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Computer algorithms for solving optimization problems for discrete-time fractional systems

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***Abstract*— Dynamic programming and discrete-time calculus of variations optimization problems for fractional discrete-time systems with quadratic performance index have been formulated and solved. A new methods for numerical computation of optimization problems have been presented. The efficiency of the methods have been demonstrated on numerical example and illustrated by graphs. Graphs also show the differences between the fractional and classical (standard) systems theory. Results for both methods have been obtained through a computer algorithms written for this purpose.**

I. INTRODUCTION

Dynamic optimization problems for integer (not fractional) order systems have been widely considered in literature (see e.g. [1]–[3]). Mathematical fundamentals of the fractional calculus are given in the monographes [4]–[6] and fractional calculus of variations is given in [7]. Fractional differential equations and their applications have been addressed in [8], [9]. The numerical simulation of the fractional order control systems has been investigated in [10]. One of the fractional discretization method has been presented in [11]. Some optimal problems for fractional order systems have been investigated in [12]–[15]. Dynamic Programming problem for discrete-time fractional systems has been formulated and solved in [16]. Variational calculus for discrete-time fractional systems have been investigated in [17]–[22]. Fractional Kalman filter and its application have been addressed in [23], [24]. Some recent interesting results in fractional systems theory and its applications for standard and positive systems can be found in [25], [26].

In this paper optimization problems for fractional discrete-time systems with quadratic performance index will be formulated. A general solutions for dynamic programming and discrete-time calculus of variations problems will be presented. A new methods for numerical computation of optimization problems will be presented. The efficiency of the methods will be demonstrated on numerical example and illustrated by graphs. Graphs also show the differences between the fractional and classical (standard) systems theory. Results for both methods will be obtained through a computer algorithm written for this purpose. Computer algorithms will be discussed in details and the block diagrams will be presented.

The paper is organized as follows. In section II some preliminaries are recalled and the optimization problem for dynamic programming method will be formulated. The general solutions of the above problem are presented also in that section. In section III the optimization problem for discrete-time calculus of variations method will be formulated. The general solutions of the above problem are presented also in that section. In section IV a computer algorithms for computation of the optimal control are proposed and numerical examples are presented. Conclusions of the paper are given in section V.

II. DYNAMIC PROGRAMMING

A. Problem Formulation

Consider a fractional discrete-time system, obtained by use of Grunwald-Letnikov'a (shifted) approximation, described by equations

$$x_{k+1} = \sum_{j=0}^{k} d_j x_{k-j} + Bu_k , \qquad k \in \mathbb{Z}_+ , \tag{1a}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ are respectively the state and control vectors, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and

$$d_0 = A_\alpha = A + \alpha I_n , \qquad 0 < \alpha < 1 , \tag{1b}$$

$$d_j = (-1)^j \binom{\alpha}{j+1} I_n , \qquad j = 1, \dots, k . \tag{1c}$$

Consider a performance index of the form

$$\begin{aligned} J_i(u) &= G(x_N) + \sum_{k=i}^{N-1} F_k(x_k, u_k) \\ &= x_N^T S x_N + \sum_{k=i}^{N-1} \left(x_k^T Q x_k + u_k^T R u_k \right) , \end{aligned} \tag{2}$$

where $R \in \mathbb{R}^{m\times m}$, $Q \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times n}$ and $S \geq 0$, $Q \geq 0$ and $R > 0$.

Optimal trajectory starting at the point x_0 and ending at the point x_k has been divided into N elementary time intervals $[0, N]$. It is desired to find optimal control sequence $u_0, u_1, \dots, u_{N-1}$, $u \in \mathbb{U}$, which minimizes the performance index (2) and satisfies the differential equation (1). The solution of this task by searching for a conditional minimum of the performance index (2) requires the solution of N equations with N unknown variables of the form

$$\frac{\partial J(u)}{\partial u_k} = 0 , \quad (k = 0, \dots, N-1) ,$$

where $J(u)$ is the performance index (2) after substituting (1) for $k = 1, 2, \dots, N-1$.

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B. Problem Solution

General solution for this problem and the procedure will be presented in [16]. Also in that paper obtained results were illustrated by numerical example. Here we present only the main results obtained by the method of dynamic programming. Identity of these results for discrete-time fractional systems with results for the classical systems of integer order has been also shown.

In the general case for q last sections of the optimal trajectory the value which minimizes performance index (2) with constraints (1) is given by the relation

$$\begin{aligned} S_{N-q}(\Sigma x_{N-q}) &= x_{N-q}^T Q x_{N-q} \\ &+ \sum_{l=0}^{q-2}\left\{\left[\sum_{j=0}^{N-q}\left(\sum_{p=0}^{q-1} V_{N-q}^{Q_{l,p+1}} d_{j+p}\right) x_{N-q-j}\right]^T Q \right. \\ &\times \left.\left[\sum_{j=0}^{N-q}\left(\sum_{p=0}^{q-1} V_{N-q}^{Q_{l,p+1}} d_{j+p}\right) x_{N-q-j}\right]\right\} \\ &+ \sum_{r=0}^{q-1}\left\{\left[\sum_{j=0}^{N-q}\left(\sum_{p=0}^{q-1} V_{N-q}^{R_{r,p+1}} d_{j+p}\right) x_{N-q-j}\right]^T R \right. \\ &\times \left.\left[\sum_{j=0}^{N-q}\left(\sum_{p=0}^{q-1} V_{N-q}^{R_{r,p+1}} d_{j+p}\right) x_{N-q-j}\right]\right\} \\ &+ \left[\sum_{j=0}^{N-q}\left(\sum_{p=0}^{q-1} V_{N-q}^{S_{p+1}} d_{j+p}\right) x_{N-q-j}\right]^T S \\ &\times \left[\sum_{j=0}^{N-q}\left(\sum_{p=0}^{q-1} V_{N-q}^{S_{p+1}} d_{j+p}\right) x_{N-q-j}\right]. \end{aligned} \tag{3}$$

where

$$V_{N-q}^{Q_{l0}} = 0\,, \quad V_{N-q}^{R_{r0}} = 0\,, \quad V_{N-q}^{S_0} = 0 \tag{4a}$$

$$\begin{aligned} V_{N-q}^{Q_{01}} &= I_n + BW_{N-q}^1\,, \; V_{N-q}^{Q_{0,p+1}} = BW_{N-q}^{p+1}\,, \\ V_{N-q}^{Q_{l,p+1}} &= \left(\sum_{v=0}^{q-2} V_{N-q+1}^{Q_{l-1,v+1}} d_v\right) V_{N-q}^{Q_{0,p+1}} + V_{N-q+1}^{Q_{l-1,p}}\,, \end{aligned} \tag{4b}$$

$$\begin{aligned} V_{N-q}^{R_{01}} &= W_{N-q}^1\,, \quad V_{N-q}^{R_{0,p+1}} = W_{N-q}^{p+1}\,, \\ V_{N-q}^{R_{r,p+1}} &= \left(\sum_{v=0}^{q-2} V_{N-q+1}^{R_{r-1,v+1}} d_v\right) V_{N-q}^{Q_{0,p+1}} + V_{N-q+1}^{R_{r-1,p}}\,, \end{aligned} \tag{4c}$$

$$\begin{aligned} V_{N-1}^{S_1} &= I_n + BW_{N-1}^1\,, \\ V_{N-q}^{S_{p+1}} &= \left(\sum_{v=0}^{q-2} V_{N-q+1}^{S_{v+1}} d_v\right) V_{N-q}^{Q_{0,p+1}} + V_{N-q+1}^{S_p}\,. \end{aligned} \tag{4d}$$

Control u_{N-q}, which minimizes the performance index $J_{N-q}(u)$, in the general case is given by the relation

$$u_{N-q} = \sum_{j=0}^{N-q}\left(\sum_{p=0}^{q-1} W_{N-q}^{p+1} d_{j+p}\right) x_{N-q-j}\,, \tag{5}$$

where

$$W_{N-1}^1 = -\left[R + R^T + B^T\left(S + S^T\right)B\right]^{-1} B^T\left(S + S^T\right), \tag{6a}$$

$$\begin{aligned} W_{N-q}^1 = &-\left\{R + R^T + B^T\left[Q + Q^T\right]B \right. \\ &+ \sum_{w=0}^{q-3}\left[\left(T_{1,w}^Q\right)^T\left[Q + Q^T\right]\left(T_{1,w}^Q\right)\right] \\ &+ \sum_{z=0}^{q-2}\left[\left(T_{1,z}^R\right)^T\left[R + R^T\right]\left(T_{1,z}^R\right)\right] \\ &+ \left.\left(T_1^S\right)^T\left[S + S^T\right]\left(T_1^S\right)\right\}^{-1} \\ &\times \left\{B^T\left[Q + Q^T\right]\right. \\ &+ \sum_{w=0}^{q-3}\left[\left(T_{1,w}^Q\right)^T\left[Q + Q^T\right]\left(T_{2,w}^Q\right)\right] \\ &+ \sum_{z=0}^{q-2}\left[\left(T_{1,z}^R\right)^T\left[R + R^T\right]\left(T_{2,z}^R\right)\right] \\ &+ \left.\left(T_1^S\right)^T\left[S + S^T\right]\left(T_2^S\right)\right\}, \end{aligned} \tag{6b}$$

$$\begin{aligned} W_{N-q}^{p+1} = &-\left\{R + R^T + B^T\left[Q + Q^T\right]B \right. \\ &+ \sum_{w=0}^{q-3}\left[\left(T_{1,w}^Q\right)^T\left[Q + Q^T\right]\left(T_{1,w}^Q\right)\right] \\ &+ \sum_{z=0}^{q-2}\left[\left(T_{1,z}^R\right)^T\left[R + R^T\right]\left(T_{1,z}^R\right)\right] \\ &+ \left.\left(T_1^S B\right)^T\left[S + S^T\right]\left(T_1^S\right)\right\}^{-1} \\ &\times \left\{\sum_{w=0}^{q-3}\left[\left(T_{1,w}^Q\right)^T\left[Q + Q^T\right]\left(V_{N-q+1}^{Q_{w,p}}\right)\right]\right. \\ &+ \sum_{z=0}^{q-2}\left[\left(T_{1,z}^R\right)^T\left[R + R^T\right]\left(V_{N-q+1}^{R_{z,p}}\right)\right] \\ &+ \left.\left(T_1^S\right)^T\left[S + S^T\right]\left(V_{N-q+1}^{S_p}\right)\right\}, \end{aligned} \tag{6c}$$

and

$$T_{1,w}^Q = \left(\sum_{v=0}^{q-2} V_{N-q+1}^{Q_{w,v+1}} d_v B\right), \tag{7a}$$

$$T_{1,z}^R = \left(\sum_{v=0}^{q-2} V_{N-q+1}^{R_{z,v+1}} d_v B\right), \tag{7b}$$

$$T_1^S = \left(\sum_{v=0}^{q-2} V_{N-q+1}^{S_{v+1}} d_v B\right), \tag{7c}$$

$$T_{2,w}^Q = \left(\sum_{v=0}^{q-2} V_{N-q+1}^{Q_{w,v+1}} d_v\right), \tag{7d}$$

$$T_{2,z}^R = \left(\sum_{v=0}^{q-2} V_{N-q+1}^{R_{z,v+1}} d_v\right), \tag{7e}$$

$$T_2^S = \left(\sum_{v=0}^{q-2} V_{N-q+1}^{S_{v+1}} d_v\right). \tag{7f}$$

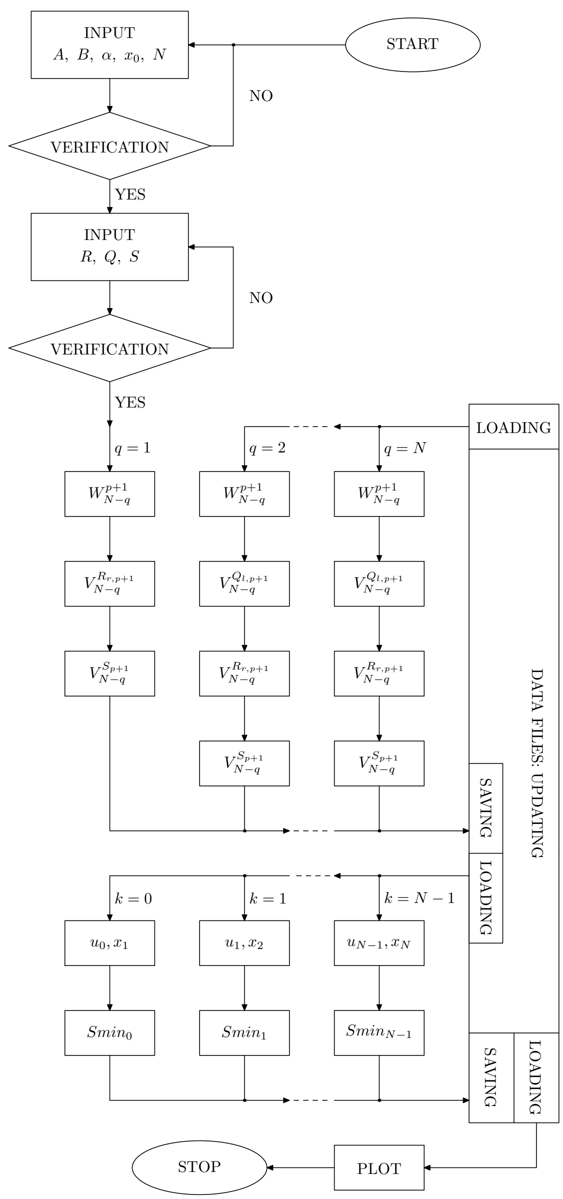

Fig. 1. Block diagram of a computer algorithm based on dynamic programming method for solving optimization problems of discrete-time fractional systems.

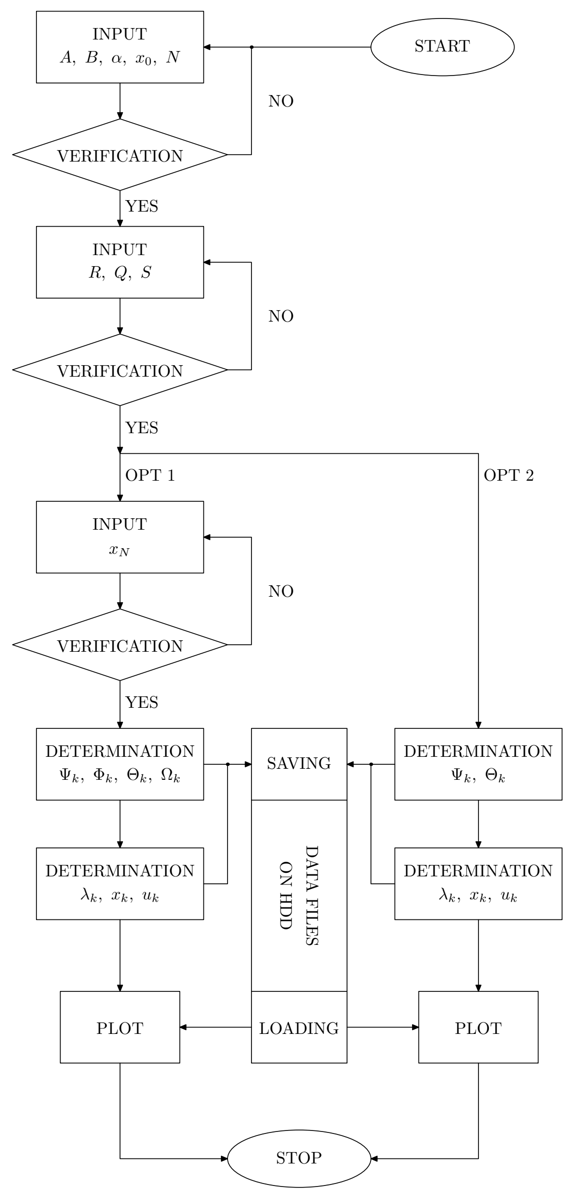

Fig. 2. Block diagram of a computer algorithm based on discrete-time calculus of variations method for solving optimization problems of discrete-time fractional systems.

Figure 1 shows a block diagram of a computer algorithm in the programming environment Matlab/Simulink implementing the above issues.

III. DISCRETE-TIME CALCULUS OF VARIATIONS

A. Problem Formulation

Lets consider again fractional discrete-time system (1). We assume that the initial value x_0 of the state vector in discrete time $k = 0$ (initial conditions) is given. The number of discrete moments $N \in \mathbb{Z}_+$, for which the final value of the state vector at discrete time $k = N$, ie $x(k = N) = x_N$ (final conditions) is also determined.

Consider a performance index of the form (2). Using the Lagrange multiplier theory we write (2) in the extended form as

$$J(u) = x_N^T S x_N + \sum_{k=i}^{N-1} \Bigg(x_k^T Q x_k + u_k^T R u_k + \left[\sum_{j=0}^{k} d_j x_{k-j} + B u_k - x_{k+1} \right]^T \lambda_{k+1} \Bigg) . \tag{8}$$

We define a scalar function $\mathcal{H}$, called the Hamiltonian, which is defined as follows

$$\mathcal{H}_k = x_k^T Q x_k + u_k^T R u_k + \left[\sum_{j=0}^{k} d_j x_{k-j} + B u_k \right]^T \lambda_{k+1} . \tag{9}$$

Using (9) and (8), we define a new performance index expressed by Hamiltonian, of the form

$$J(u) = x_N^T S x_N + \sum_{k=i}^{N-1} \left(\mathcal{H}_k - x_{k+1}^T \lambda_{k+1}\right) . \quad (10)$$

To the above equation, on the right side, add and subtract relationship $x_i^T \lambda_i$. By making a changes to indexes in the second part of the sum we get a performance index of the form

$$J(u) = x_N^T S x_N + x_i^T \lambda_i - x_N^T \lambda_N + \sum_{k=i}^{N-1} \left(\mathcal{H}_k - x_k^T \lambda_k\right) . \quad (11)$$

B. Problem Solution

General solution of this problem and the procedure in case of fixed final state and time has been presented in [17]. General solution of this problem and the procedure in case of free final state and fixed final time will be presented in [18]. Also in these papers obtained results were illustrated by numerical examples. Here we present only the main results obtained by the method of discrete-time calculus of variations. Identity of these results for discrete-time fractional systems with results for the classical systems of integer order has been also shown.

We shall now examine the increment of the performance index J due to the increments in all the variables x_k, u_k and λ_k. The increment of the performance index we write as follows

$$\begin{aligned} dJ(u) &= [(S+S^T)x_N - \lambda_N]dx_N^T + \lambda_i dx_i^T \\ &+ \sum_{k=i}^{N-1} \left[(\mathcal{H}_k - \lambda_k)\, dx_k^T + \mathcal{H}_k du_k^T \right. \\ &+ \left. (\mathcal{H}_{k-1} - x_k)\, d\lambda_k^T\right] . \end{aligned} \quad (12)$$

According to the Lagrange multiplier theory, at a constrained minimum this increment should be zero. Necessary conditions for a constrained minimum are given by

$$0 = \frac{\partial \mathcal{H}_k}{\partial u_k^T} \quad \text{dla} \quad k = i, \ldots, N-1 , \quad (13a)$$

$$\lambda_k = \sum_{k=i}^{N-1} \frac{\partial \mathcal{H}_k}{\partial x_k^T} \quad \text{dla} \quad k = i, \ldots, N-1 , \quad (13b)$$

$$x_{k+1} = \frac{\partial \mathcal{H}_k}{\partial \lambda_{k+1}^T} \quad \text{dla} \quad k = i, \ldots, N-1 , \quad (13c)$$

and

$$\lambda_N = \frac{\partial S(x_N, N)}{\partial x_N^T} , \quad (13d)$$

$$\lambda_i \in \mathbb{R} . \quad (13e)$$

Assuming $i = 0$, the conditions (13) for considered the performance index (11) and discrete-time fractional system (1) take the form

$$u_k = -\left[R + R^T\right]^{-1} B^T \lambda_{k+1} , \quad (14a)$$

$$\lambda_k = \left[Q + Q^T\right] x_k + \sum_{j=0}^{N-k-1} d_j^T \lambda_{k+j+1} , \quad (14b)$$

$$x_{k+1} = \sum_{j=0}^{k} d_j x_{k-j} + B u_k , \quad (14c)$$

and

$$\lambda_N = (S + S^T) x_N , \quad (14d)$$

$$\lambda_0 \in \mathbb{R} . \quad (14e)$$

Figure 2 shows a block diagram of a computer algorithm in the programming environment Matlab/Simulink implementing the above issues for both cases of final state.

IV. COMPUTER ALGORITHMS AND EXAMPLES

A. Computer algorithms

Computer algorithms are implemented in computer programming environment Matlab/Simulink. Using them we can obtain the numerical values for the above optimization problems for discrete-time fractional systems. Block diagram of a computer algorithm which uses dynamic programming method is shown in Figure 1. It executes an optimization problem for free final state and fixed final time. Block diagram of a computer algorithm which uses the method of discrete calculus of variations is shown in Figure 2. It executes optimization problem for two cases: the fixed (OPT1) or free (OPT2) final state and fixed final time.

In the algorithm was implemented security code to prevent the user to input incorrect data in areas such as: incorrect dimensions of the matrices of the discrete-time fractional system, the performance index and the initial state vector; not an integer or a negative value at the final time; negative value of the order of discrete-time fractional system; In case of detection any incorrectness while inputting data, the user will be asked to correct them or interrupt the work of the algorithm by the use of key combination.

B. Numerical example

Example 1: Consider a discrete-time fractional system (1) with matrices

$$A = \begin{bmatrix} 0.1 & 0.7 \\ 0.6 & 0.4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}, \quad (15)$$

and a performance index (2) with matrices

$$S = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad Q = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}, \quad R = \begin{bmatrix} 1 \end{bmatrix}. \quad (16)$$

To simplify calculations we assume that initialization functions are equal zero. We assume that the number of discrete time elementary sections are given as $N = 5$. We will consider four different cases of fractional order $\alpha = 0.5, 0.7, 0.9, 1.0$.

Using a computer algorithm for dynamic programming method we can obtain graphs of optimal trajectory (Fig. 3), optimal control (Fig. 4) and minimum values of performance index (Fig. 5). In tables I–III we present exact values of obtained graphs. In tables IV–VI we present the differences between the results obtained by using of both computer algorithms.

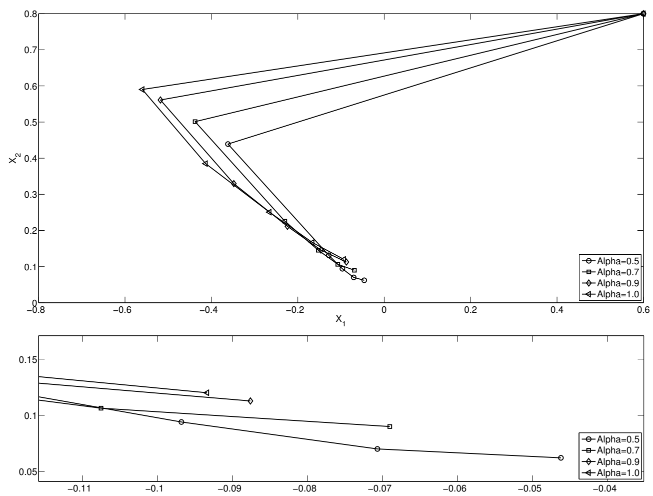

Fig. 3. Optimal trajectory for $\alpha = 0.5, 0.7, 0.9, 1.0$ and $N = 5$.

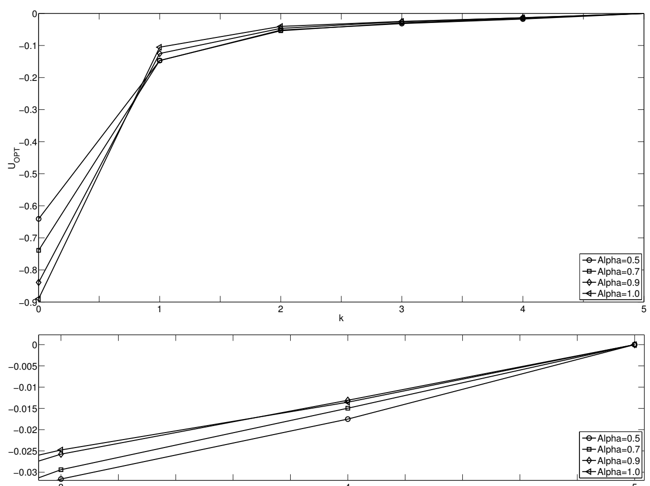

Fig. 4. Optimal control for $\alpha = 0.5, 0.7, 0.9, 1.0$ and $N = 5$.

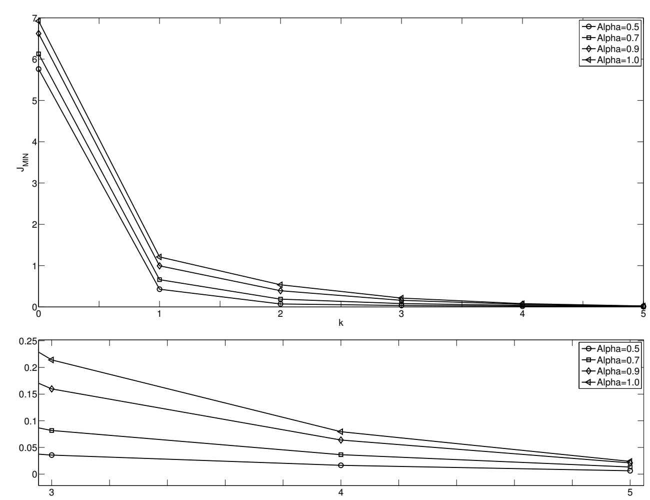

Fig. 5. The minimum values of the performance index for $\alpha = 0.5, 0.7, 0.9, 1.0$ and $N = 5$.

TABLE I

VALUES OF OPTIMAL TRAJECTORIES FROM FIG. 3.

k	$\alpha = 0.5$		$\alpha = 0.7$	
	x_{k1}	x_{k2}	x_{k1}	x_{k2}
0	0.6000	0.8000	0.6000	0.8000
1	−0.3617	0.4392	−0.4378	0.5011
2	−0.1284	0.1313	−0.2301	0.2257
3	−0.0968	0.0941	−0.1526	0.1454
4	−0.0707	0.0701	−0.1075	0.1064
5	−0.0462	0.0622	−0.0690	0.0900
k	$\alpha = 0.9$		$\alpha = 1.0$	
	x_{k1}	x_{k2}	x_{k1}	x_{k2}
0	0.6000	0.8000	0.6000	0.8000
1	−0.5179	0.5610	−0.5598	0.5901
2	−0.3481	0.3296	−0.4133	0.3849
3	−0.2239	0.2115	−0.2657	0.2507
4	−0.1464	0.1458	−0.1664	0.1668
5	−0.0876	0.1128	−0.0934	0.1201

TABLE II

VALUES OF OPTIMAL CONTROLS FROM FIG. 4.

k	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1.0$
0	−0.6408	−0.7389	−0.8390	−0.8899
1	−0.1469	−0.1469	−0.1250	−0.1053
2	−0.0520	−0.0539	−0.0466	−0.0403
3	−0.0316	−0.0294	−0.0258	−0.0248
4	−0.0175	−0.0149	−0.0131	−0.0135
5	0.0000	0.0000	0.0000	0.0000

TABLE III

THE MINIMUM VALUES OF THE PERFORMANCE INDEX FROM FIG. 5.

k	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1.0$
0	5.7601	6.1269	6.6187	6.9227
1	0.4294	0.6610	0.9948	1.2108
2	0.0722	0.1886	0.3925	0.5363
3	0.0357	0.0818	0.1598	0.2141
4	0.0165	0.0364	0.0639	0.0795
5	0.0063	0.0133	0.0210	0.0239

TABLE IV

THE DIFFERENCES BETWEEN THE OBTAINED OPTIMAL TRAJECTORIES (FIG. 3).

k	Δx_k	
	$\alpha = 0.5$	$\alpha = 0.7$
0	0.00000e − 00	0.00000e − 00
1	9.99201e − 16	1.41308e − 15
2	9.99201e − 16	2.23428e − 15
3	3.60267e − 15	2.23428e − 15
4	2.23428e − 15	3.99681e − 15
5	3.60845e − 15	1.61237e − 14
k	$\alpha = 0.9$	$\alpha = 1.0$
0	0.00000e − 00	0.00000e − 00
1	9.99201e − 16	2.82617e − 15
2	9.99201e − 16	7.06542e − 15
3	4.99601e − 15	1.01899e − 14
4	2.23428e − 15	2.24542e − 14
5	3.48198e − 14	1.20734e − 14

TABLE V

THE DIFFERENCES BETWEEN THE OBTAINED OPTIMAL CONTROLS (FIG. 4).

k	ΔU_{opt} $\alpha = 0.5$	$\alpha = 0.7$
0	$1.09912e-14$	$6.99441e-14$
1	$7.99361e-15$	$3.99958e-14$
2	$7.00134e-15$	$2.39947e-14$
3	$0.00000e-00$	$7.00134e-15$
4	$1.99841e-15$	$1.40009e-14$
5	$0.00000e-00$	$0.00000e-00$
k	$\alpha = 0.9$	$\alpha = 1.0$
0	$3.49720e-14$	$1.37002e-13$
1	$1.59872e-14$	$7.90063e-14$
2	$8.99975e-15$	$4.30003e-14$
3	$2.00187e-15$	$4.99947e-15$
4	$5.99867e-15$	$1.99996e-14$
5	$0.00000e-00$	$0.00000e-00$

TABLE VI

THE DIFFERENCES BETWEEN THE OBTAINED MINIMUM VALUES OF PERFORMANCE INDEX (FIG. 5).

k	ΔJ_{min} $\alpha = 0.5$	$\alpha = 0.7$
0	$1.68754e-14$	$1.20792e-13$
1	$9.99201e-16$	$1.99840e-14$
2	$1.99840e-15$	$5.99520e-15$
3	$1.99840e-15$	$4.99601e-15$
4	$1.00267e-15$	$2.99760e-15$
5	$9.99201e-16$	$4.00027e-15$
k	$\alpha = 0.9$	$\alpha = 1.0$
0	$6.83897e-14$	$2.57572e-13$
1	$8.99281e-15$	$1.70974e-14$
2	$6.99441e-15$	$4.99601e-15$
3	$5.99521e-15$	$0.00000e-00$
4	$1.00059e-14$	$6.00908e-15$
5	$1.09982e-14$	$4.00027e-15$

V. CONCLUSIONS

Dynamical programming and discrete-time calculus of variations problems for fractional discrete-time systems with quadratic performance index has been formulated and solved. A new method for numerical computation of optimization problems has been presented. The efficiency of the method has been demonstrated on numerical example and illustrated by graphs. The differences between the fractional and classical (standard) systems theory have been shown. A computer algorithm for solving and discrete-time calculus of variations problems with quadratic performance index for fractional discrete-time systems has been tested for different cases of coefficient alpha. Block diagrams of computer algorithms have been shown.

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