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A universal class of non-homogeneous control Lyapunov functions for linear differential inclusions

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Abstract— **The constrained stabilization of Linear Differential Inclusions (LDIs) via non-homogeneous control Lyapunov functions (CLFs) is addressed in this paper. We consider the class of "merging" CLFs, which are composite functions whose gradient is a positive combination of the gradients of two given parents CLFs. In particular, we consider the constructive merging procedure based on recently-introduced composition via R-functions, which represents a parametrized trade-off between the two given CLFs. We show that this novel class of non-homogeneous Lyapunov functions is "universal" for the stabilization of LDIs, besides some equivalence results between the control-sharing property under constraints, i.e. the existence of a single control law which makes simultaneously negative the Lyapunov derivatives of the two given CLFs, and the existence of merging CLFs. We also provide an explicit stabilizing control law based on the proposed merging CLF. The theoretical results are finally applied to a perturbed constrained double integrator system.**

I. INTRODUCTION

Lyapunov-based techniques [1] represent an effective tool for constrained and robust control design. It is well known within the control community that control-design solutions based on quadratic functions [2] are quite conservative in terms of both Domain of Attraction (DoA) [3], [4] and robustness margin [5]. Therefore many non-quadratic control Lyapunov functions (CLFs) have been suggested, for instance polyhedral functions [3], [4], smoothed-polyhedral [6], [7] and composite quadratics [8], [9]. See [10], [11], [12] for surveys.

However, there still is a fundamental issue in Lyapunov-based controllers whenever constraints, robustness and optimality are of considered simultaneously: a single Lyapunov function is usually designed for one of these goals, and hence often ineffective for the others. For instance, the size of the DoA (under constraints), can be quite large if we consider a particular CLF. On the contrary, a CLF designed only to assure local "optimality" may provide a significantly smaller DoA.

This trade-off problem can be solved via a switching control law. Namely two controllers are designed and the control system switches from the "external controller" to the locally-optimal one as long as the state reaches the "smaller" DoA of the latter controller [13], [14].

However, the control-system discontinuity can be "dangerous", since the system state and the control input could be subject to jumps which can be even persistent in the presence of noise.

It would be indeed convenient to have a "smooth" transient from the level set of the "external" CLF to the "internal" one. A CLF with this property is one possible example of *merging* CLF.

Our work is inspired by [15], where the possibility to merge two given CLFs is investigated in a setting actually related to the problem of uniting local and global controllers [16], [17]. The technique in [15] works under the assumption that there exists a suitable domain in which the two control Lyapunov function share a common control [15, Proposition 2.2]. We have indeed investigated in [18] this latter assumption, which we call "control-sharing property", namely the existence of a single control law which makes simultaneously negative the Lyapunov derivatives of two given Lyapunov functions, making connections to the existence of merging CLFs.

In this paper, we focus on the class of constrained Linear Differential Inclusions (LDIs) and on the recently-introduced composition via R-functions [19], [20], see also [21], [22], [23] for the basic definitions. We notice that R-functions are non-polynomial functions, the latter being a key tool for accurate estimates of the domain of attraction [24], [25], [26].

We show that CLRFs represents a *universal* class of (*non-homogeneous*) functions for the stabilization of LDIs, namely that stabilizability is equivalent to stabilizability by means of a CLF in that class [27]. Moreover, since the control-sharing does not always hold globally [18], we address the problem of finding a differentiable CLF, which we here call "merging-inside" CLF, whenever the control-sharing property under constraints only holds on some compact domain of the state space.

The paper is organized as follows. Sections II, III present the technical background on CLFs for constrained LDIs, on the control-sharing property under constraints and on CLRFs. Sections IV, V show the main theoretical results. Section VI presents a differential CLF together with an associated Lyapunov-based control law. In Section VII we consider the control-design problem for the perturbed constrained double integrator system as an illustrative example. We conclude the paper in Section VIII.

A. Notation

I_n denotes the $n \times n$ identity matrix. $\overline{1}_s := (1, 1, ..., 1)^\top \in \mathbb{R}^s$. The notation co$(\cdot)$ denotes the convex hull [9]. int$\mathcal{S}$ denotes the interior of a set $\mathcal{S}$ and $\partial\mathcal{S}$ denotes its boundary. For any positive (semi)definite function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, the notation $\mathcal{L}_V$ denotes its 1-level set, i.e. $\mathcal{L}_V := \{x \in \mathbb{R}^n \mid V(x) \leq 1\}$. Hence, for $\sigma \in \mathbb{R}_{\geq 0}$, we have $\mathcal{L}_{(V/\sigma)} := \{x \in \mathbb{R}^n \mid V(x) \leq \sigma\}$. A square matrix $W \in \mathbb{R}^{s \times s}$ is an $\mathcal{M}$-matrix if $W_{i,j} \geq 0 \ \forall i \neq j$. [18].

II. Technical background

Let us consider the class of linear differential inclusions

$$\dot{x} \in \text{co}\{A_i x + B_i u \mid i \in [1, N]\}, \tag{1}$$

where $x \in \mathbb{X} \subseteq \mathbb{R}^n$, $u \in \mathbb{U} \subseteq \mathbb{R}^m$, being $\mathbb{X}$ and $\mathbb{U}$ closed and convex, and, for some integer $N > 0$, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$ for all $i \in [1, N]$.

We also consider the following notion of control Lyapunov function.

Definition 1 (CLF): A positive definite, radially unbounded, smooth away from zero, function $V : \mathbb{X} \to \mathbb{R}_{\geq 0}$ is a *Control Lyapunov Function* for (1) if there exists a locally-bounded control law $u : \mathbb{X} \to \mathbb{U}$ such that for all $x \in \mathbb{X}\setminus\{0\}$ we have

$$\max_{i\in[1,N]} \nabla V(x)(A_i x + B_i u) < 0. \tag{2}$$

In the rest of the paper, V_1 and V_2 will always denote two CLFs for (1). Without loss of generality, let us assume that V_1 shapes the *desired* controlled DoA (under constraints) $\bar{\mathbb{X}} \subseteq \mathbb{X}$, namely that $\bar{\mathbb{X}} := \mathcal{L}_{V_1}$[1]. Therefore we formulate the following standing assumption.

Standing Assumption 1: Functions $V_1, V_2 : \mathbb{X} \to \mathbb{R}_{\geq 0}$ are two *homogeneous* CLFs of order 2, respectively with controlled DoA $\mathcal{L}_{V_1} = \bar{\mathbb{X}}$ and $\mathcal{L}_{V_2}$.

We notice that choosing a degree 2 is without loss of generality because, if $\dot{V}(x) \leq -\eta V(x)$, then $(\dot{V^p})(x) \leq -\eta p V^p(x)$ for any $p > 0$.

The following definition will be also addressed later on.

Definition 2 (Control-Sharing Property): Two CLFs V_1 and V_2 for (1) have the *Control-Sharing Property* (under constraints $x \in \bar{\mathbb{X}}$, $u \in \mathbb{U}$) if there exists a locally-bounded control law $u : \bar{\mathbb{X}} \to \mathbb{U}$ such that for all $x \in \bar{\mathbb{X}}\setminus\{0\}$ we have the following inequalities simultaneously satisfied.

$$\max_{i\in[1,N]} \nabla V_1(x)(A_i x + B_i u(x)) < 0. \tag{3a}$$

$$\max_{i\in[1,N]} \nabla V_2(x)(A_i x + B_i u(x)) < 0. \tag{3b}$$

III. R-composition of control Lyapunov functions

In this section we present the merging procedure between two homogeneous CLFs proposed in [19].

The "R-composition" of two CLFs consists of the following steps.

1) Define[2] $R_1, R_2 : \mathbb{R}^n \to \mathbb{R}$ as $R_i(x) = 1 - V_i(x)$, $i = 1, 2$.

2) For fixed $\phi > 0$, define $R_\wedge : \mathbb{R}^n \to \mathbb{R}$ as[3]

$$R_\wedge(x) := \rho(\phi)\left(\phi R_1(x) + R_2(x) - \sqrt{\phi^2 R_1(x)^2 + R_2(x)^2}\right), \tag{4}$$

where $\rho(\phi) := \left(\phi + 1 - \sqrt{\phi^2+1}\right)^{-1}$ is the *normalization factor* [19, Section 2].

3) Define the "R-composition" $V_\wedge : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ as

$$V_\wedge(x) := 1 - R_\wedge(x). \tag{5}$$

It follows from the properties of the "R-functions", see Appendix VIII, that $V_\wedge$ is positive definite (Lemma 6), and that $\mathcal{L}_{V_\wedge} = \mathcal{L}_{V_1} \cap \mathcal{L}_{V_2}$ for all $\phi > 0$ (Lemma 7).

Therefore, since we want $\mathcal{L}_{V_1} = \bar{\mathbb{X}}$ to be the desired domain also for $V_\wedge$, we have to scale function V_2 so that $\mathcal{L}_{V_2} \supset \mathcal{L}_{V_1}$ and hence

$$\mathcal{L}_{V_\wedge} = \mathcal{L}_{V_1} \cap \mathcal{L}_{V_2} = \mathcal{L}_{V_1} = \bar{\mathbb{X}}.$$

The function $V_\wedge$, namely the merging of V_1 and V_2 (given from Standing Assumption 1), will be used as a candidate CLF later on. Whenever $V_\wedge$ is a CLF, we will emphasize its construction via the *R-composition* by referring $V_\wedge$ as Control Lyapunov R-Function (CLRF).

We then present the expression of $\nabla V_\wedge$ as a function of ∇V_1 and ∇V_2. It follows from the proof of [19, Theorem 1] that

$$\nabla V_\wedge(x) = \rho(\phi)\left[\phi c_1(\phi, x)\nabla V_1(x) + c_2(\phi, x)\nabla V_2(x)\right], \tag{6}$$

where $c_1, c_2 : \mathbb{R}_{>0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ are defined as

$$c_1(\phi, x) := 1 + \frac{-\phi R_1(x)}{\sqrt{\phi^2 R_1(x)^2 + R_2(x)^2}} \tag{7a}$$

$$c_2(\phi, x) := 1 + \frac{-R_2(x)}{\sqrt{\phi^2 R_1(x)^2 + R_2(x)^2}}. \tag{7b}$$

Since $\phi > 0$, we notice that functions $c_i(x)$s are continuous whenever $R_1(x)$ and $R_2(x)$ are not simultaneously 0, i.e. in the set $\text{int}\mathcal{L}_{V_\wedge}$. Therefore $V_\wedge$ is differentiable in the interior of $\mathcal{L}_{V_\wedge}$ [18]. Instead, it follows from the proof of Lemma 6 that, independently from $\phi > 0$, we have $\partial\mathcal{L}_{V_\wedge} := \{x \in \mathbb{X} \mid V_\wedge(x) = 1\} = \{x \in \mathbb{X} \mid \max\{V_1(x), V_2(x)\} = 1\}$, therefore $V_\wedge$ is not differentiable on its boundary.

We also notice that in the set $\text{int}\mathcal{L}_{V_\wedge}$, the shape of the level sets of $V_\wedge$ does depend on the parameter ϕ, which plays the role of a trade-off between the shape of V_1 and the one of V_2. Namely, in light of [19, Proposition 2], we have $V_\wedge(x) \overset{\phi\to\infty}{\longrightarrow} V_2(x)$ and $V_\wedge(x) \overset{\phi\to 0^+}{\longrightarrow} V_1(x)$, pointwise in $\text{int}\mathcal{L}_{V_\wedge}$. Moreover, according to Lemmas 8, 9, 10, we have $\nabla V_\wedge(x) \overset{\phi\to\infty}{\longrightarrow} \nabla V_2(x)$ and $\nabla V_\wedge(x) \overset{\phi\to 0^+}{\longrightarrow} \nabla V_1(x)$ uniformly on compact subsets of $\text{int}\mathcal{L}_{V_\wedge}$.

IV. Universality for stabilizability of linear differential inclusions

We now show that the class of CLRFs is a *universal* class of functions for the stabilizability (stability) of LDIs (1), namely that stabilizability (or stability) of LDIs (1) is equivalent to stabilizability by means of a Lyapunov function in that class [7]. We recall that polyhedral functions [28], [29], [27], smoothed polyhedral functions of the kind

$$V_p(x) := \|Fx\|_{2p}^2, \tag{8}$$

[1] The level set 1 is taken without loss of generality.

[2] The level set 1 is taken without loss of generality. With this choice we have $R_i(x) \geq 0 \Leftrightarrow x \in \mathcal{L}_{V_i}$.

[3] For ease of reading, the dependence of $R_\wedge$ from ϕ is not made explicit in the notation.

where $F \in \mathbb{R}^{s \times n}$ is full-column rank [7], the max of quadratics [9] and the convex hull of quadratics [8] are universal classes of *homogeneous* functions for the stabilizability of LDIs (1). In the following, we indeed show that homogeneity is not necessarily required.

In the following result, we consider $V_\wedge$ as the R-composition of $V_1 = V_p$ (8) and *any* V_2 (homogeneous, of degree two).

Theorem 1: Let $\mathcal{L}_{V_2} \supset \mathcal{L}_{V_1}$. Then there exists $\phi > 0$ such that $V_\wedge$ (5) is a CLF for (1).

Since $V_\wedge$ is differentiable in $\text{int}\mathcal{L}_{V_\wedge}$ (and also including the boundary $\partial\mathcal{L}_{V_\wedge}$ whenever $\mathcal{L}_{V_2} \supset \mathcal{L}_{V_1}$), this class of functions allows the derivation of explicit formulas for the stabilizing controller [7]. Roughly speaking, for ϕ small enough, Theorem 1 presents a *universal* CLF, which is *smooth* and *non-homogeneous*. This latter degree of freedom (note that Theorem 1 is independent from V_2) can be eventually exploited to "improve" closed-loop performances, while a-priori fixing the desired domain of attraction [19].

V. Results on the control-sharing property under constraints

As shown in [19, Figure 1], the inner shape of $V_\wedge$ is a trade-off between the ones of V_1 and V_2, depending on the choice of the parameter ϕ. In this section we address (necessary and sufficient) conditions such that any ϕ is admissible for $V_\wedge$ being a CLF. Therefore we consider positive definite, radially unbounded, smooth functions $V : \mathbb{X} \to \mathbb{R}_{\geq 0}$ such that

$$\mathcal{L}_V = \mathcal{L}_{V_1} = \bar{\mathbb{X}},$$

and having the gradient ∇V of the kind

$$\nabla V(x) = \gamma_1(x)\nabla V_1(x) + \gamma_2(x)\nabla V_2(x), \tag{9}$$

for some continuous functions $\gamma_1, \gamma_2 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. We call these functions "gradient-type merging" candidates.

Remark 1: Unlike [18], we here particularize the definition of gradient-type merging candidates by introducing the property that $\mathcal{L}_V = \mathcal{L}_{V_1} = \bar{\mathbb{X}}$. Because of the state constraint $x \in \bar{\mathbb{X}}$, we impose that the external shape of the merging function V is the same of V_1, i.e. that $\mathcal{L}_V = \mathcal{L}_{V_1}$.

A. Equivalence results on the control-sharing property

We now present the main theoretical results on the equivalence between the control-sharing property under constraints and the existence of merging CLFs.

Theorem 2: If V_1 and V_2 have the control-sharing property (under constraints $x \in \bar{\mathbb{X}}$, $u \in \mathbb{U}$), then any gradient-type merging V (9) is a CLF.

Theorem 3: Assume that $B_i = B$ for all $i \in [1, N]$. Then the following statements are equivalent for (1).

1) Any gradient-type merging V (9) is a CLF.
2) V_1 and V_2 have the control-sharing property (under constraints $x \in \bar{\mathbb{X}}$, $u \in \mathbb{U}$).

It is worth mentioning that the control-sharing property does not always hold even for unconstrained LDIs [18]. The same negative result holds true for linear systems, but with the remarkable exception of two-dimensional systems [18].

B. Control-sharing via Linear Matrix Inequalities

Let us now present an LMI condition, exploited later in Section VII, which is sufficient for CLFs $V_1(x) = \left\|\left(x^\top Q_1 x, ..., x^\top Q_s x\right)\right\|_{2p}^2$, with $p > 0$ large enough, $Q_j \succcurlyeq 0$ for all $j \in [1, s]$, and $V_2(x) = x^\top P x$, with $P \succ 0$, to share an unconstrained common controller.

Theorem 4: ([18])
V_1 and V_2 have the control-sharing property if there exist $\eta > 0$, $\lambda_{i,j,k} \geq 0$, $K_k \in \mathbb{R}^{m \times n}$, for $i = 1, 2, ..., N$, and $j, k = 1, 2, ..., s$, such that

$$(A_i + B_i K_k)^\top Q_k + Q_k(A_i + B_i K_k) \preccurlyeq - 2\eta Q_k + \sum_{j=1}^{s} \lambda_{i,j,k}\left(Q_j - Q_k\right) \tag{10a}$$

$$(A_i + B_i K_k)^\top P + P(A_i + B_i K_k) \preccurlyeq - 2\eta P + \sum_{j=1}^{s} \lambda_{i,j,k}\left(Q_j - Q_k\right) \tag{10b}$$

for all $i \in [1, N]$, $k \in [1, s]$.

To include control constraints, the LMI (10) can be, for instance, augmented with linear bounds on the variables K_k.

Remark 2: Theorem 4 is less restrictive than [19, Theorem 2], because condition (10) relies on a piecewise-linear common controller, rather than a linear common controller as in [19, matrix conditions (11)].

VI. Constrained locally-optimal control design via "merging inside"

Given a robust CLF V_1 with a "large" DoA, and a "locally-optimal"[4] control law $\kappa_2 : \bar{\mathbb{X}} \to \mathbb{R}^m$, satisfying the input constraints only in a neighborhood of the origin and associated to a CLF V_2, we consider the problem to find a unique differentiable CLF V with the same DoA of V_1, together with the local optimality induced by V_2.

For simplicity, we consider (1) with $B_i = B$ for all $i \in [1, N]$. The general case can be addressed using the arguments of [7, Section V].

Let $\bar{\mathbb{X}}_1 = \mathcal{L}_{(V_1/\sigma_1)} \subseteq \bar{\mathbb{X}} = \mathcal{L}_{V_1}$, for some $\sigma_1 \in (0, 1]$, be the largest level set of V_1 in which the control-sharing property under constraints (between V_1 and V_2) holds true.

Assumption 1: Functions V_1 and V_2 have the control-sharing property under constraints $x \in \bar{\mathbb{X}}_1 = \mathcal{L}_{(V_1/\sigma_1)}$, $u \in \mathbb{U}$. Associated with V_2 there is an "optimal" control

[4] "Locally-optimal" is intended with respect to some optimality criterion not explicitly stated here for sake of generality.

law $\kappa_2 : \mathbb{X} \to \mathbb{R}^m$ such that $\kappa_2(x) \in \mathbb{U}$ for all x in a neighborhood of the origin.

Let us now scale V_2, but without relabeling it, such that $\mathcal{L}_{V_2} \supset \bar{\mathbb{X}}_1 = \mathcal{L}_{(V_1/\sigma_1)}$[5], in order to get the merging function $V_\wedge$, hence merging V_1/σ_1 and V_2, such that $\mathcal{L}_{V_\wedge} = \bar{\mathbb{X}}_1$.

It follows from Theorem 2 that, under Assumption 1, $V_\wedge$ is a CLRF for (1) under constraints $x \in \bar{\mathbb{X}}_1$, $u \in \mathbb{U}$.

We now need to consider the set $\bar{\mathbb{X}} \setminus \bar{\mathbb{X}}_1$. Since the "shape" of $V_\wedge$ is the same of V_1 on $\partial \bar{\mathbb{X}}_1$, we propose the following function $V : \bar{\mathbb{X}} \to \mathbb{R}_{\geq 0}$ as candidate "merging-inside" function.

$$V(x) := \begin{cases} V_{\text{ext}}(x) := \rho(\phi)\phi \cdot V_1(x)/\sigma_1 + c & \text{if } x \in \bar{\mathbb{X}} \setminus \bar{\mathbb{X}}_1 \\ V_{\text{int}}(x) := V_\wedge(x) & \text{if } x \in \bar{\mathbb{X}}_1 \end{cases} \tag{11}$$

where c $(:= 1 - \rho(\phi)\phi)$ is a constant such that $V_{\text{ext}}(x) = V_{\text{int}}(x)$ for all $x \in \partial \bar{\mathbb{X}}_1$.

Proposition 5: Suppose Assumption 1 holds. Then the merging-inside function V (11) is a differentiable CLF on $\bar{\mathbb{X}}$.

Since both V_1 and $V_\wedge$ grow quadratically, it follows from [19, Section 4.2] that there exists $\eta > 0$ such that the convex-valued mapping of admissible (constrained) controls

$$\mathcal{U}(x) := \left\{ u \in \mathbb{U} \mid \max_{i \in [1,N]} \nabla V(x) \left(A_i x + Bu \right) + \eta x^\top x \leq 0 \right\} \tag{12}$$

is non-empty for all $x \in \bar{\mathbb{X}}$. Therefore we propose the control law

$$\kappa(x) := \arg\min \left\{ \| v - \kappa_2(x) \| \mid v \in \mathcal{U}(x) \right\}. \tag{13}$$

Remark 3: Similarly to [18], it can be proved that the control law κ (13) associated with V (11) satisfies the constraints in $\bar{\mathbb{X}}$ and is locally optimal. Moreover, by using the arguments in [30, Chapters 2, 4], it can be proved that it is continuous provided that κ_2 is continuous as well.

VII. Application to the Perturbed Constrained Double Integrator System

As an illustrative example, we consider the perturbed linear system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + d, \tag{14}$$

where $|u| \leq 2$, $||d||_\infty \leq 0.1$.

The objective is to make the set $\mathcal{L}[\|Fx\|_\infty, 1]$, where

$$F := \begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 1 & 1 \end{bmatrix}^\top,$$

controlled invariant[6] under random time-varying disturbance $d \in \mathbb{D}$, stabilizing x whenever d vanishes, with quadratic performance cost

$$J(x, u) := \int_0^\infty \left(x(t)^\top Q x(t) + u(t)^\top R u(t) \right) dt, \tag{15}$$

characterized by weights $Q = I_2$, $R = 10$.

Three control strategies are simulated. The first is the standard LQ control, namely based on the unconstrained-optimal Quadratic CLF (QCLF) $V_2(x) = x^\top P x$ (Figure 1), where $P \succ 0$ is the unique solution of the Algebraic Riccati Equation (ARE) $A^\top P + PA + Q - PBR^{-1}B^\top P = 0$, associated with the unconstrained-optimal linear state feedback $\kappa_2(x) := -R^{-1}B^\top P x$.

The second control strategy is a slight modification of the control law (13), in order to guarantee the set $\mathbb{X}$ to be controlled invariant even under the worst-case perturbation. In particular, we indeed define the following set of admissible controls.

$$\bar{\mathcal{U}}(x) := \left\{ u \in \mathbb{U} \mid \max_{i \in [1,N]} \nabla V(x) \left(A_i x + Bu \right) + \eta x^\top x \right. \\ \left. + V_1(x) \max_{d \in \mathbb{D}} \nabla V_1(x) d \leq 0 \right\}. \tag{16}$$

We consider the merging CLRF $V_\wedge$, built by merging the previous functions V_1 and V_2 with $\phi = 10$. Since, for $d \equiv 0$, LMI (10) is feasible on $\mathcal{L}_{V_1} = \bar{\mathbb{X}}$, when the disturbance d vanishes, this choice guarantees the same controlled invariant set of $V_1(x)$ and the local optimality given by V_2.

The third control strategy consists in the control (13) with admissible controls (16), but associated with the robust smoothed PCLF $V_1(x) = \|Fx\|_{2p}^2$, for integer p sufficiently large, rather than V. This smoothed PCLF solves the robust constrained stabilization problem, but it does not address optimality. As a consequence, especially for unperturbed dynamics, the achieved closed-loop performances (in terms of closed-loop cost J (15)) are quite "far" from the constrained-optimal ones.

Figures show the controlled state trajectory, starting from an x_0 close to the "worst-case" vertex. The disturbance d used in the simulation initially takes random values in $\mathbb{D} = [-0.1, 0.1]^2$; then it vanishes.

The proposed control (13), together with the merging CLF $V_\wedge$, guarantees a larger domain of attraction with respect to the use of the optimal QCLF. Furthermore, it improves the control performances of the robust smoothed PCLF, in terms of faster convergence, smoother control signal and less closed-loop performance cost J (15).

VIII. Conclusion

The problem of merging two control Lyapunov functions is considered important because when constraints, robustness and optimality are of concern simultaneously, a single control Lyapunov function is typically suitable for one of these goals, but ineffective for the others.

For the class of constrained linear differential inclusions, we have proposed a class of universal non-homogeneous smooth control Lyapunov functions. Since previous results show how to combine control Lyapunov functions if these

[5]Practically, for any given "tolerance" $\varepsilon > 0$, we have to impose the set inclusion: $\mathcal{L}_{V_2} \supseteq \bar{\mathbb{X}}_1 + \varepsilon \mathbb{B}$.

[6]We use such F for sake of simplicity. The largest controlled invariant set (up to an arbitrary precision) can be computed by using the algorithm proposed in [31], [32].

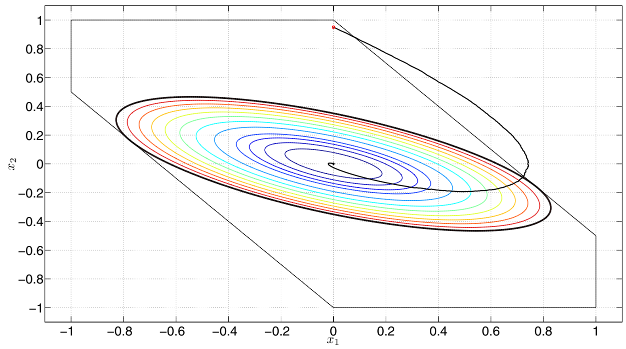

Fig. 1. The LQ control does not guarantee the fulfillment of the state constraints.

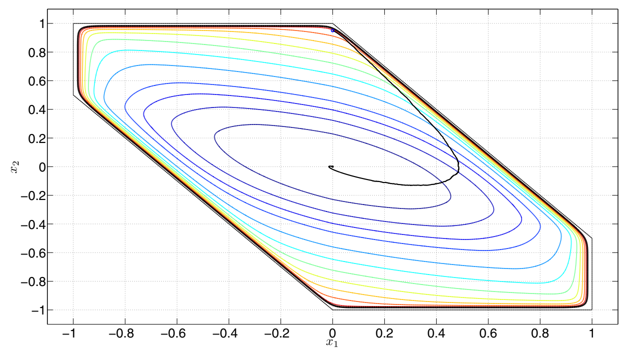

Fig. 2. A merging CLF solves the robust constrained stabilization problem with locally-optimal performance (for vanishing disturbances).

share a common control in a suitable region of the state space, we have shown the equivalence between the control-sharing property under constraints and the existence of merging control Lyapunov functions. We also have shown a constructive procedure, which we call "merging inside", to handle the case in which the control-sharing property holds only in a restricted domain of the state space.

The constrained, locally-optimal, stabilization of perturbed linear systems under vanishing disturbances has been addressed as an application of the provided theoretical results.

Appendix: Technical properties of the R-composition

The following results, established in [18], are further exploited in this paper.

Lemma 6: ([18])
$V_\wedge$ is positive definite.

Lemma 7: ([18])
$\mathcal{L}_{V_\wedge} = \mathcal{L}_{V_1} \cap \mathcal{L}_{V_2}$.

Lemma 8: ([18])
$\nabla V_\wedge$ converges to ∇V_2 uniformly on compact sets in int$\mathcal{L}_{V_\wedge}$, as $\phi \to \infty$. Namely, for any $\delta \in (0,1)$ we have

$$\lim_{\phi\to\infty} \max_{x\in\mathcal{L}_{(V_\wedge/\delta)}} \|\nabla V_\wedge(x) - \nabla V_2(x)\| = 0.$$

Lemma 9: ([18])
$\nabla V_\wedge$ converges to ∇V_1 uniformly on compact sets in int$\mathcal{L}_{V_\wedge}$, as $\phi \to 0^+$. Namely, for any $\delta \in (0,1)$ we have

$$\lim_{\phi\to 0^+} \max_{x\in\mathcal{L}_{(V_\wedge/\delta)}} \|\nabla V_\wedge(x) - \nabla V_1(x)\| = 0.$$

Lemma 10: ([18])
Assume that $\mathcal{L}_{V_2} \supset \mathcal{L}_{V_1}$. Then $\nabla V_\wedge$ converges to ∇V_1 uniformly on $\mathcal{L}_{V_\wedge}$, as $\phi \to 0^+$. Namely, we have

$$\lim_{\phi\to 0^+} \max_{x\in\mathcal{L}_{V_\wedge}} \|\nabla V_\wedge(x) - \nabla V_1(x)\| = 0.$$

Appendix: Proofs

A. Proof of Theorem 1

Proof: Since V_2 has been scaled so that $\mathcal{L}_{V_2} \supset \mathcal{L}_{V_1}$, we have $\mathcal{L}_{V_\wedge} = \mathcal{L}_{V_1}$ from Lemma 7.

As V_p is a (homogeneous) CLF, there exists $\eta > 0$ and a piecewise-linear $u : \mathbb{X} \to \mathbb{U}$ such that for all $x \in \mathbb{X}$ we have $\max_{i\in[1,N]} \nabla V_p(x)\,(A_i x + B_i u(x)) \leq -\eta x^\top x$.

According to Lemma 10, the gradient $\nabla V_\wedge$ converges to ∇V_p uniformly on $\mathcal{L}_{V_\wedge} = \mathcal{L}_{V_p}$, namely we can claim that for any $\epsilon > 0$ there exists $\phi > 0$ such that $\nabla V_p(x) = \nabla V_\wedge(x) + v(x)^\top$, where $v : \mathbb{R}^n \to \mathbb{R}^n$ is such that $\max_{x\in\mathcal{L}_{V_p}} \|v(x)\| \leq \epsilon$. We can indeed write

$$\begin{aligned} &\max_{i\in[1,N]} \{\nabla V_\wedge(x)\,(A_i x + B_i u(x))\} = \\ &\max_{i\in[1,N]} \left\{\nabla V_p(x)\,(A_i x + B_i u(x)) - v(x)^\top (A_i x + B_i u(x))\right\} \\ &\quad\leq \max_{i\in[1,N]} \{\nabla V_p(x)\,(A_i x + B_i u(x))\} + \\ &\qquad\qquad \max_{i\in[1,N]} \left\{v(x)^\top (A_i x + B_i u(x))\right\}. \end{aligned}$$

We can now choose $\epsilon \geq \|v(x)\|$ such that for all $x \in \mathcal{L}_{V_\wedge}$ we have $\max_{i\in[1,N]} \left\{v(x)^\top (A_i x + B_i u(x))\right\} \leq \frac{\eta}{2} x^\top x$, therefore we finally get $\max_{i\in[1,N]} \nabla V_\wedge(x)\,(A_i x + B_i u(x)) \leq -\frac{\eta}{2} x^\top x$ for all $x \in \mathcal{L}_{V_\wedge}$. ■

B. Proof of Theorem 2

Proof: By assumption, V_1 and V_2 have the control-sharing under constraints $x \in \bar{\mathbb{X}}$, $u \in \mathbb{U}$. Since the controlled set $\bar{\mathbb{X}}$ is a level set of the candidate CLF V, i.e. $\bar{\mathbb{X}} = \mathcal{L}_{V_1} = \mathcal{L}_V$, then the proof immediately follows from [18]. ■

C. Proof of Theorem 3

Proof: The implication (2) $\Rightarrow$ (1) follows from Theorem 2, therefore we have to prove the claim (1) $\Rightarrow$ (2). Fix arbitrary $\gamma_1, \gamma_2 > 0$ and define

$$i^*(x) := \arg\max_{i\in[1,N]} \left(\gamma_1 \nabla V_1(x) + \gamma_2 \nabla V_2(x)\right)(A_i x).$$

Now, by assumption we have the Lyapunov condition $\max_{i\in[1,N]} \left(\gamma_1 \nabla V_1(x) + \gamma_2 \nabla V_2(x)\right)(A_i x + Bu) < 0$, that, in view of the definition of $i^*(x)$, is equivalent to

$$\left(\gamma_1 \nabla V_1(x) + \gamma_2 \nabla V_2(x)\right)\left(A_{i^*(x)} x + Bu\right) < 0 \qquad (17)$$

Finally, we define $f(x) := A_{i^*(x)} x$ and we prove that (17) is equivalent to the fact that V_1 and V_2 have the control-sharing property.

We notice that (17) is equivalent to

$$\max_{(\alpha_1,\alpha_2)\in\mathcal{A}} \inf_{u\in\mathbb{U}} \left(\alpha_1 \nabla V_1(x) + \alpha_2 \nabla V_2(x)\right)\left(f(x)+Bu\right) < 0 \quad (18)$$

where $\mathcal{A} := \{(\alpha,\beta) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \mid \alpha+\beta = 1\}$.

Since $\mathcal{A}$ is compact and $\mathbb{U}$ is closed, and the function in (18) is linear in both (α_1,α_2) and u, we can exchange "max" and "min" [33, Corollary 37.3.2] to get the following equivalent condition.

$$\begin{aligned}
&\max_{(\alpha_1,\alpha_2)\in\mathcal{A}} \inf_{u\in\mathbb{U}} (\alpha_1 \nabla V_1(x) + \alpha_2 \nabla V_2(x))(f(x)+Bu) = \\
&\inf_{u\in\mathbb{U}} \max_{(\alpha_1,\alpha_2)\in\mathcal{A}} (\alpha_1 \nabla V_1(x) + \alpha_2 \nabla V_2(x))(f(x)+Bu) = \\
&\quad \inf_{u\in\mathbb{U}} \max_{(\alpha_1,\alpha_2)\in\mathcal{A}} \{\alpha_1 \nabla V_1(x)(f(x)+Bu) + \\
&\qquad \alpha_2 \nabla V_2(x)(f(x)+Bu)\} < 0 \iff \\
&\qquad \inf_{u\in\mathbb{U}} \max\{\nabla V_1(x)(f(x)+Bu), \\
&\qquad\qquad \nabla V_2(x)(f(x)+Bu)\} < 0. \quad (19)
\end{aligned}$$

The last inequality, which follows from the fact that $\alpha_1, \alpha_2 \in [0,1]$ and $\alpha_1 + \alpha_2 = 1$, is equivalent to the existence of a common controller, which is constrained in $\mathbb{U}$. ■

D. Proof of Proposition 5

Proof: From Theorem 2 we have that, under Assumption 1, $V_\wedge$ merging V_1/σ_1 and V_2, is a CLF for (1) under constraints $x \in \bar{\mathbb{X}}_1$, $u \in \mathbb{U}$.

Now, to exploit the fact that V_1 is a CLF on the whole $\mathbb{X}$ (Standing Assumption 1) and hence also on $\mathbb{X} \setminus \mathbb{X}_1$, we notice that $\partial\mathbb{X}_1 = \{x \in \mathbb{R}^n \mid V_1(x) = \sigma_1\} = \{x \in \mathbb{R}^n \mid R_1(x) := 1 - V_1(x)/\sigma_1 = 0\}$. So for $R_1(x) = 0$ we have $c_1(\phi, x) = 1$ and $c_2(\phi, x) = 0$ from (7). Therefore, from (6), we get that $\nabla V_\wedge(x) = \rho(\phi)\phi\nabla V_1(x)/\sigma_1$ for all $x \in \partial\mathbb{X}_1$. This means that ∇V (11) is continuous on $\mathbb{X}$. ■

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