2013 European Control Conference (ECC)
July 17-19, 2013, Zürich, Switzerland.

The Method of Integro-Differential Relations for Control of Spatially Two-Dimensional Heat Transfer Processes

Andreas Rauh, Christina Dittrich, and Harald Aschemann

Abstract— **The design of control strategies for distributed parameter systems is an important field of current research. To design control laws and state estimation procedures for this class of systems, it is essential to find approximations that represent the system dynamics with good accuracy and simultaneously allow for an evaluation in real time. Moreover, possibilities for the quantification of approximation errors are useful to determine reliable finite-dimensional models which are sufficiently accurate for the control task at hand. Approximation errors result from the replacement of the original system model that is given in terms of partial differential equations by a finite-dimensional system representation. In this paper, the method of integro-differential relations is used for the derivation of a finite-dimensional approximation of a spatially two-dimensional heat transfer process. Simulation results for control and state estimation employing the before-mentioned modeling approach are presented for a test rig that is available at the Chair of Mechatronics at the University of Rostock.**

Index Terms— **Multi-dimensional distributed parameter systems, integro-differential relations, control design.**

I. Introduction

A large number of technical processes involves system components that are characterized by a dependency on both time and, at least, one space coordinate. Such distributed parameter properties can be found, for example, in heat and mass transfer processes, diffusion processes, or mechanical systems with elasticity, where the system models can be represented by partial differential equations (PDEs).

To derive control and state estimation procedures for such systems, two fundamentally different approaches can be distinguished. On the one hand, it is possible to replace a PDE in early design stages by finite-dimensional approximations [1]–[4]. Such techniques, often referred to as *early lumping approaches*, are based on finite difference schemes [5] (such as the Crank-Nicolson method [6]), finite volume methods, or finite element approaches. Typically, finite volume methods are characterized by the fact that physical conservation properties (such as the conservation of internal energy in heat transfer processes) are represented by integral balances over finitely large domains in which the storage variables are assumed to be homogeneously distributed. For systems with large spatial gradients of the storage variables, this modeling procedure usually involves the use of a large number of finite volume elements. In contrast, finite element representations are based on local ansatz functions describing the spatial dependency of the storage variables with improved accuracy. For all early lumping procedures, however, the quantification of approximation errors is usually a tedious task because explicit error estimates cannot be derived in a straightforward way.

On the other hand, so-called *late lumping procedures* are often discussed in the literature. In contrast to the early lumping case, specific control design techniques like flatness-based approaches are often tightly interwoven with the modeling. This is underlined by the fact that typically infinite series approximations are used for the description of the system dynamics and the control laws that are both truncated to a finite number of terms in the latest possible design stages [7], [8]. However, it is not trivial in this case — especially for spatially higher-dimensional systems — to fulfill boundary conditions for all possible disturbances and control inputs exactly, i.e., for both boundary control and spatially distributed control.

Due to the difficulties summarized above, the authors have employed the method of integro-differential relations (MIDR) for the approximation of various distributed parameter systems in previous work [9]–[12]. The MIDR is based on an integro-differential formulation of the system model. For this purpose, the underlying PDE is split up into (i) a conservation law which has to be fulfilled exactly by the finite-dimensional approximation and into (ii) a system of constitutive relations which have to be approximated in an optimal way. This approximation can either be determined by a projection approach — as an extension of the Galerkin method — or by an optimization-based formulation. Applications to which this procedure has already been applied successfully are spatially one-dimensional heat transfer processes [9], [10], the transport of liquid in long tubes [12] as well as mechanical systems with elasticities consisting of one dominant space coordinate [11]. An extension to spatially two-dimensional applications is described in this paper for a controlled heat transfer in a plate-like structure.

This paper is structured as follows: In Sec. II, the integro-differential formulation of spatially two-dimensional heat transfer processes is derived. Sec. III summarizes the approximation procedure for a PDE by a finite-dimensional system of ordinary differential equations (ODEs) using an optimization-based version of the MIDR. Moreover, differences between the MIDR formulation and classical finite element approximations are highlighted. In Sec. IV, this system of ODEs is used for the design of tracking control strategies and state observers. Simulation results are presented which demonstrate specific properties of the

A. Rauh and H. Aschemann are with the Chair of Mechatronics, University of Rostock, Justus-von-Liebig-Weg 6, D-18059 Rostock, Germany. {Andreas.Rauh, Harald.Aschemann}@uni-rostock.de

Ch. Dittrich was with the same institute while this work was performed.

MIDR. These are the quantification of approximation errors in space and time and the possibility to directly quantify worst-case bounds for the temperature distribution as well as for the heat flux density on the basis of a novel finite element approximation with Bernstein polynomials as basis functions. Finally, conclusions and an outlook on future work are given in Sec. V.

II. Integro-Differential Formulation

In this paper, heat transfer processes in plate-like structures are investigated. A dedicated test rig is shown in Figs. 1 and 2, which consists of an aluminum plate that can be heated or cooled from below with an array of 15 equally large Peltier elements serving either as control or disturbance inputs.

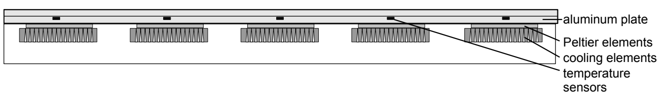


Fig. 1: Test rig: Distributed heating system.

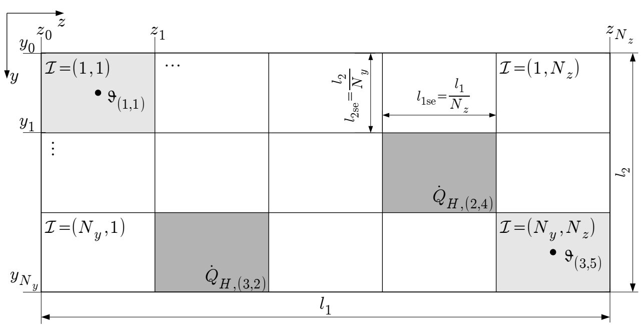


Fig. 2: Test rig: Visualization of input and output variables with $N_y = 3$ and $N_z = 5$.

This system can be described by a PDE for the temperature distribution $\vartheta := \vartheta(y, z, t)$, depending on two position coordinates y and z. Due to the small thickness of the considered aluminum plate, the temperature dependency in the third position coordinate can be neglected. To enable a system representation based on the MIDR, the PDE is split up into the constitutive relations

$$\xi_y\Big(\vartheta, q_y\Big) := q_y(y,z,t) + \lambda \frac{\partial \vartheta(y,z,t)}{\partial y} = 0 \qquad (1)$$

and

$$\xi_z\Big(\vartheta, q_z\Big) := q_z(y,z,t) + \lambda \frac{\partial \vartheta(y,z,t)}{\partial z} = 0 \ , \qquad (2)$$

representing Fourier's law in the corresponding space coordinate. Here, λ is the heat conductivity in the plate, which is assumed to be independent of the space direction, and $q_y := q_y(y, z, t)$ and $q_z := q_z(y, z, t)$ are the heat flux densities in the y- and z-direction, respectively.

The system formulation is completed by the energy conservation law (first law of thermodynamics)

$$\frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} + \kappa_1 \frac{\partial \vartheta}{\partial t} + \kappa_2 \vartheta = \mu(y,z,t) \ , \qquad (3)$$

which is coupled with the constitutive relations (1) and (2). The parameters $\kappa_1 = \rho c$ and $\kappa_2 = \frac{\alpha}{h}$ in (3) consist of the volume density ρ and the specific heat capacity c of aluminum, the convective heat transfer coefficient α and the plate thickness h. Further geometric data are summarized in Fig. 2, where $N_y = 3$ and $N_z = 5$ are the numbers of Peltier elements (corresponding to the numbers of finite elements in the following sections) in the y- and z-direction, respectively. Moreover, $\mu := \mu(y, z, t)$ represents the influence of distributed disturbances onto the system — free convection on top of the plate as well as disturbance heat flows provided by selected Peltier elements — and the influence of Peltier elements acting as control inputs. Here, the elements $\mathcal{I} := (i, j) = (3, 2)$ and $\mathcal{I} = (2, 4)$ are considered as control inputs with the heat flows $\dot{Q}_{H,(3,2)}$ and $\dot{Q}_{H,(2,4)}$. The system outputs are the temperatures $\vartheta_{(1,1)}$ and $\vartheta_{(3,5)}$ in the midpoints of the corresponding elements. If an exact solution to the system model (1)–(3) can be found, the equalities (1) and (2) are fulfilled exactly. Otherwise, the integral relations

$$\int_0^{l_1}\int_0^{l_2} \xi_y^2\Big(\vartheta, q_y\Big) dy dz \quad \text{and} \quad \int_0^{l_1}\int_0^{l_2} \xi_z^2\Big(\vartheta, q_z\Big) dy dz \qquad (4)$$

are minimized by the MIDR procedure described in the following section.

III. MIDR Formulation of the Heating Process

A. Bernstein Polynomials as a Basis for a Finite Element Discretization

Both the heat flux density and the temperature distribution are described by space- and time-dependent functions. To represent the space dependency, the bivariate Bernstein polynomial vectors

$$\boldsymbol{b}_{\mathcal{I},\mathcal{M}}(y,z) = \boldsymbol{b}_{i,M_y}(y) \otimes \boldsymbol{b}_{j,M_z}(z) \qquad (5)$$

of order $\mathcal{M} = (M_y, M_z)$ are introduced for each finite element $\mathcal{I} = (i, j)$ as the Kronecker product of two univariate polynomial vectors defined independently for each space coordinate. Here, $\boldsymbol{b}_{i,M_y}(y)$ consists of all polynomials

$$b_{i,k,M_y}(y) = \binom{M_y}{k} \cdot \left(\frac{y - y_{i-1}}{y_i - y_{i-1}}\right)^k \cdot \left(\frac{y_i - y}{y_i - y_{i-1}}\right)^{M_y - k} \qquad (6)$$

$0 \le k \le M_y$ of order M_y for the coordinate y with $i \in \{1, \ldots, N_y\}$, and $\boldsymbol{b}_{j,M_z}(z)$ is a vector of the polynomials

$$b_{j,k,M_z}(z) = \binom{M_z}{k} \cdot \left(\frac{z - z_{j-1}}{z_j - z_{j-1}}\right)^k \cdot \left(\frac{z_j - z}{z_j - z_{j-1}}\right)^{M_z - k} \qquad (7)$$

with the corresponding expressions for the coordinate z with $0 \le k \le M_z$ and $j \in \{1, \ldots, N_z\}$. The polynomials (6) and (7) are defined in such a way that their function values are identical to zero if they are evaluated outside their domains $y \in [y_{i-1}\ ;\ y_i]$ and $z \in [z_{j-1}\ ;\ z_j]$, respectively.

In (6) and (7), the terms

$$\binom{M_y}{k} := \frac{M_y!}{k!(M_y - k)!} \quad \text{and} \quad \binom{M_z}{k} := \frac{M_z!}{k!(M_z - k)!} \qquad (8)$$

denote the binomial coefficients of appropriate orders. Note that the orders M_y and M_z in both space dimensions do not necessarily have to be identical. In the following sections, the partial derivatives of (5) with respect to y and z are required. By introducing $\mathcal{M}_y = (M_y+1, M_z)$ and $\mathcal{M}_z = (M_y, M_z+1)$, these derivatives can be written in a compact notation with

$$\frac{\partial \boldsymbol{b}_{\mathcal{I},\mathcal{M}_y}(y,z)}{\partial y} = \frac{\partial \boldsymbol{b}_{i,M_y+1}(y)}{\partial y} \otimes \boldsymbol{b}_{j,M_z}(z) \tag{9}$$

and

$$\frac{\partial \boldsymbol{b}_{\mathcal{I},\mathcal{M}_z}(y,z)}{\partial z} = \boldsymbol{b}_{i,M_y}(y) \otimes \frac{\partial \boldsymbol{b}_{j,M_z+1}(z)}{\partial z} \ , \tag{10}$$

where the partial derivative of $\boldsymbol{b}_{i,M_y+1}(y)$ with respect to y is given by

$$\frac{\partial \boldsymbol{b}_{i,M_y+1}}{\partial y} = \frac{M_y+1}{y_i - y_{i-1}} \left[\begin{bmatrix} b_{i,-1,M_y} \\ b_{i,0,M_y} \\ \vdots \\ b_{i,M_y,M_y} \end{bmatrix} - \begin{bmatrix} b_{i,0,M_y} \\ b_{i,1,M_y} \\ \vdots \\ b_{i,M_y+1,M_y} \end{bmatrix} \right] \tag{11}$$

and the corresponding derivative of $\boldsymbol{b}_{j,M_z+1}(z)$ with respect to z by

$$\frac{\partial \boldsymbol{b}_{j,M_z+1}}{\partial z} = \frac{M_z+1}{z_j - z_{j-1}} \left[\begin{bmatrix} b_{j,-1,M_z} \\ b_{j,0,M_z} \\ \vdots \\ b_{j,M_z,M_z} \end{bmatrix} - \begin{bmatrix} b_{j,0,M_z} \\ b_{j,1,M_z} \\ \vdots \\ b_{j,M_z+1,M_z} \end{bmatrix} \right] \ . \tag{12}$$

Here, the following equalities hold: $b_{i,-1,M_y} \equiv 0$, $b_{j,-1,M_z} \equiv 0$, $b_{i,M_y+1,M_y} \equiv 0$, and $b_{j,M_z+1,M_z} \equiv 0$.

B. Finite Element Representation of the Temperature Distribution and the Heat Flux Density

Using the time-dependent coefficient vectors

$$\boldsymbol{\theta}_{\mathcal{I}} = \begin{bmatrix} \vartheta_{\mathcal{I},(0,0)} & \dots & \vartheta_{\mathcal{I},(0,M_z)} & \vartheta_{\mathcal{I},(1,0)} & \dots & \vartheta_{\mathcal{I},\mathcal{M}} \end{bmatrix}^T \tag{13}$$

and

$$\dot{\boldsymbol{\theta}}_{\mathcal{I}} = \begin{bmatrix} \dot\vartheta_{\mathcal{I},(0,0)} & \dots & \dot\vartheta_{\mathcal{I},(0,M_z)} & \dot\vartheta_{\mathcal{I},(1,0)} & \dots & \dot\vartheta_{\mathcal{I},\mathcal{M}} \end{bmatrix}^T , \tag{14}$$

the approximation of the temperature distribution and its corresponding time derivative can be parameterized according to

$$\vartheta_{\mathcal{I}}(y,z,t) = \boldsymbol{\theta}_{\mathcal{I}}^T(t) \cdot \boldsymbol{b}_{\mathcal{I},\mathcal{M}}(y,z) \tag{15}$$

and

$$\dot\vartheta_{\mathcal{I}}(y,z,t) = \dot{\boldsymbol{\theta}}_{\mathcal{I}}^T(t) \cdot \boldsymbol{b}_{\mathcal{I},\mathcal{M}}(y,z) \tag{16}$$

with the bivariate vectors of Bernstein polynomials $\boldsymbol{b}_{\mathcal{I},\mathcal{M}}(y,z)$ introduced above.

For the sake of a compact notation of the MIDR, the stack vectors

$$\boldsymbol{\Theta}_{\mathcal{M}} = \begin{bmatrix} \boldsymbol{\theta}_{(1,1)}^T & \dots & \boldsymbol{\theta}_{(1,N_z)}^T & \boldsymbol{\theta}_{(2,1)}^T & \dots & \boldsymbol{\theta}_{(N_y,N_z)}^T \end{bmatrix}^T \tag{17}$$

and

$$\dot{\boldsymbol{\Theta}}_{\mathcal{M}} = \begin{bmatrix} \dot{\boldsymbol{\theta}}_{(1,1)}^T & \dots & \dot{\boldsymbol{\theta}}_{(1,N_z)}^T & \dot{\boldsymbol{\theta}}_{(2,1)}^T & \dots & \dot{\boldsymbol{\theta}}_{(N_y,N_z)}^T \end{bmatrix}^T \tag{18}$$

are introduced that contain the coefficients of the temperature distribution and their time derivatives for all finite elements.

Moreover, the constitutive relations

$$\xi_y\left(\vartheta_{\mathcal{I}}, q_{y,\mathcal{I}}\right) := q_{y,\mathcal{I}}(y,z,t) + \lambda \frac{\partial \vartheta_{\mathcal{I}}(y,z,t)}{\partial y} \tag{19}$$

and

$$\xi_z\left(\vartheta_{\mathcal{I}}, q_{z,\mathcal{I}}\right) := q_{z,\mathcal{I}}(y,z,t) + \lambda \frac{\partial \vartheta_{\mathcal{I}}(y,z,t)}{\partial z} \tag{20}$$

for the heat flux density have to be parameterized for each finite element. In analogy to the parameterization of the temperature distribution, the expressions[1]

$$q_{y,\mathcal{I}}(y,z,t) = \boldsymbol{q}_{y,\mathcal{I}}^T(t) \cdot \boldsymbol{b}_{\mathcal{I},\mathcal{M}_y}(y,z) \tag{21}$$

and

$$q_{z,\mathcal{I}}(y,z,t) = \boldsymbol{q}_{z,\mathcal{I}}^T(t) \cdot \boldsymbol{b}_{\mathcal{I},\mathcal{M}_z}(y,z) \tag{22}$$

are introduced for the heat flux density in both space coordinates, where the corresponding time-dependent coefficient vectors are denoted by

$$\boldsymbol{q}_{y,\mathcal{I}}(t) = \begin{bmatrix} q_{y,\mathcal{I},(0,0)} & \dots & q_{y,\mathcal{I},(0,M_z)} & \dots & q_{y,\mathcal{I},\mathcal{M}_y} \end{bmatrix}^T \tag{23}$$

and

$$\boldsymbol{q}_{z,\mathcal{I}}(t) = \begin{bmatrix} q_{z,\mathcal{I},(0,0)} & \dots & q_{z,\mathcal{I},(0,M_z+1)} & \dots & q_{z,\mathcal{I},\mathcal{M}_z} \end{bmatrix}^T . \tag{24}$$

Moreover, these element-wise vectors are again summarized in stack vectors

$$\boldsymbol{Q}_{y,\mathcal{M}_y}(t) = \begin{bmatrix} \boldsymbol{q}_{y,(1,1)}^T(t) & \dots & \boldsymbol{q}_{y,(N_y,N_z)}^T(t) \end{bmatrix}^T \tag{25}$$

and

$$\boldsymbol{Q}_{z,\mathcal{M}_z}(t) = \begin{bmatrix} \boldsymbol{q}_{z,(1,1)}^T(t) & \dots & \boldsymbol{q}_{z,(N_y,N_z)}^T(t) \end{bmatrix}^T \ , \tag{26}$$

respectively, to parameterize the heat flux density for the complete plate. Using these definitions, the PDE

$$\frac{\partial q_{y,\mathcal{I}}}{\partial y} + \frac{\partial q_{z,\mathcal{I}}}{\partial z} + \kappa_1 \frac{\partial \vartheta_{\mathcal{I}}}{\partial t} + \kappa_2 \vartheta_{\mathcal{I}} = \mu_{\mathcal{I}}(y,z,t) \ , \tag{27}$$

with the piecewise defined function

$$\mu_{\mathcal{I}}(y,z,t) = a_c \cdot \dot{Q}_{H,\mathcal{I}} + \kappa_2 \cdot \vartheta_A = \boldsymbol{\mu}_{\mathcal{I}}^T(t) \cdot \boldsymbol{b}_{\mathcal{I},\mathcal{M}}(y,z) \tag{28}$$

of external control inputs and disturbances can be rewritten in the form

$$\begin{aligned} &\boldsymbol{q}_{y,\mathcal{I}}^T(t) \cdot \frac{\partial \boldsymbol{b}_{\mathcal{I},\mathcal{M}_y}(y,z)}{\partial y} + \boldsymbol{q}_{z,\mathcal{I}}^T(t) \cdot \frac{\partial \boldsymbol{b}_{\mathcal{I},\mathcal{M}_z}(y,z)}{\partial z} \\ &+ \left(\kappa_1 \dot{\boldsymbol{\theta}}_{\mathcal{I}}^T(t) + \kappa_2 \boldsymbol{\theta}_{\mathcal{I}}^T(t)\right) \boldsymbol{b}_{\mathcal{I},\mathcal{M}}(y,z) = \mu_{\mathcal{I}}(y,z,t) \ . \end{aligned} \tag{29}$$

[1] The increase of the order M_y to M_y+1 in q_y is necessary to fulfill eq. (3) exactly by the corresponding approximations. The same holds for the order increase from M_z to M_z+1 in q_z.

The term (28) can be split up into the heat flows provided by the Peltier elements, characterized by the gain value

$$a_c = \frac{N_y \cdot N_z}{l_1 \cdot l_2 \cdot h} \ , \tag{30}$$

and the free convection on the top of the plate, where the ambient temperature is denoted by $\vartheta_A := \vartheta_A(t)$.

C. Specification of Initial and Boundary Conditions

To fully specify the dynamics of the heat transfer process, the spatial boundary conditions for the heat flux density in y-direction

$$\begin{aligned} q_{y,\mathcal{I}}(0,z,t) &= \bar{q}_y^0(z,t) \quad \text{and} \\ q_{y,\mathcal{I}}(l_2,z,t) &= \bar{q}_y^{l_2}(z,t) \end{aligned} \tag{31}$$

and for the heat flux density in z-direction

$$\begin{aligned} q_{z,\mathcal{I}}(y,0,t) &= \bar{q}_z^0(y,t) \quad \text{and} \\ q_{z,\mathcal{I}}(y,l_1,t) &= \bar{q}_z^{l_1}(y,t) \end{aligned} \tag{32}$$

have to be specified. For the given case of adiabatic insulation of all edges of the aluminum plate depicted in Fig. 2, these boundary conditions simplify to

$$\begin{aligned} &\bar{q}_y^0(z,t) = 0 \ , \quad \bar{q}_y^{l_2}(z,t) = 0 \\ &\bar{q}_z^0(y,t) = 0 \ , \quad \text{and} \quad \bar{q}_z^{l_1}(y,t) = 0 \ . \end{aligned} \tag{33}$$

Due to the representation of the heat flux densities q_y and q_z by the Bernstein polynomial approximations (21) and (22), these boundary conditions are equal to the following equalities for the coefficient vectors

$$\boldsymbol{q}_{y,(1,j),(0,k)} = 0 \ , \quad \boldsymbol{q}_{y,(N_y,j),(M_y+1,k)} = 0 \ , \tag{34}$$

$j \in \{1,\ldots,N_z\}$, $k \in \{0,\ldots,M_z\}$, and

$$\boldsymbol{q}_{z,(i,1),(k,0)} = 0 \ , \quad \boldsymbol{q}_{z,(i,N_z),(k,M_z+1)} = 0 \ , \tag{35}$$

$i \in \{1,\ldots,N_y\}$, $k \in \{0,\ldots,M_y\}$.

Without loss of generality, it is assumed in the following that the initial temperature distribution in the aluminum plate is given by

$$\vartheta_{\mathcal{I}}(y,z,0) = \bar{\vartheta}_{\mathcal{I}}^0(y,z) \ , \tag{36}$$

which is set equal to the thermodynamic equilibrium that is specified by the ambient temperature ϑ_A according to

$$\bar{\vartheta}_{\mathcal{I}}^0(y,z) = \vartheta_A \ . \tag{37}$$

D. Specification of Inter-Element Conditions

According to physical considerations, the temperature distribution $\vartheta(y,z,t)$ as well as the heat flux densities $q_y(y,z,t)$ and $q_z(y,z,t)$ have to fulfill certain continuity conditions.

The temperature distribution has to be continuous in both space coordinates, which can be stated as

$$\begin{aligned} \vartheta_{(i,j),(M_y,k)}(t) &= \vartheta_{(i+1,j),(0,k)}(t) \ , \\ \dot{\vartheta}_{(i,j),(M_y,k)}(t) &= \dot{\vartheta}_{(i+1,j),(0,k)}(t) \end{aligned} \tag{38}$$

with $i \in \{1,\ldots,N_y-1\}$, $j \in \{1,\ldots,N_z\}$, $k \in \{0,\ldots,M_z\}$, and as

$$\begin{aligned} \vartheta_{(i,j),(k,M_z)}(t) &= \vartheta_{(i,j+1),(k,0)}(t) \ , \\ \dot{\vartheta}_{(i,j),(k,M_z)}(t) &= \dot{\vartheta}_{(i,j+1),(k,0)}(t) \end{aligned} \tag{39}$$

with $i \in \{1,\ldots,N_y\}$, $j \in \{1,\ldots,N_z-1\}$, $k \in \{0,\ldots,M_y\}$, for the coefficients and time derivatives of the Bernstein polynomial approximation in the y- and z-directions, respectively.

Continuity of the heat flux density $q_y(y,z,t)$ in y-direction is given by the equalities

$$q_{y,(i,j),(M_y+1,k)}(t) = q_{y,(i+1,j),(0,k)}(t) \tag{40}$$

for all $i \in \{1,\ldots,N_y-1\}$, $j \in \{1,\ldots,N_z\}$, $k \in \{0,\ldots,M_z\}$. Analogously,

$$q_{z,(i,j),(k,M_z+1)}(t) = q_{z,(i,j+1),(k,0)}(t) \tag{41}$$

must hold for the heat flux density $q_z(y,z,t)$ in z-direction with $i \in \{1,\ldots,N_y\}$, $j \in \{1,\ldots,N_z-1\}$, $k \in \{0,\ldots,M_y\}$.

E. Optimization-Based Solution of the MIDR

With the help of the approximations for both the temperature distribution and the heat flux density introduced above, an optimization-based version of the MIDR can be formulated for the spatially two-dimensional heat transfer process. Here, the constitutive relations $\xi_y\Big(\vartheta, q_y\Big)$ and $\xi_z\Big(\vartheta, q_z\Big)$, introduced in (1) and (2), are replaced by quadratic error functionals[2]

$$\begin{aligned} J_1 &= \int_0^{l_1}\int_0^{l_2} \left(\lambda \frac{\partial \vartheta_{\mathcal{I}}(y,z,t)}{\partial y} + q_{y,\mathcal{I}}(y,z,t)\right)^2 dydz \\ &= \sum_{\mathcal{I}=(0,0)}^{(N_y,N_z)} \Bigg\{ \lambda^2 \boldsymbol{\theta}_{\mathcal{I}}^T \left(\int_{y_{i-1}}^{y_i}\int_{z_{j-1}}^{z_j} \frac{\partial \boldsymbol{b}_{\mathcal{I},\mathcal{M}}}{\partial y}\frac{\partial \boldsymbol{b}_{\mathcal{I},\mathcal{M}}^T}{\partial y} dzdy \right) \boldsymbol{\theta}_{\mathcal{I}} \\ &\quad + 2\lambda \boldsymbol{\theta}_{\mathcal{I}}^T \left(\int_{y_{i-1}}^{y_i}\int_{z_{j-1}}^{z_j} \frac{\partial \boldsymbol{b}_{\mathcal{I},\mathcal{M}}}{\partial y} \boldsymbol{b}_{\mathcal{I},\mathcal{M}_y}^T dzdy \right) \boldsymbol{q}_{y,\mathcal{I},\mathcal{M}_y} \\ &\quad + \boldsymbol{q}_{y,\mathcal{I},\mathcal{M}_y}^T \left(\int_{y_{i-1}}^{y_i}\int_{z_{j-1}}^{z_j} \boldsymbol{b}_{\mathcal{I},\mathcal{M}_y} \boldsymbol{b}_{\mathcal{I},\mathcal{M}_y}^T dzdy \right) \boldsymbol{q}_{y,\mathcal{I},\mathcal{M}_y} \Bigg\} \\ &= \sum_{\mathcal{I}=(0,0)}^{(N_y,N_z)} \Big\{ \lambda^2 \boldsymbol{\theta}_{\mathcal{I}}^T \boldsymbol{K}_{y,\mathcal{I},1} \boldsymbol{\theta}_{\mathcal{I}} + 2\lambda \boldsymbol{\theta}_{\mathcal{I}}^T \boldsymbol{K}_{y,\mathcal{I},2} \boldsymbol{q}_{y,\mathcal{I},\mathcal{M}_y} \\ &\quad + \boldsymbol{q}_{y,\mathcal{I},\mathcal{M}_y}^T \boldsymbol{K}_{y,\mathcal{I},3} \boldsymbol{q}_{y,\mathcal{I},\mathcal{M}_y} \Big\} \end{aligned} \tag{42}$$

and

$$\begin{aligned} J_2 &= \int_0^{l_1}\int_0^{l_2} \left(\lambda \frac{\partial \vartheta_{\mathcal{I}}(y,z,t)}{\partial z} + q_{z,\mathcal{I}}(y,z,t)\right)^2 dydz \\ &= \sum_{\mathcal{I}=(0,0)}^{(N_y,N_z)} \Bigg\{ \lambda^2 \boldsymbol{\theta}_{\mathcal{I}}^T \left(\int_{y_{i-1}}^{y_i}\int_{z_{j-1}}^{z_j} \frac{\partial \boldsymbol{b}_{\mathcal{I},\mathcal{M}}}{\partial z}\frac{\partial \boldsymbol{b}_{\mathcal{I},\mathcal{M}}^T}{\partial z} dzdy \right) \boldsymbol{\theta}_{\mathcal{I}} \end{aligned}$$

[2]In contrast to usual finite element techniques, a global optimality criterion is used by the MIDR instead of a local criterion only taking into account the approximation quality for directly neighboring elements.

$$+ 2\lambda\boldsymbol{\theta}_{\mathcal{I}}^T \left(\int_{y_{i-1}}^{y_i} \int_{z_{j-1}}^{z_j} \frac{\partial \boldsymbol{b}_{\mathcal{I},\mathcal{M}}}{\partial z} \boldsymbol{b}_{\mathcal{I},\mathcal{M}_z}^T dz dy \right) \boldsymbol{q}_{z,\mathcal{I},\mathcal{M}_z}$$

$$+ \boldsymbol{q}_{z,\mathcal{I},\mathcal{M}_z}^T \left(\int_{y_{i-1}}^{y_i} \int_{z_{j-1}}^{z_j} \boldsymbol{b}_{\mathcal{I},\mathcal{M}_z} \boldsymbol{b}_{\mathcal{I},\mathcal{M}_z}^T dz dy \right) \boldsymbol{q}_{z,\mathcal{I},\mathcal{M}_z} \Bigg\}$$

$$= \sum_{\mathcal{I}=(0,0)}^{(N_y,N_z)} \Big\{ \lambda^2 \boldsymbol{\theta}_{\mathcal{I}}^T \boldsymbol{K}_{z,\mathcal{I},1} \boldsymbol{\theta}_{\mathcal{I}} + 2\lambda \boldsymbol{\theta}_{\mathcal{I}}^T \boldsymbol{K}_{z,\mathcal{I},2} \boldsymbol{q}_{z,\mathcal{I},\mathcal{M}_z}$$

$$+ \boldsymbol{q}_{z,\mathcal{I},\mathcal{M}_z}^T \boldsymbol{K}_{z,\mathcal{I},3} \boldsymbol{q}_{z,\mathcal{I},\mathcal{M}_z} \Big\} \quad . \tag{43}$$

They have to be minimized by parameterizations for the temperature distribution as well as for the heat flux densities that satisfy the boundary conditions (34) and (35), the initial temperature distribution (36), and the inter-element conditions (38)–(41) exactly. Note that the integrals in (42) and (43) can be computed analytically according to the procedures given in [13]. This leads to constant matrices $\boldsymbol{K}_{y,\mathcal{I},1}$, $\boldsymbol{K}_{y,\mathcal{I},2}$, $\boldsymbol{K}_{y,\mathcal{I},3}$, $\boldsymbol{K}_{z,\mathcal{I},1}$, $\boldsymbol{K}_{z,\mathcal{I},2}$ and $\boldsymbol{K}_{z,\mathcal{I},3}$ in the cost functions defined above. Using these expressions, the global cost function — defined over the complete space and time domain — can be defined as

$$J = J_1 + J_2 \overset{!}{=} \min \quad . \tag{44}$$

The minimization of (44) has to be performed in such a way that the energy conservation law (29) is fulfilled exactly. The relation (29) can be stated as a vector-valued equation $\boldsymbol{N}$, purely depending on $\boldsymbol{Q}_{y,\mathcal{M}_y}$, $\boldsymbol{Q}_{z,\mathcal{M}_y}$, $\boldsymbol{\Theta}_{\mathcal{M}}$, $\dot{\boldsymbol{\Theta}}_{\mathcal{M}}$, $\boldsymbol{M}_{\mathcal{M}}$, where

$$\boldsymbol{M}_{\mathcal{M}} = \begin{bmatrix} \boldsymbol{\mu}_{(1,1)}^T & \cdots & \boldsymbol{\mu}_{(1,N_z)}^T & \boldsymbol{\mu}_{(2,1)}^T & \cdots & \boldsymbol{\mu}_{(N_y,N_z)}^T \end{bmatrix}^T \tag{45}$$

is a Bernstein polynomial-based stack vector containing coefficients of the external disturbances and control inputs. The corresponding procedure is summarized in detail in [9] for a one-dimensional heat transfer process. Summarizing the resulting equations for the coefficients in a vector $\boldsymbol{N}$, which becomes equal to zero if (29) is fulfilled exactly, the modified cost function

$$\widetilde{J} = J + \boldsymbol{\beta}^T \boldsymbol{N} \overset{!}{=} \min \tag{46}$$

can be defined, where $\boldsymbol{\beta}$ is a vector of Lagrange multipliers of appropriate dimension.

Using the cost function (46), the necessary optimality conditions

$$\frac{\partial \widetilde{J}}{\partial \boldsymbol{\Theta}_{\mathcal{M}}} = 0 \ , \quad \frac{\partial \widetilde{J}}{\partial \boldsymbol{\beta}} = 0 \ , \quad \frac{\partial \widetilde{J}}{\partial \boldsymbol{Q}_{\mathcal{M}_y}} = 0 \ , \text{and} \quad \frac{\partial \widetilde{J}}{\partial \boldsymbol{Q}_{\mathcal{M}_z}} = 0 \tag{47}$$

can be derived. The corresponding set of linear algebraic equations is usually underdetermined. It can be solved according to the following procedure

$$\frac{\partial \widetilde{J}}{\partial \boldsymbol{v}} = \widetilde{\boldsymbol{A}} \cdot \bar{\boldsymbol{v}} + \widetilde{\boldsymbol{b}} \overset{!}{=} \boldsymbol{0} \ , \quad \bar{\boldsymbol{v}} = \widetilde{\boldsymbol{A}}^+ \cdot \left(\frac{\partial \widetilde{J}}{\partial \boldsymbol{v}} - \widetilde{\boldsymbol{b}} \right) = -\widetilde{\boldsymbol{A}}^+ \cdot \widetilde{\boldsymbol{b}} \ , \tag{48}$$

where $\widetilde{\boldsymbol{A}}^+$ denotes the left pseudo inverse of the corresponding matrix. In (48), the vectors $\boldsymbol{v}$ and $\bar{\boldsymbol{v}}$ are defined as

$$\begin{aligned} \boldsymbol{v} &= \begin{bmatrix} \boldsymbol{\Theta}_{\mathcal{M}}^T, & \boldsymbol{Q}_{y,\mathcal{M}_y}^T, & \boldsymbol{Q}_{z,\mathcal{M}_z}^T, & \boldsymbol{\beta}^T \end{bmatrix}^T \quad \text{and} \\ \bar{\boldsymbol{v}} &= \begin{bmatrix} \dot{\boldsymbol{\Theta}}_{\mathcal{M}}^T, & \boldsymbol{Q}_{y,\mathcal{M}_y}^T, & \boldsymbol{Q}_{z,\mathcal{M}_z}^T, & \boldsymbol{\beta}^T \end{bmatrix}^T \quad , \end{aligned} \tag{49}$$

where the equality constraints resulting from the boundary and inter-element conditions are already taken into account explicitly by elimination of the corresponding vector entries.

Finally, the entries of $\bar{\boldsymbol{v}}$ that correspond to the time derivatives $\dot{\boldsymbol{\Theta}}_{\mathcal{M}}$ represent the state equations

$$\dot{\boldsymbol{\Theta}}_{\mathcal{M}} = \boldsymbol{A}\boldsymbol{\Theta}_{\mathcal{M}} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{E}\boldsymbol{z} \ , \tag{50}$$

which are used later on in the control design. The input vector $\boldsymbol{u}$ consists of the heat flows $\dot{Q}_{H,(3,2)}$ and $\dot{Q}_{H,(2,4)}$. All remaining heat flows $\dot{Q}_{H,\mathcal{I}}$ as well as the ambient temperature ϑ_A are included in the disturbance vector $\boldsymbol{z}$ which has to be compensated by the control strategy derived in the following section.

IV. Simulation Results for the Observer-Based Closed-Loop Control Design

The MIDR formulation of the heat transfer process leads to a set of 160 ODEs for a parameterization of the temperature distribution with $M_y = M_z = 3$. Since these state variables are not directly measurable, they have to be reconstructed together with the disturbances $\boldsymbol{z}$ by means of a suitable observer. The control design is based on the minimization of the cost function

$$J_C = \frac{1}{2} \int_0^\infty \Big(\boldsymbol{\Theta}_{\mathcal{M}}^T \boldsymbol{Q} \boldsymbol{\Theta}_{\mathcal{M}} + \boldsymbol{u}^T \boldsymbol{R} \boldsymbol{u} \Big) dt \ . \tag{51}$$

Furthermore, the control input can be defined according to

$$\boldsymbol{u} = -\boldsymbol{K}\boldsymbol{\Theta}_{\mathcal{M}} - \left(\boldsymbol{C}\boldsymbol{A}_C^{-1}\boldsymbol{B}\boldsymbol{S} \right)^{-1} \boldsymbol{C}\boldsymbol{A}_C^{-1}\boldsymbol{E}\boldsymbol{z} + \boldsymbol{u}_V \ , \tag{52}$$

in which the controller gain matrix results from

$$\boldsymbol{K} = \boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P} \ . \tag{53}$$

In (52), $\boldsymbol{A}_C$ is defined as $\boldsymbol{A}_C = \boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}$ and $\boldsymbol{S}$ denotes a feedforward gain. The second term in (52) is used to compensate the influence of constant disturbances in the resulting steady state. Finally, the dynamic feedforward control signal $\boldsymbol{u}_V$ (aiming at accurate trajectory tracking) is included, which makes use of a reduced-order model of the system in which the steady-state gain of the system dynamics and five dominant eigenvalues are accounted for. Using the weighting matrices $\boldsymbol{Q} = \boldsymbol{I} \cdot 1000$ for the states and $\boldsymbol{R} = \boldsymbol{I}$ for the control inputs, the matrix $\boldsymbol{P}$ results from the solution of the algebraic Riccati equation

$$\boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P} - \boldsymbol{A}^T\boldsymbol{P} - \boldsymbol{P}\boldsymbol{A} - \boldsymbol{Q} \overset{!}{=} \boldsymbol{0} \ . \tag{54}$$

Since the state variables of the system model (50) are not directly measurable, the following observer is implemented. Using the temperatures at the midpoints of the elements $\mathcal{I} = (1,1)$ and $\mathcal{I} = (3,5)$, the ambient temperature ϑ_A, and the sum of the disturbance heat flows as measured data, an observer can be built for the extended system model

$$\begin{bmatrix} \dot{\boldsymbol{\Theta}}_{\mathcal{M}} \\ \dot{\boldsymbol{x}}_z \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{E}\boldsymbol{C}_z \\ \boldsymbol{0} & \boldsymbol{A}_z \end{bmatrix} \begin{bmatrix} \boldsymbol{\Theta}_{\mathcal{M}} \\ \boldsymbol{x}_z \end{bmatrix} + \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{u}$$
$$\boldsymbol{y} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{c}_H^T \\ \boldsymbol{0} & \boldsymbol{c}_A^T \end{bmatrix} \begin{bmatrix} \boldsymbol{\Theta} \\ \boldsymbol{x}_z \end{bmatrix} , \quad (55)$$

where the integrator disturbance model is implemented according to

$$\dot{\boldsymbol{x}}_z = \boldsymbol{A}_z \boldsymbol{x}_z \qquad \text{with} \qquad \boldsymbol{z} = \boldsymbol{C}_z \boldsymbol{x}_z \ , \quad (56)$$

$\boldsymbol{A}_z = \boldsymbol{0}$, $\boldsymbol{C}_z = \boldsymbol{I}$, and the disturbance vector $\boldsymbol{z}$. The observer approach which is described in [9], [10] exploits the duality between the above-mentioned control and the corresponding optimal estimation of $\boldsymbol{\Theta}_{\mathcal{M}}$ and $\boldsymbol{z}$ which are both included in the before-mentioned control law.

Simulation results for the observer-based control design as well as for the quantification of the approximation quality by means of the expression

$$\bar{\xi}_y^2(y,z) := \left(\int_0^{t_f} \xi_y^2(y,z,t)dt \right) \left(\int_0^{t_f} J_1(t)dt \right)^{-1} \quad (57)$$

resulting from the MIDR system formulation with $M_y = M_z = 3$ are summarized in Figs. 3 and 4.

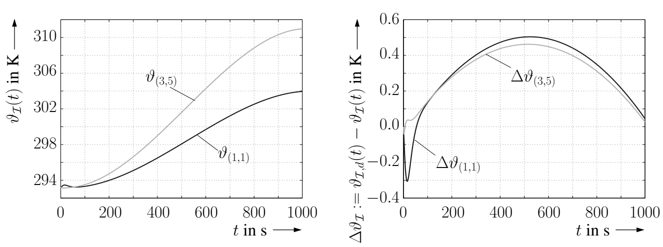
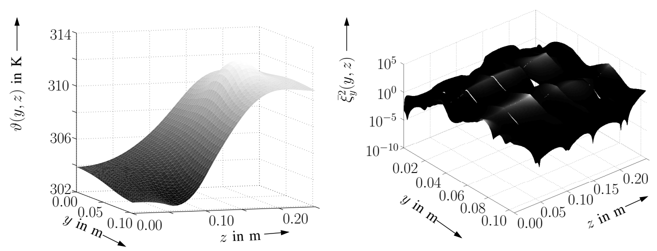


Fig. 3: Simulation of tracking control for two selected output temperatures $\vartheta_{(1,1)}(t)$ and $\vartheta_{(3,5)}(t)$.



(a) Temperature distribution for $t = t_f = 1000\,\text{s}$.

(b) Error distribution.

Fig. 4: Temperature and approximation error distributions.

V. Conclusions and Outlook on Future Work

In this paper, procedures for modeling as well as control and observer design have been presented which are based on an integro-differential formulation of spatially two-dimensional heat transfer processes. The advantage of the MIDR formulation is the inherent quantification of approximation errors which can be expressed in terms of deviations in the corresponding constitutive relations. These constitutive relations represent Fourier's law in the case of heat transfer systems. In addition, the finite-dimensional approximation has been designed in such a way that the energy conservation law is fulfilled exactly. The simulations presented in this paper for the control and observer approaches can be evaluated in real time on suitable rapid control prototyping hardware. Therefore, future work aims at the implementation of the presented procedures on a test rig that is currently being built up at the Chair of Mechatronics at the University of Rostock. Moreover, predictive and adaptive controllers as well as an online parameter identification will be studied during this experimental investigation.

References

[1] A. Rauh, I. Menn, and H. Aschemann, "Robust Control with State and Disturbance Estimation for Distributed Parameter Systems," in *Proc. of 15th Intl. Workshop on Dynamics and Control*, J. Rodellar and E. Reithmeier, Eds., Tossa de Mar, Spain, 2009, pp. 135–142.

[2] O. Zienkiewicz, R. Taylor, and J. Zhu, *The Finite Element Method - Its Basis and Fundamentals*. Oxford: Elsevier Butterworth-Heinemann, 2000.

[3] A. Rauh, Ch. Dittrich, H. Aschemann, N. S. Nedialkov, and J. D. Pryce, "A Differential-Algebraic Approach for Robust Control Design and Disturbance Compensation of Finite-Dimensional Models of Heat Transfer Processes," in *Proc. of IEEE International Conference on Mechatronics ICM 2013*, Vicenza, Italy, 2013.

[4] Y. Yener and S. Kakaç, *Heat Conduction*. Bristol, PA: Taylor & Francis Group, 2008.

[5] T. Utz, T. Meurer, and A. Kugi, "Trajectory Planning for a Two-Dimensional Quasi-Linear Parabolic PDE Based on Finite Difference Semi-Discretizations," in *Proc. of 18th IFAC World Congress*, Milano, Italy, 2011.

[6] J. Crank and P. Nicolson, "A Practical Method for Numerical Evaluation of Solutions of Partial Differential Equations of the Heat-Conduction Type," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 43, no. 1, pp. 50–67, 1947.

[7] A. Kharitonov and O. Sawodny, "Flatness-Based Disturbance Decoupling for Heat and Mass Transfer Processes with Distributed Control," in *Proc. of the IEEE International Conference on Control Applications CCA*, Munich, Germany, 2006, pp. 674–679.

[8] F. Malchow and O. Sawodny, "Feedforward Control of Inhomogeneous Linear First Order Distributed Parameter Systems," *American Control Conference on ACC 2011, San Francisco, CA, USA*, 2011.

[9] A. Rauh, L. Senkel, H. Aschemann, G. V. Kostin, and V. V. Saurin, "Reliable Finite-Dimensional Control Procedures for Distributed Parameter Systems with Guaranteed Approximation Quality," in *Proc. of IEEE Multi-Conference on Systems and Control*, Dubrovnik, Croatia, 2012.

[10] V. V. Saurin, G. V. Kostin, A. Rauh, L. Senkel, and H. Aschemann, "An Integrodifferential Approach to Adaptive Control Design for Heat Transfer Systems with Uncertainties," in *Proc. of 7th Vienna Intl. Conference on Mathematical Modelling MATHMOD 2012*, Vienna, Austria, 2012.

[11] G. V. Kostin, A. Rauh, V. V. Saurin, and H. Aschemann, "Optimal Real-Time Control of Flexible Rack Feeders Using the Method of Integro-Differential Relations," in *Proc. of 7th Vienna Intl. Conference on Mathematical Modelling MATHMOD 2012*, Vienna, Austria, 2012.

[12] G. V. Kostin, V. V. Saurin, H. Aschemann, and A. Rauh, "Modeling and Optimization of Control Processes for Compressible Liquid Flow in Pipeline Systems," in *Proc. of 7th Vienna Intl. Conference on Mathematical Modelling MATHMOD 2012*, Vienna, Austria, 2012.

[13] G. G. Lorentz, Ed., *Bernstein Polynomials*. Chelsea Publishing Company, New York, N. Y., 1986.