t) = \xi_2^r\frac{\partial}{\partial\xi_1}\left[\bar{d}_{11}^{-1}(\varphi_2(\xi_1,\xi_3))\right](\xi_1^r,\xi_3^r), \quad (57)$$

$$\mu_2(t) = \bar{d}_{11}^{-1}(\varphi_2(\xi_1^r,\xi_3^r)), \quad (58)$$

$$\mu_3(t) = \xi_2^r\frac{\partial}{\partial\xi_3}\left[\bar{d}_{11}^{-1}(\varphi_2(\xi_1,\xi_3))\right](\xi_1^r,\xi_3^r). \quad (59)$$

The tracking error dynamics (56) can be stabilized by designing suitable state feedback for w, see [1] for LMI application, or [2],[6] for various kinds of the time varying feedback.

Yet another method to stabilize the tracking error dynamics (56) will be presented here. This is the new result of the current paper being significantly better than those just mentioned results. It is given as the following theorem.

Theorem 3: Let $\tilde{e} = (\tilde{e}_1,\ldots,\tilde{e}_4)^\top$ be a new error variable related to $e = (e_1,\ldots,e_4)^\top$ in (56) as follows

$$\tilde{e}_1 = \frac{e_1 - \mu_3 e_2}{\mu_1\mu_3 - \dot{\mu}_3 + \mu_2}, \tilde{e}_2 = e_2, \tilde{e}_3 = e_3, \tilde{e}_4 = e_4. \quad (60)$$

Then the "new" error dynamics of $\tilde{e}$ is as follows

$$\begin{aligned}\dot{\tilde{e}}_1 &= \tilde{\mu}_1(t)\tilde{e}_1 + \tilde{e}_2\\ \dot{\tilde{e}}_2 &= \tilde{e}_3, \quad \dot{\tilde{e}}_3 = \tilde{e}_4, \quad \dot{\tilde{e}}_4 = w,\end{aligned} \quad (61)$$

$$\tilde{\mu}_1(t) = \mu_1 - \frac{\dot{\mu}_1\mu_3 + \mu_1\dot{\mu}_3 - \ddot{\mu}_3 + \dot{\mu}_2}{\mu_1\mu_3 - \dot{\mu}_3 + \mu_2}. \quad (62)$$

Furthermore, assume that there exists $M_1 \in \mathbb{R}^+$ such that $|\tilde{\mu}_1(t)| \le M_1$, $\forall t \in [0,T]$, then there exists a linear feedback law $w = K_1\tilde{e}_1 + K_2\tilde{e}_2 + K_3\tilde{e}_3 + K_4\tilde{e}_4$ that globally exponentially stabilizes the system (61). Moreover, if in addition there exists $M_2, M^2 \in \mathbb{R}^+$, such that $M^2 > \mu_1(t)\mu_3(t) - \dot{\mu}_3(t) + \mu_2(t) \ge M_2$, $\forall t \ge 0$, then the feedback

$$w = K_1\frac{e_1 - \mu_3 e_2}{\mu_1\mu_3 - \dot{\mu}_3 + \mu_2} + K_2 e_2 + K_3 e_3 + K_4 e_4$$

globally exponentially stabilizes the system (56).

Proof: First, note that (61,62) follow easily from (56) by (60). To stabilize (61), choose $\hat{K}_1, \hat{K}_2, \hat{K}_3, \hat{K}_4 \in \mathbb{R}$ defining the Hurwitz matrix $\hat{A}$ and $S = S^\top > 0$ such that

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hat{K}_1 & \hat{K}_2 & \hat{K}_3 & \hat{K}_4 \end{bmatrix}, \quad \begin{aligned} & S\hat{A} + \hat{A}^\top S = -I, \\ & S = \begin{bmatrix} s_{11} & \dots & s_{14} \\ \vdots & \ddots & \vdots \\ s_{41} & \dots & s_{44} \end{bmatrix}. \end{aligned}$$

Consider $\forall \theta \geq 1$ in (61) the feedback $w = \hat{K}_1\theta^4\tilde{e}_1 + \hat{K}_2\theta^3\tilde{e}_2 + \hat{K}_3\theta^2\tilde{e}_3 + \hat{K}_4\theta\tilde{e}_4$ and the positive definite function $V(\tilde{e}) := \hat{e}^\top S\hat{e}$, $\hat{e} := [\tilde{e}_1, \theta^{-1}\tilde{e}_2, \theta^{-2}\tilde{e}_3, \theta^{-3}\tilde{e}_4]^\top$. Note that

$$\dot{\hat{e}} = \theta\hat{A}\hat{e} + [\tilde{\mu}_1(t)\hat{e}_1, 0, 0, 0]^\top.$$

As a consequence, along trajectories of (61) it holds

$$\dot{V} = \theta[\hat{e}^\top(S\hat{A} + \hat{A}^\top S)\hat{e}] + 2\tilde{\mu}_1(t)\hat{e}_1 \sum_{i=1}^{4} s_{1i}\hat{e}_i \leq -\theta\hat{e}^\top\hat{e}$$
$$+2|\tilde{\mu}_1(t)\sum_{i=1}^{4} s_{1i}\hat{e}_1\hat{e}_i| \leq -\hat{e}^\top\hat{e}\left[\theta - M_1(2|s_{11}| + \sum_{i=2}^{4}|s_{1i}|)\right].$$

The last inequality is due to $2|\hat{e}_1\hat{e}_i| \leq \hat{e}^\top\hat{e}\ \forall i = 2, 3, 4$. As a consequence, the feedback $w = \theta^4[\hat{K}_1, \hat{K}_2, \hat{K}_3, \hat{K}_4]\hat{e} = \hat{K}_1\theta^4\tilde{e}_1 + \hat{K}_2\theta^3\tilde{e}_2 + \hat{K}_3\theta^2\tilde{e}_3 + \hat{K}_4\theta\tilde{e}_4$ stabilizes globally exponentially the error dynamics (61,62) if $\theta > M_1(2|s_{11}| + |s_{12}| + |s_{13}| + |s_{14}|)$. Note that $s_{11}, \dots, s_{14}$ depend on choice of $\hat{K}_1, \hat{K}_2, \hat{K}_3, \hat{K}_4 \in \mathbb{R}$ only, *i.e* such $\theta > 0$ always exists. Introducing $K_1 = \hat{K}_1\theta^4, K_2 = \hat{K}_2\theta^3, K_3 = \hat{K}_3\theta^2, K_4 = \hat{K}_4\theta$ and using (60) completes the proof. □

B. General n-link walking design using GA embedding

Combining Theorems 1, 2, 3 provides the constructive way how to track the desired reference for the n-link. Actually, Theorem 1 shows how to extend local exponential tracking control law to the larger system where the former one is linearly embedded into. Then, by Theorem 2, generalized Acrobot is globally linearly embedded into n-link, while Theorem 3 provides locally exponentially tracking control law for the generalized Acrobot. Note, that the condition $|\tilde{\mu}_1(t)| \leq M_1,\ \forall t \in [0, T]$ of Theorem 3 can be easily verified and it is valid during computations later on.

To obtain the walking-like reference for the n-link, it remains to choose virtual constraints $q_3 = \phi_3(q_2), \dots, q_n = \phi_n(q_2)$ in some suitable way. Such a selection schedules the overall course of the step given by the time dependencies of all n variables via the single variable q_2. The following example illustrates this idea for the case of 4-link.

Example 2: In the case of 4-link in Fig. 2, the following virtual constraints have been chosen

$$\begin{aligned} \phi_3 &= 16b_{stance}\frac{(q_2-q_{2_0})^2(q_2-q_{2_T})^2}{(-q_{2_0}+q_{2_T})^4} + q_{3_0}, \\ \phi_4 &= 16b_{swing}\frac{(q_2-q_{2_0})^2(q_2-q_{2_T})^2}{(-q_{2_0}+q_{2_T})^4} + q_{4_0}, \end{aligned} \qquad (63)$$

where q_{2_0} (respectively, q_{2_T}) is value of the angle q_2 at the beginning (respectively, at the end) of the step, q_{3_0} (respectively, q_{4_0}) is initial value of the angle q_3 (respectively, q_4) and b_{stance} (respectively, b_{swing}) is maximal value of the

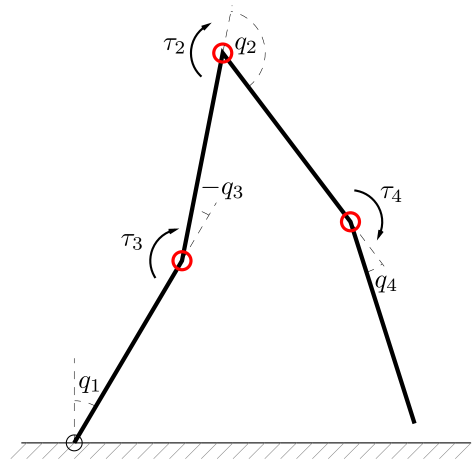

Fig. 2. The 4-link.

stretching of the stance (respectively bending of the swing) leg. Namely, straightforfard computations show that

$$\max_{q_2 \in [q_{2_0}, q_{2_T}]} \Phi_3 = b_{stance} + q_{3_0}, \quad \max_{q_2 \in [q_{2_0}, q_{2_T}]} \Phi_4 = b_{swing} + q_{4_0},$$
$$\Phi_3(q_{2_0}) = \Phi_3(q_{2_T}) = q_{3_0}, \quad \Phi_4(q_{2_0}) = \Phi_4(q_{2_T}) = q_{4_0}.$$

This also explains the coefficient 16 used in (63). Note also, that in Fig. 2 angle q_3 is defined to be negative during the step, so its growing indeed corresponds to stretching the "knee" of the stance leg, while q_4 is defined to be positive during the step, so its growing indeed corresponds to the bending of the swing leg "knee". One can see that the stance leg is stretching until the middle of the step, then it is bending back to the original value. The swing leg is doing other way around. Parameters b_{stance}, b_{swing} can be used to adjust those bendings and stretchings, so that hitting the ground during the step is eliminated.

This approach has been succesfully tested in simulations, see Fig. 3 for the tracking errors in linearized coordinates and Fig. 4 for the step animation.

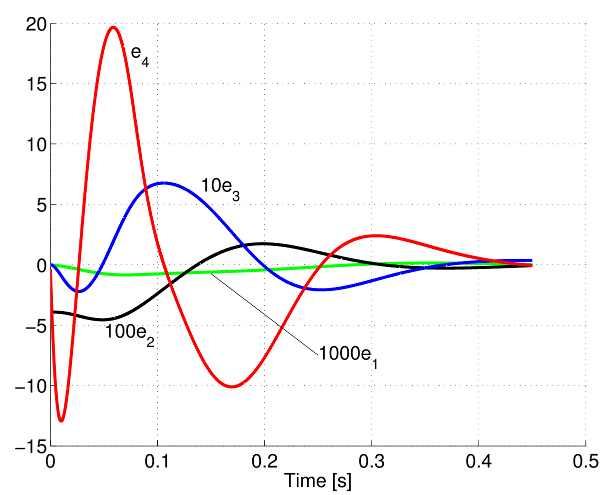

Fig. 3. Errors in linearized coordinates e_1, e_2, e_3, e_4 for 4-link one step using the new derived tracking feedback.

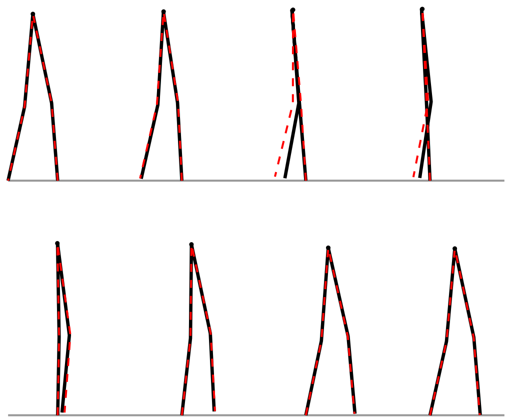

Fig. 4. The animation of the single step shown in time moments with gaps $\Delta t = 0.08$ s between them. Dashed line is the reference, the solid one represents the actual 4-link model

VI. CONCLUSIONS AND OUTLOOKS

The embedding of the generalized Acrobot into the general n-link having unactuated cyclic variable has been obtained and some control techniques for the generalized Acrobot have been developed. Based on that, a novel design for that n-link walking was presented. The nice property here is that the tracking error decays exponentially even during the swing phase of a single step only. Nevertheless, multi-step walking with impact effects using the generalized Acrobot embedding has yet to be developed.

The crucial peculiarity here is how to obtain also the "embedded impact" to reduce completely the n-link cyclic trajectory design to that of the generalized Acrobot. The motivation here is that the latter design can again mimic the cyclic trajectory design for the Acrobot [3]. The current research indicates that the impact embedding might be possible by special selection of the virtual constraints functions $\phi_3, \ldots, \phi_n$. This leads to the notion of the so-called **hybrid embedding**, *i.e.* the generalized Acrobot embedding is invariant also for the multi-step hybrid system with impacts that represents multi-step walking. Such a hybrid embedding would therefore be a direct generalization of the well-known hybrid zero dynamics notion and it would facilitate both the target walking trajectory planning and its tracking design.

Another interesting area of the current and future research is the walking model state estimation using the limited available measurements where the generalized Acrobot embedding may also find a natural implementation.

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