

On the Weight Controllability of Consensus Algorithms*

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Abstract—In this paper we consider controllability of leader-follower networks running a consensus algorithm. We allow the communication links between the agents to be weighted and give the necessary and sufficient conditions for a the system to be controllable generically, i.e. for almost all choices of weights. We call such a network weight controllable, a new notion that is introduced in this paper. A new descriptor formulation for the leader-follower consensus algorithm is derived that allows modelling communication weights as free parameters without changing the system structure. This leads us to obtaining necessary and sufficient conditions for weight controllability of leader-follower consensus algorithms.

I. INTRODUCTION

A multi-agent consensus system is a system comprised of n communicating mobile agents that follow a consensus protocol in order to achieve agreement. The consensus protocol is usually given by

$$\dot{x}_i(t) = - \sum_{j=1}^n w_{ij}(x_i(t) - x_j(t)), \quad i = 1, \dots, n, \quad (1)$$

where $x_i : \mathbb{R}_0^+ \rightarrow \mathbb{R}$ is the state of the i th agent, which is denoted by x_i . The weight of the communication channel from agent x_j to agent x_i is given by w_{ij} , where $w_{ij} = 0$ means that there is no communication. Such systems have received a lot of interest in the last decade, see [1] and references therein. One of the questions raised is whether a consensus system can be controlled by adding one or several external agents, u_1, \dots, u_k , resulting in the system

$$\begin{aligned} \dot{x}_i(t) = & - \sum_{j=1}^n w_{ij}(x_i(t) - x_j(t)) - \\ & \sum_{j=1}^k (w_{i,n+j}(x_i(t) - u_j(t))), \quad i = 1, \dots, n, \end{aligned} \quad (2)$$

where $u_i : \mathbb{R}_0^+ \rightarrow \mathbb{R}$ is the state of u_i that is understood as a free control input that is assumed to be differentiable sufficiently many times. This model was first introduced in [2] and is usually referred to as a leader-follower consensus network, with leader agents u_i and follower agents x_i . Writing $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$, $u(t) = (u_1(t), u_2(t), \dots, u_k(t))^T$ we can express (2) in matrix form as

$$\dot{x}(t) = A_c x(t) + B_c u(t), \quad (3)$$

where $B_c \in \mathbb{R}^{n \times k}$, $(B_c)_{ij} = w_{i,n+j}$, $i = 1, \dots, n$, $j = 1, \dots, k$, $A_c \in \mathbb{R}^{n \times n}$, $(A_c)_{ij} = w_{ij}$, $i, j = 1, \dots, n$, if $i \neq j$ and $(A_c)_{ii} = - \sum_{j=1}^n w_{ij} - \sum_{j=1}^k (B_c)_{ij}$.

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One of the most commonly studied problems is to assume that $w_{ij} \in \{0, 1\}$, $i \in \{1, \dots, n\}$, $j \in \{1, \dots, n+k\}$ and then relate controllability properties to the underlying communication graph, e.g. [2], [3], [4]. One of the results obtained is that not every network that achieves consensus is also controllable in the classical sense, for example [2] shows that a complete network (i.e. $w_{ij} = 1$ for all weights) is not controllable by less than n leaders. Furthermore, available results seem to indicate ([5], [6]) that system (A_c, B_c) becomes uncontrollable if certain symmetries are present in the graph. These symmetries can be disrupted by choosing different edge weights for the system.

It is thus a natural extension of the problem to study whether (A_c, B_c) can be made controllable by allowing the communication links to have different weights. In [7], [8], [9] and other related work this question has been linked with the notion of structural controllability of consensus systems. However this existing work suffers from one major drawback. Structural controllability is the property that almost all systems are controllable that have the same zero/nonzero structure as (A_c, B_c) under the assumption that all nonzero entries of (A_c, B_c) are algebraically independent over \mathbb{R} . This is not the case with consensus systems, where the diagonal entries of the system matrix A_c in (3) are dictated by the row sums of the other entries, and symmetries might be present when the communication between two agents is bidirectional. Thus, though the system might be structurally controllable in the case that all entries are “free“, it is yet to be shown that it is controllable without destroying the particular consensus structure of the system.

For this reason, in this paper we introduce a different controllability concept for leader-follower consensus systems. We say that a system is weight controllable if we allow each communication channel to be weighted and the resulting system (3) is controllable for almost all combinations of weights. To achieve our ends we use mixed polynomial matrices, developed in [10], [11]. We introduce a mixed matrix descriptor formulation of the system (3) and, following an algorithm in [10], derive conditions that allow us to link controllability properties to graph properties. Our contribution is first that we give a formulation of (3) that allows to model the communication weights as free parameters. Second, we outline a proof for weight controllability of consensus systems. Third, the methods we use are applicable to both directed and undirected communication topologies, whereas the results available so far only treat undirected communication among follower agents. Finally, for systems where bidirectional communication is present, we can easily impose the condition $w_{ij} = w_{ji}$ between followers. The

results available until now did not allow this symmetry preservation. A related work is [12], who consider the same problem independently with different methods.

This paper is organized as follows. In Section II we give a brief overview of algebraic graph theory, which is the basis of multi-agent network modelling. We then give a graph-theoretic definition of system (3) in Section III and introduce some assumptions that allow us to better state our results. In Section IV we formally define the notion of weight controllability of the leader-follower consensus system. In Section V the notion of mixed matrices is introduced, a model that allows us to derive the main result. The main result is presented in Section VI, particularly in VI-D. It is followed by a discussion and outlook in Section VII.

We use the following notation. For a finite set V , $|V|$ denotes its cardinality. For a $n \times m$ matrix A , $(A)_{ij}$ denotes its (i, j) th entry. For two matrices A, B of same row size, $[A, B]$ denotes the composite matrix. I^n denotes the $n \times n$ identity matrix, while 0^n resp. $0^{n \times m}$ is a $n \times n$ resp. $n \times m$ zero matrix.

II. GRAPH THEORY

In this section we give a brief overview of the graph theoretic notions used in this paper. For a more thorough introduction see, e.g. [13]. With a slight abuse of notation we use the variables x_i to denote graph nodes, while $x_i(t)$ denotes agent states. As we will show in Section III there is a one to one correspondence between agents and nodes, thus we believe that this choice makes the notation less cluttered without introducing confusion.

A graph $G = (V, E, \mathcal{W})$ is given on the set of nodes $V = \{x_1, \dots, x_n\}$ with the edge set $E = \{e_1, \dots, e_m\}$ and the weight set $\mathcal{W} = \{w_{ij}\}$, $i, j \in \{1, \dots, n\}$. The edge $e_k \in E$ between two nodes $x_i, x_j \in V$ is directed, denoted $e_k = x_i \rightarrow x_j$ if $w_{ji} \neq 0$ and $w_{ji} \neq w_{ij}$. Then x_i is called the initial and x_j the final node of e_k , and e_k is an incoming edge for x_j and an outgoing edge for x_i . If $w_{ij} = w_{ji} \neq 0$, the corresponding edge e_k is called undirected, denoted $e_k = x_i \leftrightarrow x_j$. Then x_i, x_j are simultaneously initial and final nodes of the edge and the edge is incoming and outgoing for both nodes. This definition of edges is unconventional, but its advantage will become apparent in Section III. We only consider simple graphs, i.e. there are no parallel edges and no self-loops (edges $x_i \rightarrow x_i$).

A graph is called undirected if it contains no directed edges. A path is a finite sequence of nodes such that every node is the initial node of an edge and the next node in the sequence is the final node of the edge. The length of a path is the number of edges in the path. A node x_2 is said to be reachable from x_1 if there is a path from x_1 to x_2 . An undirected graph is connected if every node is reachable from every node. Otherwise it is said to consist of connected components. A directed graph is called simply connected if it is connected ignoring edge orientation. A directed tree is a graph where one node, the root, has no incoming edges, and there is exactly one path from the root to every other node in the graph. A directed graph contains a directed spanning

tree if there is a node such that every other node in the graph is reachable from it.

Given a node set $V^* \subset V$, an induced subgraph of G on V^* is given by taking the nodes V^* and all the edges between nodes in V^* , including their weights.

An algebraic representation of a graph G is the $|V| \times |V|$ Laplacian matrix $L(G)$, with $(L)_{ij} = -w_{ij}$, $i, j \in \{1, \dots, n\}$, $i \neq j$ and $(L)_{ii} = \sum_{j \neq i} w_{ij}$.

Another notion associated with graphs is the $|E| \times |V|$ incidence matrix $L_I(G)$. Assign an (arbitrary) orientation to every undirected edge of the graph. Then each row of L_I corresponds to an edge of the graph, namely $(L_I)_{ij} = 1$ ($(L_I)_{ij} = -1$) if x_j is the initial (final) node of the i th edge e_i . The incidence matrix is unique up to multiplication of rows that correspond to an undirected edge by -1 , however, due to the arbitrary assignment of orientation to undirected edges, it is not always possible to reconstruct the original graph from it.

Note that if the graph is a directed tree or a set of node-disjoint directed trees on p nodes, then there is an ordering of edges in the graph such that the first p columns and rows of L_I form an upper triangular submatrix.

III. LEADER-FOLLOWER CONSENSUS SYSTEM

The meaning behind system (3) is that it represents the information flow between leader and follower agents. This information flow follows a communication network that determines which agents transmit and receive information. Using the methods of Section II we can use graph theory to describe the network. In order to exclude trivial systems, we introduce the following assumption.

Assumption 1: There is at least one leader ($k \geq 1$) and every leader is connected to at least one follower.

If there are no leaders, the system evidently is not controllable. If there is a leader that is not connected to a follower, then the corresponding column of B_c is 0 and the leader can be neglected.

Associated with the system (3) is the so-called leader-follower (LF) graph $G = (V, E, \mathcal{W})$, where $V = V_x \cup V_u$ is the node set, containing n follower nodes $V_x = \{x_1, \dots, x_n\}$ and k leader nodes $V_u = \{u_1, \dots, u_k\}$, $E = E_u \cup E_x$ is the edge set with $E_u = \{u_i \rightarrow x_j : \mathfrak{x}_j \text{ obtains information from } u_i\}$ and $E_x = E_D \cup E_{UD}$, where $E_{UD} = \{x_i \leftrightarrow x_j : \text{there is bidirectional communication between the agents } \mathfrak{x}_i \text{ and } \mathfrak{x}_j \text{ and } w_{ij} = w_{ji}\}$, $E_D = \{x_i \rightarrow x_j : x_i \leftrightarrow x_j \notin E_{UD} \text{ and } \mathfrak{x}_j \text{ obtains information from } \mathfrak{x}_i\}$. The edge weights w_{ij} , given by the nonzero weights of the communication channels, are collected in the set \mathcal{W} . We call the subgraph induced by V_x the follower graph. The LF-graph is directed, and is equipped with the following connectivity notion.

Definition 1 (Leader-follower connected): An LF-graph is said to be LF-connected if every node $x_i \in V_x$ is reachable from at least one node $u_j \in V_u$.

The Laplacian matrix of the leader-follower graph is given by $\begin{pmatrix} -A_c & -B_c \\ 0^{k \times n} & 0^{k \times k} \end{pmatrix}$, while its incidence matrix is the matrix

$$L_I = \begin{pmatrix} K_I & K_B \end{pmatrix}, \quad (4)$$

where the $m \times n$ matrix K_I represents V_x and the $m \times k$ matrix K_B represents V_u . Let K be an $n \times m$ matrix with

$$(K)_{ij} = \begin{cases} 1 & \text{if } (K_I)_{ji} = -1, \\ -1 & \text{if } (K_I)_{ji} = 1 \text{ and } e_j \in E_{UD} \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

Then the following decomposition holds: $A_c = KK_I$ and $B_c = KK_B$. Note that by construction $\text{rank}(K) = \text{rank}(K_I)$.

In order to make the paper easier to read it is useful to introduce the following assumptions.

Assumption 2: The LF-graph is simply connected.

If the LF-graph is simply connected then there are at least $n+k-1$ edges in the graph. Therefore Assumption 2 implies that

$$m \geq n + k - 1. \quad (6)$$

Assumption 3: Every follower node has at least one incoming edge.

If this assumption does not hold, there are obviously follower nodes that cannot be influenced by the leaders.

We will need the following result on the rank of K_I of equation (4).

Lemma 1: Suppose that the LF-graph is LF-connected. Then $\text{rank}(K_I) = n$.

Proof: The matrix K_I is $m \times n$, and under Assumption 2 it holds that $m \geq n$. There are n follower nodes in the graph and due to Assumption 1 and as the graph is LF-connected, there is a path from every leader to at least one follower node and a path to every follower node from a leader. Consider leader u_1 and let there be k_1 followers that are connected to it by a directed path. Together u_1 and the k_1 followers form a tree that consists of k_1 edges. As the graph is leader-follower connected, the followers that are not reached from node u_1 must be reached from u_2, \dots, u_k . Thus there is a node-disjoint tree rooted at u_2 . Without loss of generality (w.l.o.g.) let it reach k_2 followers and thus consist of k_2 edges. Analogously, node-disjoint trees of size $k_3, \dots, k_l, l \leq k$ are rooted at u_3, \dots, u_l until every node is contained in exactly one tree and $\sum_{i=1}^l k_i = n$. Thus there is a reordering of the rows of L_I (cf. Section II) such that the edges contained in the trees are in the first n columns of the matrix and L_I contains a $n \times n$ upper triangular submatrix with ± 1 on the diagonal. As the first n columns of L_I correspond with follower nodes, we see that this submatrix is also contained in K_I . Thus $\text{rank}(K_I) = n$. ■

IV. STRUCTURAL AND WEIGHT CONTROLLABILITY

Structural controllability of LF-systems was first defined in [8]. Remember that a structured matrix (see [14] and references therein for an overview) is a matrix containing fixed zero entries and “free” nonzero parameters that are algebraically independent over the chosen field.

Definition 2 (Structural Controllability): We say that the consensus system (3) is structurally controllable, if almost all pairs of matrices $(\tilde{A}_c, \tilde{B}_c)$ that have the same zero/nonzero structure as A_c, B_c are controllable.

Here, almost all means all but for a set of Lebesgue measure 0. In [8] the authors obtain the following result:

Lemma 2: Assume that the follower graph is undirected. The structured system $(\tilde{A}_c, \tilde{B}_c)$ is structurally controllable if and only if the LF-graph is LF-connected.

While the obtained result is correct, its applicability to consensus systems is questionable. Particularly, structural controllability results require that all entries of the system matrix are independent over the chosen field (\mathbb{R} in [8]). This is certainly not the case with A_c in system (3). Here, the diagonal entries are sums of entries of A_c and B_c and therefore cannot be chosen freely. This is an inherent property of the system. By assuming that all nonzero entries of A_c are mutually independent, we might obtain a system that is controllable but does not represent a consensus algorithm given by equations (2). Furthermore, the set of all matrices $\begin{pmatrix} A_c & B_c \\ 0 & 0 \end{pmatrix}$ that are Laplacian matrices has Lebesgue measure 0 in the set of all $n+k \times n+k$ matrices with the same structure. Thus it could occur, that while $(\tilde{A}_c, \tilde{B}_c)$ is structurally controllable, it is not possible to choose the entries such that the obtained system $\dot{x}(t) = A_c x(t) + B_c u(t)$ represents the dynamic (2). Finally, the authors of [8] assume that the follower graph is undirected. The use of structural controllability requires however that the weights w_{ij} and w_{ji} associated with the same edge are treated as two independent variables. Therefore it cannot be guaranteed that a LF-system can be made controllable by a choice of weights that preserves the symmetries in the follower graph.

It remains therefore an open question whether or not system (3) is structurally controllable without changing the system’s principal characteristics. To our best knowledge this problem has not been addressed in the literature. Clearly there are no methods within structural controllability theory to tackle this problem. Therefore in this paper we turn to matroid theory ([10], [11]). This theory has been expressly developed in order to deal with matrices that have interdependent and free entries.

In order to formally distinguish between structural controllability and controllability of the consensus problem we introduce

Definition 3 (Weight Controllability): We say that the consensus system (3) with the graph $G = (V, E, \mathcal{W})$ is weight controllable if for almost all sets of weights \mathcal{W} of G the corresponding system (A_W, B_W) is controllable.

V. MIXED MATRIX CONTROLLABILITY

As we have explained in Section IV, system (3) cannot be studied with structural methods, as the assumption that all nonzero entries of A_c and B_c are algebraically independent fails for the diagonal entries of A_c . Thus, the system equations are governed by both “free” parameters (the edge weights) and other parameters that depend upon them. In order to treat such systems, mixed matrices have been developed in the eighties, see [10] and references therein. Formally, a mixed matrix is given by

$$M_0 = Q_0 + T_0, \quad (7)$$

where Q_0 is an $n \times n$ matrix over \mathbb{R} and T_0 is an $n \times n$ generic matrix, such that its entries are either 0 or algebraically independent over \mathbb{R} . Given a set of mixed matrices $M_0 = Q_0 + T_0, \dots, M_t = Q_t + T_t$, a mixed polynomial matrix is a rational function matrix in s given by

$$M(s) = Q(s) + T(s) = \sum_{i=0}^t s^i (Q_i + T_i). \quad (8)$$

A. Finding an Appropriate Model

The matrix A_c does not fit the definition of a mixed matrix, as it cannot be straightforwardly separated into a fixed and a free part. In order to apply mixed matrix theory to system (3) we first need to define an equivalent mixed polynomial matrix formulation for the system.

A dynamical system is said to be a descriptor system if it is given by

$$\mathcal{F}\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t), \quad (9)$$

where $\mathcal{F}, \mathcal{A} \in \mathbb{R}^{n \times n}$, $\mathcal{B} \in \mathbb{R}^{n \times m}$, $x(t)$ is an unknown real vector and $u(t)$ is a vector function differentiable sufficiently many times. The initial condition $\dot{x}(t=0)$ is admissible if equation (9) has a solution $(x(t=0), u(t=0))$. A state x_1 is reachable from a state x_0 if and only if there exists an $u(t)$ such that $x(t=0) = x_0$ and $x(t=t_1) = x_1$ for some $t_1 > 0$.

Definition 4 (R-controllability, [15]): The system (9) is said to be R-controllable if one can reach any state in the set of reachable states from any admissible initial state.

Assuming that $\mathcal{F} = Q_{\mathcal{F}} + T_{\mathcal{F}}$, $\mathcal{A} = Q_{\mathcal{A}} + T_{\mathcal{A}}$, $\mathcal{B} = Q_{\mathcal{B}} + T_{\mathcal{B}}$ are mixed matrices in equation (9), then we obtain a family of descriptor systems parametrised by the free entries in $T_{\mathcal{F}}, T_{\mathcal{A}}, T_{\mathcal{B}}$. A particular system in this form is said to be structurally R-controllable if and only if almost all of the systems in this family are R-controllable.

We now derive a mixed matrix descriptor formulation of the LF-network that is closely related to system (3). The m algebraically independent weights are the free parameters of the mixed system matrices, where each weight corresponds to a directed or an undirected edge. The algebraic relations are expressed by introducing m additional states $y_1(t), \dots, y_m(t)$ and the LF-consensus algorithm on graph $G = (V, E, \mathcal{W})$ is given by

$$\underbrace{\begin{pmatrix} I^n & 0 \\ 0 & 0^{m \times m} \end{pmatrix}}_{:=F} \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & K \\ K_I & W \end{pmatrix}}_{:=A} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ K_B \end{pmatrix}}_{:=B} u(t). \quad (10)$$

Here, $x(t)$ is the vector of agent states, $u(t)$ is the control input provided by leader agents, $W \in \mathbb{R}^{m \times m}$ is a diagonal matrix, where $(W)_{ii} = -1/w_{e_i}$ and $w_{e_i} \in \mathcal{W}$ is the weight of edge e_i , $K_I \in \mathbb{R}^{m \times n}$, $K_B \in \mathbb{R}^{m \times k}$, and $[K_I, K_B] = L_I$, the incidence matrix of G introduced in equation (4). Finally, K is the $n \times m$ matrix introduced in equation (5).

In this formulation F and B are fixed matrices, i.e. $F = Q_F$, $B = Q_B$, while $A = Q_A + T_A$ with $Q_A = \begin{pmatrix} 0 & K \\ K_I & 0 \end{pmatrix}$, $T_A = \begin{pmatrix} 0 & 0 \\ 0 & W \end{pmatrix}$.

Lemma 3: System (A_W, B_W) is weight-controllable if and only if (10) is structurally R-controllable.

Proof: The lemma follows directly from the fact that for all $x(t=0) \in \mathbb{R}^n$ there exists $y(t=0) \in \mathbb{R}^m$ such that $(\bar{x}^T, \bar{y}^T)^T$ is admissible for the system (10). ■

Example 1: Let the controlled consensus system be given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -w_{1u} & 0 & 0 \\ w_{21} & -(w_{21} + w_{23}) & w_{23} \\ 0 & w_{32} & -w_{32} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} w_{1u} \\ 0 \\ 0 \end{pmatrix} u. \quad (11)$$

Here $n = m = 3$, $k = 1$. Assume that $w_{23} = w_{32}$ must hold. A corresponding descriptor formulation is then $W = \text{diag}(-1/w_{1u}, -1/w_{21}, -1/w_{23})$, $F = \text{blockdiag}(I^3, 0^3)$,

$$K_I = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, K_B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}. \quad (12)$$

B. Mixed Matrix Controllability Conditions

Mixed matrix systems that represent physical systems further have certain properties in regard to the powers of s they may contain.

Definition 5 (Dimensioned Matrix, [10], p. 145): An $n \times m$ mixed polynomial matrix $M(s) = Q(s) + T(s)$ is dimensioned if there are integers $\{r_1, \dots, r_n\}$, $\{c_1, \dots, c_{n+m}\}$ such that $Q(s) =$

$$\text{diag}(s^{r_1}, \dots, s^{r_n}) Q(s=1) \text{diag}(s^{-c_1}, \dots, s^{-c_{n+m}}). \quad (13)$$

The following conditions for R-controllability of (10) can be found in [10], P. 233:

Theorem 1: Suppose that the matrix $[Q_{\mathcal{A}} - sQ_{\mathcal{F}}, Q_{\mathcal{B}}]$ of system (9) is dimensioned. System (10) is structurally R-controllable if and only if the following conditions hold for almost all choices of the free entries in $T_{\mathcal{A}}, T_{\mathcal{B}}, T_{\mathcal{F}}$:

- 1) $\det(\mathcal{A} - s\mathcal{F}) \neq 0$ for some $s \in \mathbb{C}$,
- 2) $\text{rank}[\mathcal{A}, \mathcal{B}] = n$,
- 3) $\text{rank}[\mathcal{A} - s\mathcal{F}, \mathcal{B}] = n$ for all $s \in \mathbb{C} \setminus \{0\}$.

Condition 1 ensures that for every $u(t)$ there is a unique solution $x(t)$ of system (9), while Conditions 2 and 3 ensure the controllability of the zero resp. nonzero mode of (9).

It is straightforward to verify that for system (10) $[Q_{\mathcal{A}} - sQ_{\mathcal{F}}, Q_{\mathcal{B}}]$ is a dimensioned $n+m \times n+m+k$ matrix with $\{r_1, \dots, r_n\} = 1$, $\{r_{n+1}, \dots, r_{n+m}\} = 0$, $\{c_1, \dots, c_n\} = 0$, $\{c_{n+1}, \dots, c_{n+m}\} = 1$, $\{c_{n+m+1}, \dots, c_{n+m+k}\} = 0$.

VI. WEIGHT CONTROLLABILITY OF THE SYSTEM

In this section we provide conditions for the LF graph such that Conditions 1-3 are satisfied. We can consider Conditions 1 and 2 in a straightforward way. In order to establish when Condition 3 is satisfied, we will resort to a graph-based algorithm presented in [10].

A. Unique Solvability

Lemma 4: For system (10) there is an $s \in \mathbb{C}$ such that $\det(A - sF) \neq 0$.

Proof: For a block matrix $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ with quadratic diagonal blocks it holds that $\det \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \det(A_{22}) \det(A_{11} - A_{12}A_{22}^{-1}A_{21})$, provided that A_{22} is

invertible. As W is diagonal with nonzero diagonal entries, it is invertible. We can therefore write $\det(A - sF) =$

$$\det \begin{pmatrix} -sI^n & K \\ K_I & W \end{pmatrix} = -\det(W) \det(KW^{-1}K_I + sI^n). \quad (14)$$

i.e. $\det(A - sF) = 0$ if and only if $-s$ is an eigenvalue of $KW^{-1}K_I$. This is an $n \times n$ matrix, therefore it has at most n distinct eigenvalues. Thus $\det(A - sF) \neq 0$ for almost all $s \in \mathbb{C}$. ■

B. Controllability of the Zero Mode

For system (10) Condition 2 reads $\text{rank}[A, B] = n + m$. In this section we give a sufficient condition for this. One way to determine the rank of a mixed matrix is to put it into layered mixed matrix form.

Definition 6 (Layered Mixed Matrix) [10], p.133: A mixed matrix M is called a layered mixed (LM) matrix if the nonzero rows of Q and T are disjoint, i.e., if there exists a row permutation such that after the permutation

$$M = \begin{pmatrix} Q \\ T \end{pmatrix} = \begin{pmatrix} Q \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ T \end{pmatrix}. \quad (15)$$

Theorem 2 ([11], Theorem 4.2.2): For an LM matrix $M = \begin{pmatrix} Q \\ T \end{pmatrix}$, let C be the index set of its columns, R_Q be the index set of the rows of submatrix Q , and R_T of the rows of submatrix T . Let $M[I, J]$ denote the submatrix of M with row-set I and column-set J . Then

$$\text{rank}(M) = \max_{J \subseteq C} (\text{rank}(Q[R_Q, J]) + \text{rank}(T[R_T, C \setminus J])). \quad (16)$$

Here $\text{rank}(T)$ of a ‘‘free’’ matrix is its term rank, defined as the maximal set of independent columns J_T of T . In particular for a $n \times n$ matrix T with free parameters, $\text{rank}(T) = n$ if T has a nonzero diagonal.

Every mixed matrix can be expressed as an LM matrix. Particularly, following [10] we have

$$\text{rank} \begin{pmatrix} 0 & K & 0 \\ K_I & W & K_B \end{pmatrix} = \text{rank} \begin{pmatrix} 0 & 0 & K & 0 \\ I^m & K_I & 0 & K_B \\ P & 0 & W & 0 \end{pmatrix} - m, \quad (17)$$

where P is a $m \times m$ diagonal matrix of free parameters.

Lemma 5: Consider system (10). If the LF-graph is LF-connected, then $\text{rank}[A, B] = n + m$.

Proof: If the follower graph is simply connected, then by Lemma 1 $\text{rank}(K_I) = n$ holds. W.l.o.g. let the first n rows of K_I linearly independent. Then by construction the first n columns of K are also linearly independent. Write $K_I^T = (K_{I1}^T \ K_{I2}^T)^T$, $K = (K_1 \ K_2)$. Let P_1, P_2 be a $n \times n$ resp. $m-n \times m-n$ diagonal matrix with free parameters and consider the LM matrix

$$\begin{pmatrix} n & J_1 & J_2 & J_3 & J_4 & J_5 & J_6 \\ & \frac{n}{n} & m-n & \frac{n}{n} & \frac{n}{n} & m-n & k \\ n & \begin{pmatrix} 0 & 0 & 0 & -K_1 & -K_2 & 0 \\ I^n & 0 & K_{I1} & 0 & 0 & K_{B1} \\ 0 & I^{m-n} & K_{I2} & 0 & 0 & K_{B2} \\ P_1 & 0 & 0 & W_1 & 0 & 0 \\ 0 & P_2 & 0 & 0 & W_2 & 0 \end{pmatrix} \end{pmatrix}. \quad (18)$$

This matrix is $(n+2m) \times (n+2m+k)$, therefore it has at most rank $n+2m$. Following Theorem 2 choose $J = J_2 \cup J_3 \cup J_4 \cup J_6$. Then $T[R_T, C \setminus J] = \begin{pmatrix} P_1 & 0 \\ 0 & W_2 \end{pmatrix}$ and $\text{rank}(T[R_T, J]) = m$. It remains to show that

$$\text{rank} \begin{pmatrix} 0 & 0 & K_1 & 0 \\ 0 & K_{I1} & 0 & K_{B1} \\ I^m & K_{I2} & 0 & K_{B2} \end{pmatrix} \geq 2n + m. \quad (19)$$

This is satisfied because $\text{rank}(K_{I1}) = \text{rank}(K_1) = n$. ■

C. Controllability of the Nonzero Mode

Condition 3 of Theorem 1 is satisfied for system (10) if $\forall s \in \mathbb{C} \setminus \{0\}$ $\text{rank}[A - sF, B] = n + m$. Here we use the following structure, which allows us to relate the properties of the LF-network to the weight controllability of the system in a straightforward way.

Definition 7 (Auxiliary LF-graph): Given the LF-graph $G = (V, E, \mathcal{W})$, $V = V_x \cup V_u = \{x_1, \dots, x_n, u_1, \dots, u_k\}$, $E = \{e_1, \dots, e_m\}$, the auxiliary LF-graph $G_{\text{aux}} = (V_{\text{aux}}, E_{\text{aux}}, V_{\text{aux}}^M)$ is given by the node set $V_{\text{aux}} = \{e_1, \dots, e_m\}$, the marked nodes $V_{\text{aux}}^M = \{e_i : e_i \text{ is an edge from a leader to a follower node in } G\}$, and the edge set $E_{\text{aux}} = \{e_i \rightarrow e_j : e_i \notin V_{\text{aux}}^M \wedge \exists x \in V \text{ such that } x \text{ is either initial or final node of } e_i \text{ and } x \text{ is the final node of } e_j\}$.

An example of an LF-graph and the corresponding auxiliary graph is given in Figure 1.

Lemma 6: For system (10), $\text{rank}[A - sF, B] = n + m$, $s \in \mathbb{C} \setminus \{0\}$, if and only if V_{aux}^M is reachable to every unmarked node in the auxiliary LF-graph.

Proof of Lemma 6 can be obtained by straight application of the algorithm in Chapter 29, p. 241 of [10]. We leave it out due to the space limitation, it is available from the authors upon request. Using Lemma 6, we can verify the following result.

Lemma 7: For system (10), $\text{rank}[A - sF, B] = n + m$, $s \in \mathbb{C} \setminus \{0\}$, if and only if the LF-graph is LF-connected.

Proof: By definition of G_{aux} there is an edge from a node e_i to a node e_j if there is a node x , such that x is either initial or final node of e_i and final node of e_j . Suppose that the LF-graph is LF-connected. Then every follower can be reached by a directed path from at least one leader agent. Clearly, then there is also a directed path in the auxiliary graph, from the edge that connects the follower agent to the path to the edge that connects the leader to the follower nodes. Thus using Lemma 6 the forward direction holds true.

It remains to show the only if part. For this, suppose that the graph is not LF-connected and denote by V_0 the set of all follower nodes that are not reached by any leader node. Let G_0 be the subgraph of G induced by V_0 and G' the subgraph of G induced by $V' = V \setminus V_0$. There are no incoming edges from a leader to any node in V_0 . But as the LF-graph is simply connected by Assumption 2, there must be at least one outgoing edge from V_0 to V' . w.l.o.g. let this edge be $e_2 = x_3 \rightarrow x_2$, where $x_3 \in V_0$, $x_2 \in V'$. As x_2 is reachable from V_u , there is an edge, say e_1 from a node in V' to x_2 that is part of a path from some leader node to x_2 . Furthermore, x_3 has at least one incoming edge by Assumption 3, w.l.o.g.

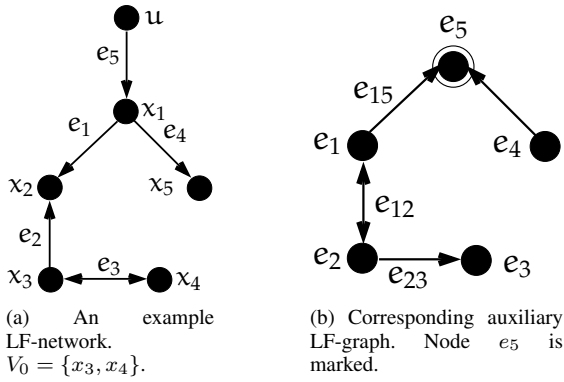


Fig. 1: An illustration for Lemma 7.

let it be called e_3 and let its initial node be x_4 . Clearly $x_4 \in G_0$. For simplicity assume that e_3 is undirected, i.e. it is also an incoming edge for x_4 . An example of this construction is shown in Figure 1.

The hereby introduced edges e_1, e_2, e_3 of G are nodes of G_{aux} . As x_2 is part of e_2 and the final node of e_1 , there is a directed edge $e_{12} : e_2 \rightarrow e_1$ in G_{aux} and therefore V_{aux}^M is reachable to e_2 . Consider now e_3 : x_3 is part of e_3 but not the final node of e_2 , which means there is no edge $e_3 \rightarrow e_2$ in G_{aux} and V_{aux}^M is not reachable to e_3 by a path through e_2 . This process is visualised in Figure 1b.

The same argument applies to any nodes of G_{aux} that are edges of G_0 . Therefore we have just shown that V_{aux}^M is not reachable to these nodes, i.e. the condition of Lemma 6 is not satisfied. This concludes the proof. ■

D. Main Result

Theorem 3: Suppose that Assumptions 1-3 are satisfied. The LF-consensus system (3) is weight controllable if and only if the LF-graph is LF-connected.

Proof: According to Lemma 3, system (3) is weight controllable if and only if system (10) is structurally R-controllable. If system (10) is structurally R-controllable, it satisfies Conditions 1-3 of Theorem 1. Particularly, it satisfies Condition 3. By Lemma 7 it follows that the graph is LF-connected.

On the other hand if the graph is LF-connected, then by Lemma 7 Condition 3 is satisfied. By Lemma 5 Condition 2 is also satisfied. By Lemma 4 Condition 1 is satisfied regardless of the LF-connectivity. Thus, if the graph is LF-connected, then the LF-consensus system is weight controllable. ■

VII. DISCUSSION AND OUTLOOK

In this paper, we study controllability problems for leader-follower systems, where communication links (edges) maybe uni- or bidirectional. All edges are associated with a free weight. We ask whether the resulting LF consensus system is generically controllable, i.e. controllable for almost all combinations of weights. This property is called "weight controllability". Note that a related study can be found

in [8]. However, [8] touches only systems with bidirected communication links (i.e., undirected graphs). Furthermore, [8] uses methods from the field of structural controllability, which assume that all nonzero entries in the corresponding system matrices are algebraically independent. Clearly, as these entries may consist of combinations of weights, this assumption does not hold for our problem. Somewhat surprisingly, we found that a necessary and sufficient condition for weight controllability is LF connectedness of the LF graph, i.e., the same condition as in [8]. While it is clear from the problem setup that LF connectedness will be necessary for any kind of LF controllability, we did not expect it to be sufficient as well. Our study also shows that for a weight controllable system, introducing additional communication links will not change its controllability property.

It is an interesting question to investigate whether it is possible to limit the number of weights to be assigned, for example to identify a minimal set of edges with free weight that would guarantee weight controllability. This in turn would shed more light on "classical" controllability of LF-systems as it would identify nodes and edges in graphs that are critical in order to make the whole system controllable.

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