

On-line Switching Signal Estimation of Switched Linear Systems with Measurement Noise

Chanhwa Lee, Zhaowu Ping, and Hyungbo Shim

Abstract—In this paper, we study a switching signal estimation problem for continuous time switched linear control systems with measurement noise. Inspired by the work of [8], we first propose a generalized minimum distance criterion to estimate the active mode of the plant with inputs. Then we propose an implementable on-line robust switching signal estimation algorithm to detect the switching time with guaranteed precision, where some update condition and threshold condition are checked all the time. Under the update and threshold conditions, we detect the switching time within a predetermined time interval after the switching occurred.

I. INTRODUCTION

Estimation problem for switched linear systems has attracted much attention over the past decade, and depending on the cases, various observability notions and observer design methods are already available in the literature. In particular, when the switching signal (or, the mode) is known or can be measured while the states of the system are to be estimated, observability has been studied in various perspectives such as in [12], [15], [20], [21], and an asymptotic observer is constructed for quite a general class of switched systems. For example, the observer proposed in [15] can estimate the states even when the system switches to unobservable subsystems as long as a certain accumulative observability holds for a certain time window. With the knowledge of switching signals, observability and observer design have also been extended to the nonlinear case in [14].

On the other hand, when the switching signal is not measurable and unknown, the estimation problem of both the switching signal and the states becomes more challenging, and many results are available under certain restrictions. In the case of discrete-time switched linear systems, [2], [4], [5], and [18] have studied observability, and [1] and [9] have proposed estimation algorithms in various situations depending on the existence of external inputs, process disturbance, and/or measurement noise. For continuous-time switched linear systems, [3], [11], and [19] have characterized observability for the state and the mode in various situations depending on the existence of external inputs and/or process disturbance. In particular, [19] systematically presented various observability conditions for switched linear systems without inputs by a simple rank test. In addition, for the estimation problem of the state and the mode, [13]

and [16] have considered the estimation of the switching signal for unforced systems by the distribution theory, but the states are assumed to be known. In particular, [16] only investigated two-mode cases and two system matrices are assumed to be square commuting. The approach in [16] was further extended to more general cases in [17] where the plant consists of more than two subsystems and only partial state is known. For switched systems with inputs, [10] have characterized singular inputs and presented an estimation algorithm for the switching signal and the states based on ‘strong distinguishable’ assumption, but their result is limited to single-input-single-output case and relies on numerical estimation of the time derivatives of inputs and outputs. [6] and [7] have proposed a hybrid observer to estimate both the switching signal and the states, which is composed of a location observer and a continuous (state) observer. The location observer together with a decision function was used to detect the mode switching with short delay. However, the mode is not uniquely determined in [6] due to the existence of singular inputs, and no systematic construction of the location observer is given in [7].

In this paper, we propose a hybrid-type observer to estimate the switching signal for continuous-time switched linear systems. Emphasis is given to generality and implementability. In fact, the system may have external inputs, and/or measurement noise. Because of them, immediate detection of switching time is not possible, and some delay of detection is unavoidable. The condition that the proposed design is based upon characterizes the relationship among the amount of delay, the strength of the joint observability, and the sizes of state and noise. The proposed design is summarized as follows. When there is no mode switching in a δ time interval, where δ is the minimum delay for mode estimation, we estimate the plant’s mode exactly and the state approximately by an ‘estimation algorithm’. This estimation algorithm is inspired by the ‘minimum distance criterion’ proposed by [8]. When the algorithm finishes, we run another ‘detection algorithm’ to detect the mode switching within Δ delay, where Δ is our desired accuracy of detection. Compared with [6] and [7], our results are more general because the measurement noise is considered. Moreover, our results ensure the unique mode estimation and present a systematic construction of the location observer.

The organization of this paper is as follows. Section II introduces the problem formulation and assumptions which are used in the following sections. Section III introduces how to estimate the active mode and discusses the implementation problem. Section IV introduces how to detect the switching

This work was supported by National Research Foundation Grant funded by the Korean Government (MEST) via (NRF-2011- 220-D00043) and (No. 2012-0000921).

C. Lee, Z. Ping, and H. Shim are with ASRI, School of Electrical Engineering and Computer Science, Seoul National University, Korea. chlee@cdsl.kr, zwping@cdsl.kr, hshim@snu.ac.kr

time on-line with Δ precision. Section V summarizes the operation of the whole switching signal estimation algorithm, and some concluding remarks and the future works are given in Section VI.

II. PRELIMINARIES

We consider a switched linear system given by

$$\begin{aligned} \dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) &= C_{\sigma(t)}x(t) + v(t) \end{aligned} \quad (1)$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state, $u(t) \in \mathbb{R}^{n_u}$ is the input, $y(t) \in \mathbb{R}^{n_y}$ is the measured output, $v(t) \in \mathbb{R}^{n_y}$ is the measurement noise, and $\sigma : [0, \infty) \mapsto \mathcal{N}$ is the switching signal with $\mathcal{N} := \{1, 2, \dots, N\}$. It is supposed that both the state $x(t)$ and the switching signal $\sigma(t)$ are unknown while $\sigma(\cdot)$ is assumed to be piecewise constant and right continuous. We assume that all subsystems of (1) be observable, i.e., the pair (A_i, C_i) , $\forall i \in \mathcal{N}$, is observable. Moreover, we assume a stronger notion of joint observability given as follows.

Assumption 1: For any $i, j \in \mathcal{N}$ such that $i \neq j$,

$$\text{rank} \begin{bmatrix} O_i^{(2n_x)} & O_j^{(2n_x)} \end{bmatrix} = 2n_x$$

where $O_i^{(k)} = [C_i^\top (C_i A_i)^\top \dots (C_i A_i^{k-1})^\top]^\top$. \diamond

Assumption 1 is equivalent to the fact that the extended system, whose system matrix is $\text{blockdiag}(A_i, A_j)$ and its output matrix is $[C_i, C_j]$, is observable for every $i \neq j$ [19]. It is also equivalently said that the joint observability Gramian

$$W_{i,j}(t) := \int_0^t \begin{bmatrix} \phi_i^\top(\tau) \\ \phi_j^\top(\tau) \end{bmatrix} \begin{bmatrix} \phi_i(\tau) & \phi_j(\tau) \end{bmatrix} d\tau$$

where $\phi_i(t) := C_i e^{A_i t}$, is nonsingular for any $t \neq 0$ and $i \neq j$. Therefore, with

$$\omega_{\min}(t) := \min_{i,j \in \mathcal{N}, i \neq j} \lambda_{\min}(W_{i,j}(t))$$

where λ_{\min} indicates the minimum eigenvalue, it is seen that $\omega_{\min}(t) > 0$ for $t > 0$.

Remark 1: Joint observability is necessary for mode detection in the following sense. Let $y_i(t; t_0, x_0, u)$ be the output at time t , which is generated from the initial condition x_0 at time t_0 under the input function $u(\cdot)$ and no measurement noise, for the mode i . Then it can be shown that, for two plant modes $i, j \in \mathcal{N}$ and zero input,

$$\begin{aligned} & \int_0^t |y_i(\tau; 0, x_0, 0) - y_j(\tau; 0, x'_0, 0)|^2 d\tau \\ &= \begin{bmatrix} x_0^\top & -x'_0{}^\top \end{bmatrix} W_{i,j}(t) \begin{bmatrix} x_0 \\ -x'_0 \end{bmatrix}. \end{aligned}$$

It is clear that, if $\omega_{\min}(t) = 0$ with $t > 0$, then there are x_0 and x'_0 (not both zero), and two different modes i and j such that $y_i(\tau; 0, x_0, 0) = y_j(\tau; 0, x'_0, 0)$ for $0 \leq \tau \leq t$. Hence, the modes i and j cannot be distinguished from the output. \diamond

In addition, we present a condition which will enable the estimation of the switching signal even under measurement noise and external input. For this, let $\check{\mu} \geq 1$ and $\check{\lambda} \geq 0$ be

such that $\|e^{A_i t}\| \leq \check{\mu} e^{\check{\lambda} t}$, $\forall t \geq 0$, $\forall i \in \mathcal{N}$. In addition, let L_i be such that $(A_i - L_i C_i)$ is Hurwitz. Let $\hat{\mu} \geq 1$ and $\hat{\lambda} > 0$ be such that $\|e^{(A_i - L_i C_i)t}\| \leq \hat{\mu} e^{-\hat{\lambda} t}$, $\forall t \geq 0$, $\forall i \in \mathcal{N}$. Define $L_{\max} := \max_{i \in \mathcal{N}} \|L_i\|$ and $C_{\max} := \max_{i \in \mathcal{N}} \|C_i\|$.

Assumption 2: The input u and the measurement noise v are uniformly bounded, i.e.,

$$|u(t)| \leq u_{\max}, |v(t)| \leq v_{\max}, \forall t \geq 0. \quad (2)$$

Moreover, there exist positive constants δ and Δ such that

$$\omega_{\min}(\delta) |x(t)|^2 > \left(u_{\max} \sqrt{N_u(\delta)} + 2v_{\max} \sqrt{\delta} \right)^2, \quad (3)$$

$$\begin{aligned} \omega_{\min}(\Delta) |x(t)|^2 &> \frac{1}{2} \left(u_{\max} \sqrt{N_u(\Delta)} + 2v_{\max} \sqrt{\Delta} \right. \\ &\quad \left. + 2v_{\max} C_{\max} (\check{\mu} e^{\check{\lambda} \Delta} + 1)^2 \hat{E}(\delta) \sqrt{\Delta} \right)^2, \end{aligned} \quad (4)$$

for all $t \geq 0$ where, with $h_i(t) := C_i e^{A_i t} B_i$ and $U_i(t) := \int_0^t \phi_i^\top(\tau) \phi_i(\tau) d\tau$ (observability Gramian for mode i),

$$N_u(\delta) := \max_{i,j \in \mathcal{N}, i \neq j} \int_0^\delta \left(\int_0^\tau \|h_i(s) - h_j(s)\| ds \right)^2 d\tau,$$

$$M_{\max}(\delta) := \max_{i \in \mathcal{N}} M_i(\delta) := \max_{i \in \mathcal{N}} \int_0^\delta \|U_i^{-1}(\delta) \phi_i^\top(\tau)\| d\tau,$$

$$\hat{E}(\delta) := \hat{\mu} \max \left\{ M_{\max}(\delta), \frac{L_{\max}}{\hat{\lambda}} \right\}.$$

\diamond

A few comments should be added to Assumption 2. When there is neither measurement noise nor input (so that $v_{\max} = u_{\max} = 0$), the conditions (3) and (4) simply become $|x(t)| > 0$ because $\omega_{\min}(t) > 0$ for any $t > 0$. Then this condition is necessary because, if $|x(t)| = 0$, then $x(t) = 0$ and $y(t) = 0$ for all time for any mode $i \in \mathcal{N}$, so that the estimation of mode is not possible. Similarly, when the measurement noise is present, the state norm $|x(t)|$ should not be too small because a certain norm-bounded noise $v(t)$ may make the output $y(t) = C_{\sigma(t)}x(t) + v(t)$ identically zero while $x(t)$ is not. In this case, the estimation of mode is not possible either. A similar observation can be made when the input u is present. Consider a two-mode switched system $\Sigma_1 : \dot{x} = -x + u$, $y = x$ and $\Sigma_2 : \dot{x} = -2x + 2u$, $y = x$ which satisfies Assumption 1. With $u(t) \equiv 1$ and $x(0) = 1$, the output $y(t) = 1$, $\forall t \geq 0$, for any mode switching, which disables the estimation of mode. Thus, the conditions (3) and (4) can be understood as a way to avoid this pathological case. Moreover, these conditions somehow illustrate the relationship among many factors for the detection and the estimation to become easier, in the sense that the condition is more likely to hold if $|x(t)|$ gets larger, if u_{\max} and v_{\max} get smaller, or if $\omega_{\min}(\cdot)$ gets larger (i.e., joint observability gets stronger).

It will be clarified that δ of (3) indicates the time required to estimate the active mode correctly in Section III, and that Δ of (4) represents the maximum delay to detect the switching time in Section IV. With them, a dwell time condition is given as follows.

Assumption 3: The switching signal $\sigma(t)$ has the dwell time of $\Delta + \delta$, and there is no switching for the initial time interval $[0, \delta]$. \diamond

Finally, for convenience, we also assume that the computation is fast enough, so that the time for numerical integration and calculation is ignored in this paper.

III. ESTIMATION OF ACTIVE MODE

Based on the assumptions in Section II, the idea of estimating the active mode can be briefly described as follows. From now on, $\hat{\sigma}(t)$ is the estimate of the mode $\sigma(t)$, and \hat{t}_j denotes the estimated j -th switching time. For convenience, we let $\hat{t}_0 = 0$. Suppose that, at time $\hat{t}_j + \delta$, the input and output signals (i.e., $u(\tau)$ and $y(\tau)$) for the past time interval $[\hat{t}_j, \hat{t}_j + \delta]$ (with $\delta > 0$) are available, and that no switching has occurred during the interval $[\hat{t}_j, \hat{t}_j + \delta]$. Let the estimation of the mode at time $\hat{t}_j + \delta$ be given by

$$\hat{\sigma}(\hat{t}_j + \delta) = \arg \min_{i \in \mathcal{N}} J_i(X_i^*) \quad (5)$$

where

$$J_i(X_i) := \int_{\hat{t}_j}^{\hat{t}_j + \delta} |y(\tau) - y_i(\tau; \hat{t}_j, X_i, u)|^2 d\tau, \quad (6)$$

with

$$X_i^* = U_i^{-1}(\delta) \int_{\hat{t}_j}^{\hat{t}_j + \delta} \phi_i^\top(s - \hat{t}_j) \times \left(y(s) - \int_{\hat{t}_j}^s h_i(s - \tau) u(\tau) d\tau \right) ds. \quad (7)$$

By differentiating $J_i(X_i)$ with respect to X_i , it is seen that the above X_i^* minimizes $J_i(X_i)$. With $\hat{\sigma}(\hat{t}_j + \delta)$ at hand, the state estimate $\hat{x}(\hat{t}_j)$ is obtained at time $\hat{t}_j + \delta$ by

$$\hat{x}(\hat{t}_j) = X_{\hat{\sigma}(\hat{t}_j + \delta)}^*. \quad (8)$$

Then the following theorem asserts that the active mode is exactly recovered and the initial state can be approximately estimated.

Theorem 1: Under Assumptions 1 and 2, if there is no switching from \hat{t}_j to $\hat{t}_j + \delta$, then the estimates by (5) and (8) have the property that

$$\hat{\sigma}(t) = \sigma(t) \quad (9)$$

$$\hat{x}(\hat{t}_j) = x(\hat{t}_j) + U_{\hat{\sigma}}^{-1}(\delta) \int_{\hat{t}_j}^{\hat{t}_j + \delta} \phi_{\hat{\sigma}}^\top(s - \hat{t}_j) v(s) ds \quad (10)$$

for all $t \geq \hat{t}_j + \delta$ until the next mode switching occurs. \diamond

Proof: This proof is mostly inspired by the ‘minimum distance criterion,’ proposed in [8]. First, because $y(s) = \phi_\sigma(s - \hat{t}_j)x(\hat{t}_j) + \int_{\hat{t}_j}^s h_\sigma(s - \tau)u(\tau)d\tau + v(s)$, it follows from (7) that

$$X_\sigma^* = x(\hat{t}_j) + U_\sigma^{-1}(\delta) \int_{\hat{t}_j}^{\hat{t}_j + \delta} \phi_\sigma^\top(s - \hat{t}_j)v(s)ds. \quad (11)$$

On the other hand, by a routine computation with $w_\sigma(\hat{t}_j) := U_\sigma^{-1}(\delta) \int_{\hat{t}_j}^{\hat{t}_j + \delta} \phi_\sigma^\top(s - \hat{t}_j)v(s)ds$, it can be shown that

$$\begin{aligned} & \sqrt{J_\sigma(X_\sigma^*)} \\ &= \left(\int_{\hat{t}_j}^{\hat{t}_j + \delta} |y_\sigma(t; \hat{t}_j, x(\hat{t}_j), u) + v(t) \right. \\ & \quad \left. - y_\sigma(t; \hat{t}_j, x(\hat{t}_j) + w_\sigma(\hat{t}_j), u)|^2 dt \right)^{\frac{1}{2}} \\ &= \left(\int_{\hat{t}_j}^{\hat{t}_j + \delta} |v(t) - y_\sigma(t; \hat{t}_j, w_\sigma(\hat{t}_j), 0)|^2 dt \right)^{\frac{1}{2}} \\ &= \left(\int_{\hat{t}_j}^{\hat{t}_j + \delta} v^\top(t)v(t)dt - w_\sigma^\top(\hat{t}_j)U_\sigma(\delta)w_\sigma(\hat{t}_j) \right)^{\frac{1}{2}} \\ &\leq \left(\int_{\hat{t}_j}^{\hat{t}_j + \delta} v^\top(t)v(t)dt \right)^{\frac{1}{2}} \leq v_{\max}\sqrt{\delta} \end{aligned}$$

and

$$\begin{aligned} & \sqrt{J_i(X_i^*)} \\ &= \left(\int_{\hat{t}_j}^{\hat{t}_j + \delta} |y_\sigma(t; \hat{t}_j, x(\hat{t}_j), u) + v(t) - y_i(t; \hat{t}_j, X_i^*, u)|^2 dt \right)^{\frac{1}{2}} \\ &= \left(\int_{\hat{t}_j}^{\hat{t}_j + \delta} | (y_\sigma(t; \hat{t}_j, x(\hat{t}_j), 0) - y_i(t; \hat{t}_j, X_i^*, 0)) \right. \\ & \quad \left. + (y_\sigma(t; \hat{t}_j, 0, u) - y_i(t; \hat{t}_j, 0, u)) + v(t) |^2 dt \right)^{\frac{1}{2}} \\ &\geq (\lambda_{\min}(W_{\sigma,i}(\delta))|x(\hat{t}_j)|^2)^{\frac{1}{2}} \\ & \quad - \left(\int_{\hat{t}_j}^{\hat{t}_j + \delta} \left| \int_{\hat{t}_j}^t (h_\sigma(t - \tau) - h_i(t - \tau))u(\tau)d\tau \right|^2 dt \right)^{\frac{1}{2}} \\ & \quad - \left(\int_{\hat{t}_j}^{\hat{t}_j + \delta} v^\top(t)v(t)dt \right)^{\frac{1}{2}} \\ &\geq \sqrt{\omega_{\min}(\delta)}|x(\hat{t}_j)| - u_{\max}\sqrt{N_u(\delta)} - v_{\max}\sqrt{\delta} \end{aligned}$$

for all $i \neq \sigma$. Then, by (3), we obtain that $J_i(X_i^*) > J_\sigma(X_\sigma^*)$ for all $i \neq \sigma$, which ensures that $\hat{\sigma}(t) = \sigma(t)$ in (9). Finally, (10) is easily verified from (9) and (11). \blacksquare

Remark 2: The estimation given by (5) and (8) can be implemented on-line as follows. First, compute $U_i(\delta)$ a priori for each $i \in \mathcal{N}$ with δ satisfying (3). Second, start integrating the following systems (12) and (13) at $t = \hat{t}_j$ for all $i \in \mathcal{N}$, with the initial conditions $\tilde{x}_i(\hat{t}_j) = 0$ and $z_i(\hat{t}_j) = 0$, which continues until $t = \hat{t}_j + \delta$ real time:

$$\dot{\tilde{x}}_i = A_i \tilde{x}_i + B_i u \quad (12)$$

$$\dot{z}_i = -A_i^\top z_i + C_i^\top (y - C_i \tilde{x}_i). \quad (13)$$

Then we obtain the estimate X_i^* of the time \hat{t}_j by

$$X_i^* = U_i^{-1}(\delta)e^{A_i^\top \delta} z_i(\hat{t}_j + \delta). \quad (14)$$

Third, to compute $J_i(X_i^*)$, simply run the following systems (15) and (16) for the interval $[\hat{t}_j, \hat{t}_j + \delta]$ and for all $i \in \mathcal{N}$ in computer, with the initial condition $\tilde{x}_i(\hat{t}_j) = X_i^*$, $s_i(\hat{t}_j) = 0$:

$$\dot{\tilde{x}}_i = A_i \tilde{x}_i + B_i u \quad (15)$$

$$\dot{s}_i = |y - C_i \tilde{x}_i|^2. \quad (16)$$

(Here we suppose that the computation process is performed instantly at time $\hat{t}_j + \delta$, with the stored data of $u(\tau)$ and $y(\tau)$ on the interval $[\hat{t}_j, \hat{t}_j + \delta]$.) Then we have

$$J_i(X_i^*) = s_i(\hat{t}_j + \delta). \quad (17)$$

Choose i such that $J_i(X_i^*)$ is the minimum for $i \in \mathcal{N}$, and let $\hat{\sigma}(\hat{t}_j + \delta) = i$ and $\hat{x}(\hat{t}_j) = X_i^*$. \diamond

IV. DETECTION OF SWITCHING TIME

By the algorithm of the previous section, the mode and the state are estimated if the mode switching does not occur for a time interval of the length δ . In this section, we present an on-line algorithm that detects the mode switching within Δ time after the system is switched to another mode so that the precision of the detection is less than or equal to Δ , where the value of Δ is determined by (4) in Assumption 2. Suppose that, at time $\hat{t}_j + \delta$, the correct mode estimate $\hat{\sigma}(\hat{t}_j + \delta)$ and an approximate state estimate $\hat{x}(\hat{t}_j)$ are obtained through the algorithm of the previous section. With them at time $\hat{t}_j + \delta$, call the following Detection Algorithm which monitors the occurrence of mode switching. From now on, define $S_\Delta := v_{\max}^2(1 + C_{\max}(\tilde{\mu}e^{\lambda\Delta} + 2)\hat{E}(\delta))^2\Delta$, and denote $\bar{t}_j := \hat{t}_j + \delta$.

Detection Algorithm

Require: $\tilde{x}(\hat{t}_j) \leftarrow \hat{x}(\hat{t}_j)$, $s(\hat{t}_j) \leftarrow 0$, $\hat{\sigma}_j \leftarrow \hat{\sigma}(\hat{t}_j + \delta)$, $t_{\text{up}} \leftarrow \hat{t}_j$

1: run the following systems, using stored $u(\tau)$ and $y(\tau)$,

$$\dot{\tilde{x}} = A_{\hat{\sigma}_j} \tilde{x} + B_{\hat{\sigma}_j} u \quad (18)$$

$$\dot{\hat{x}} = (A_{\hat{\sigma}_j} - L_{\hat{\sigma}_j} C_{\hat{\sigma}_j}) \hat{x} + B_{\hat{\sigma}_j} u + L_{\hat{\sigma}_j} y \quad (19)$$

$$\dot{s} = |y - C_{\hat{\sigma}_j} \tilde{x}|^2 \quad (20)$$

for the interval $[\hat{t}_j, \hat{t}_j + \delta]$ in an instant

2: **repeat**

3: integrate (18), (19), and (20) real time

4: **if**

$$t - t_{\text{up}} \geq \Delta \text{ and } |\tilde{x}(t) - \hat{x}(t)| > v_{\max}(\tilde{\mu}e^{\lambda\Delta} + 1)\hat{E}(\delta) \quad (21)$$

5: **then**

$$\tilde{x}(t) \leftarrow \hat{x}(t), \quad t_{\text{up}} \leftarrow t \quad (22)$$

6: **end if**

7: **until** either one of the following holds:

$$t - t_{\text{up}} < \Delta \text{ and } |\tilde{x}(t) - \hat{x}(t)| > v_{\max}(\tilde{\mu}e^{\lambda\Delta} + 1)\hat{E}(\delta), \quad (23)$$

$$\text{or } s(t) - s(t - \Delta) > S_\Delta. \quad (24)$$

8: $j \leftarrow j + 1$, mark the estimated switching time $\hat{t}_j \leftarrow t$.

Remark 3: When (24) is evaluated, we regard that $s(t) = 0$ for $t < \bar{t}_j$. \diamond

We now prove that Detection Algorithm detects the switching within the maximum delay of Δ . For this, suppose

a switching occurs at $t^* > \hat{t}_j + \delta = \bar{t}_j$, and let $\check{e} := x - \tilde{x}$ and $\hat{e} := x - \hat{x}$. Then, with $\hat{\sigma}(t) = \sigma(t)$ for the interval $[\hat{t}_j, t^*]$, we have the error dynamics given by

$$\dot{\check{e}} = A_{\sigma} \check{e},$$

$$\dot{\hat{e}} = (A_{\sigma} - L_{\sigma} C_{\sigma}) \hat{e} - L_{\sigma} v.$$

From Theorem 1 and the initialization of the algorithm, it is seen that

$$\begin{aligned} \check{e}(\hat{t}_j) &= \hat{e}(\hat{t}_j) = x(\hat{t}_j) - \hat{x}(\hat{t}_j) \\ &= -U_{\sigma}^{-1}(\delta) \int_{\hat{t}_j}^{\hat{t}_j + \delta} \phi_{\sigma}^{\top}(s - \hat{t}_j) v(s) ds \end{aligned}$$

so that, $|\check{e}(\hat{t}_j)| = |\hat{e}(\hat{t}_j)| \leq v_{\max} M_{\max}(\delta)$. From this, we note that

$$\begin{aligned} |\hat{e}(t)| &= \left| e^{(A_{\sigma} - L_{\sigma} C_{\sigma})(t - \hat{t}_j)} \hat{e}(\hat{t}_j) \right. \\ &\quad \left. - \int_{\hat{t}_j}^t e^{(A_{\sigma} - L_{\sigma} C_{\sigma})(t - \tau)} L_{\sigma} v(\tau) d\tau \right| \\ &\leq \hat{\mu} e^{-\hat{\lambda}(t - \hat{t}_j)} |\hat{e}(\hat{t}_j)| + \int_{\hat{t}_j}^t \hat{\mu} e^{-\hat{\lambda}(t - \tau)} d\tau L_{\max} v_{\max} \\ &\leq \hat{\mu} e^{-\hat{\lambda}(t - \hat{t}_j)} M_{\sigma}(\delta) v_{\max} \\ &\quad + \frac{\hat{\mu}}{\hat{\lambda}} (1 - e^{-\hat{\lambda}(t - \hat{t}_j)}) L_{\max} v_{\max} \\ &\leq v_{\max} \hat{\mu} \max \{ M_{\max}(\delta), L_{\max} / \hat{\lambda} \} = v_{\max} \hat{E}(\delta) \end{aligned}$$

for $\bar{t}_j \leq t \leq t^*$. Furthermore, because of the conditions (21) and (23), when $|\tilde{x}(t) - \hat{x}(t)| > v_{\max}(\tilde{\mu}e^{\lambda\Delta} + 1)\hat{E}(\delta)$ either $\tilde{x}(t)$ is reset to $\hat{x}(t)$ or Detection Algorithm terminates. Therefore, we have that

$$|\tilde{x}(t) - \hat{x}(t)| \leq v_{\max}(\tilde{\mu}e^{\lambda\Delta} + 1)\hat{E}(\delta),$$

and

$$\begin{aligned} |\check{e}(t)| &\leq |\hat{e}(t)| + |\tilde{x}(t) - \hat{x}(t)| \\ &\leq v_{\max}(\tilde{\mu}e^{\lambda\Delta} + 2)\hat{E}(\delta) \end{aligned} \quad (25)$$

for $\bar{t}_j \leq t \leq t^*$.

Lemma 1: The conditions (23) and (24) are never activated during the interval $[\bar{t}_j, t^*]$. \diamond

Proof: When $t - t_{\text{up}} < \Delta$, because $\hat{\sigma}(t) = \sigma(t)$ for the interval $[\hat{t}_j, t^*]$, we have that

$$\begin{aligned} |\tilde{x}(t) - \hat{x}(t)| &\leq |x(t) - \tilde{x}(t)| + |x(t) - \hat{x}(t)| \\ &\leq \tilde{\mu} e^{\hat{\lambda}(t - t_{\text{up}})} |\check{e}(t_{\text{up}})| + |\hat{e}(t)| \\ &\leq \tilde{\mu} e^{\hat{\lambda}(t - t_{\text{up}})} v_{\max} \hat{E}(\delta) + v_{\max} \hat{E}(\delta) \\ &< v_{\max}(\tilde{\mu}e^{\lambda\Delta} + 1)\hat{E}(\delta). \end{aligned}$$

Thus, (23) is not activated for the interval. For (24), we note

that

$$\begin{aligned}
& (s(t) - s(t - \Delta))^{\frac{1}{2}} \\
&= \left(\int_{\max\{\bar{t}_j, t - \Delta\}}^t |C_\sigma x(\tau) + v(\tau) - C_{\hat{\sigma}} \check{x}(\tau)|^2 d\tau \right)^{\frac{1}{2}} \\
&= \left(\int_{\max\{\bar{t}_j, t - \Delta\}}^t |v(\tau) + C_{\hat{\sigma}} \check{e}(\tau)|^2 d\tau \right)^{\frac{1}{2}} \\
&\leq \left(\int_{\max\{\bar{t}_j, t - \Delta\}}^t |v(\tau)|^2 d\tau \right)^{\frac{1}{2}} \\
&\quad + \left(\int_{\max\{\bar{t}_j, t - \Delta\}}^t |C_{\hat{\sigma}} \check{e}(\tau)|^2 d\tau \right)^{\frac{1}{2}} \\
&\leq v_{\max} \sqrt{\Delta} + v_{\max} C_{\max} (\check{\mu} e^{\check{\lambda} \Delta} + 2) \hat{E}(\delta) \sqrt{\Delta} = \sqrt{S_\Delta}
\end{aligned}$$

as long as $\hat{\sigma}(t) = \sigma(t)$, which proves the claim. \blacksquare

Lemma 2: Under Assumptions 2 and 3, within Δ after the switching time t^* , either (23) or (24) is activated, so that Detection Algorithm completes at time $t = \hat{t}_{j+1} \leq t^* + \Delta$. \diamond

Proof: If (23) becomes active in the interval $[t^*, t^* + \Delta]$, the proof is complete. Hence, assuming that

$$t - t_{\text{up}} \geq \Delta \text{ or } |\check{x}(t) - \hat{x}(t)| \leq v_{\max} (\check{\mu} e^{\check{\lambda} \Delta} + 1) \hat{E}(\delta), \quad (26)$$

for $t^* \leq t \leq t^* + \Delta$, we show that (24) becomes active in $[t^*, t^* + \Delta]$. Because $t - t_{\text{up}} \geq \Delta$ and $|\check{x}(t) - \hat{x}(t)| > v_{\max} (\check{\mu} e^{\check{\lambda} \Delta} + 1) \hat{E}(\delta)$ never happen simultaneously by the updating condition (21), we can simply assume that

$$|\check{x}(t) - \hat{x}(t)| \leq v_{\max} (\check{\mu} e^{\check{\lambda} \Delta} + 1) \hat{E}(\delta) \quad (27)$$

instead of (26). Since $(s(t^*) - s(t^* - \Delta)) \leq S_\Delta$ by Lemma 1 and $(s(t) - s(t - \Delta))$ is continuous, it is enough to show that $(s(t^* + \Delta) - s(t^*)) > S_\Delta$, which implies activation of (24) in the interval $[t^*, t^* + \Delta]$.

Let us define a virtual error variable

$$\tilde{e}_i(t) := \check{x}(t) - \left(e^{A_i(t-t^*)} x(t^*) + \int_{t^*}^t e^{A_i(t-s)} B_i u(s) ds \right)$$

for $t \geq t^*$, where $\check{x}(t)$ is the solution of (18) while the parenthesis is the solution of the plant (1) from $x(t^*)$ at $t = t^*$ as if the mode is i for $t \geq t^*$. Note that $\check{x}(t)$ obeys (18) with $\hat{\sigma}(t) = \hat{\sigma}(\bar{t}_j)$ for $t \geq t^* > \bar{t}_j$ (in spite of the switching at $t = t^*$). We claim that

$$|\tilde{e}_{\hat{\sigma}}(t)| \leq v_{\max} \check{\mu} e^{\check{\lambda} \Delta} (2\check{\mu} e^{\check{\lambda} \Delta} + 3) \hat{E}(\delta), \quad (28)$$

for $t^* \leq t \leq t^* + \Delta$. To see this, we consider two cases (keeping in mind that $\check{x}(t)$ is updated to $\hat{x}(t)$ at most once in Δ time period by (21)). First, if there is no update between t^* and $t^* + \Delta$, then

$$\begin{aligned}
|\tilde{e}_{\hat{\sigma}}(t)| &= |e^{A_{\hat{\sigma}}(t-t^*)} (\check{x}(t^*) - x(t^*))| \\
&\leq v_{\max} \check{\mu} e^{\check{\lambda} \Delta} (\check{\mu} e^{\check{\lambda} \Delta} + 2) \hat{E}(\delta)
\end{aligned}$$

by (25), for $t^* \leq t \leq t^* + \Delta$, which proves the claim. Second, suppose that there is one update between t^* and $t^* + \Delta$, and

denote by t_{up^*} the time of the update (22) in $[t^*, t^* + \Delta]$. For $t^* \leq t < t_{\text{up}^*}$, we have $\tilde{e}_{\hat{\sigma}}(t) = e^{A_{\hat{\sigma}}(t-t^*)} (\check{x}(t^*) - x(t^*))$. For $t_{\text{up}^*} \leq t \leq t^* + \Delta$, we have by linearity that

$$\begin{aligned}
\tilde{e}_{\hat{\sigma}}(t) &= e^{A_{\hat{\sigma}}(t-t_{\text{up}^*})} (\check{x}(t_{\text{up}^*}) - \check{x}(t_{\text{up}^*}^-)) + \tilde{e}_{\hat{\sigma}}(t_{\text{up}^*}^-) \\
&= e^{A_{\hat{\sigma}}(t-t_{\text{up}^*})} (\hat{x}(t_{\text{up}^*}) - \check{x}(t_{\text{up}^*}^-)) \\
&\quad + e^{A_{\hat{\sigma}}(t-t^*)} (\check{x}(t^*) - x(t^*)).
\end{aligned}$$

We note that, $|\hat{x}(t_{\text{up}^*}) - \check{x}(t_{\text{up}^*}^-)| \leq v_{\max} (\check{\mu} e^{\check{\lambda} \Delta} + 1) \hat{E}(\delta)$ by (27), and that $|\check{x}(t^*) - x(t^*)| \leq v_{\max} (\check{\mu} e^{\check{\lambda} \Delta} + 2) \hat{E}(\delta)$ by (25), for $t^* \leq t \leq t^* + \Delta$. Combining them, the claim (28) follows.

Finally,

$$\begin{aligned}
& (s(t^* + \Delta) - s(t^*))^{\frac{1}{2}} \\
&= \left(\int_{t^*}^{t^* + \Delta} |(\phi_\sigma(\tau - t^*) - \phi_{\hat{\sigma}}(\tau - t^*)) x(t^*) + v(\tau) \right. \\
&\quad \left. + \int_{t^*}^\tau (h_\sigma(\tau - s) - h_{\hat{\sigma}}(\tau - s)) u(s) ds - C_{\hat{\sigma}} \tilde{e}_{\hat{\sigma}}(\tau) \right|^2 d\tau \Big)^{\frac{1}{2}} \\
&\geq \left(\int_{t^*}^{t^* + \Delta} |[\phi_\sigma(\tau - t^*), \phi_{\hat{\sigma}}(\tau - t^*)][x(t^*); -x(t^*)]|^2 d\tau \right)^{\frac{1}{2}} \\
&\quad - \left(\int_{t^*}^{t^* + \Delta} \left| \int_{t^*}^\tau (h_\sigma(\tau - s) - h_{\hat{\sigma}}(\tau - s)) u(s) ds \right|^2 d\tau \right)^{\frac{1}{2}} \\
&\quad - \left(\int_{t^*}^{t^* + \Delta} |v(\tau)|^2 d\tau \right)^{\frac{1}{2}} - \left(\int_{t^*}^{t^* + \Delta} |C_{\hat{\sigma}} \tilde{e}_{\hat{\sigma}}(\tau)|^2 d\tau \right)^{\frac{1}{2}} \\
&\geq \left(\lambda_{\min}(W_{\sigma, \hat{\sigma}}(\Delta)) |x(t^*); -x(t^*)|^2 \right)^{\frac{1}{2}} \\
&\quad - \left(\int_0^\Delta \left(\int_0^\tau \|h_\sigma(s) - h_{\hat{\sigma}}(s)\| ds \right)^2 d\tau u_{\max}^2 \right)^{\frac{1}{2}} \\
&\quad - \left(\int_{t^*}^{t^* + \Delta} |v(\tau)|^2 d\tau \right)^{\frac{1}{2}} - \left(\int_{t^*}^{t^* + \Delta} |C_{\hat{\sigma}} \tilde{e}_{\hat{\sigma}}(\tau)|^2 d\tau \right)^{\frac{1}{2}} \\
&\geq \sqrt{2\omega_{\min}(\Delta)} |x(t^*)| - u_{\max} \sqrt{N_u(\Delta)} \\
&\quad - v_{\max} \sqrt{\Delta} - v_{\max} C_{\max} \check{\mu} e^{\check{\lambda} \Delta} (2\check{\mu} e^{\check{\lambda} \Delta} + 3) \hat{E}(\delta) \sqrt{\Delta}.
\end{aligned}$$

By (4), we have $s(t^* + \Delta) - s(t^*) > S_\Delta$, which completes the proof. \blacksquare

Combining Lemmas 1 and 2 with Assumption 3, the analysis leads to the following main result.

Theorem 2: Under Assumptions 1, 2 and 3, the proposed Detection Algorithm guarantees the detection of the switching time by \hat{t}_j within the delay of Δ . \diamond

V. OPERATION OF WHOLE ALGORITHM

With all discussed so far, the operation of the whole algorithm is summarized in this section. As an initialization, let $j = 0$ and $\hat{t}_j = 0$. At time \hat{t}_j (which is 0 when $j = 0$), run the estimation algorithm in Remark 2. This algorithm completes immediately after the time $\hat{t}_j + \delta$ when it updates \hat{x} and $\hat{\sigma}_j$ of (19). After the completion of the estimation algorithm (i.e., at $t = \hat{t}_j + \delta$), Detection Algorithm starts running and monitors the occurrence of the mode switching.

If the switching signal jumps to another mode, Detection Algorithm will detect it soon and reports the switching. At this moment, increase j , mark the time as \hat{t}_j , and repeat (i.e., start the estimation algorithm because the time is $t = \hat{t}_j$). Note that these algorithms are performed as a background operation of the computer.

Whole Algorithm of Switching Signal Estimation

Require: $j \leftarrow 0, \hat{t}_j \leftarrow 0$

```

1: while  $t \geq 0$  do
2:   if  $t < \hat{t}_j + \delta$  then
3:     integrate (12), (13) with zero initial state,  $\forall i \in \mathcal{N}$ 
4:   else if  $t = \hat{t}_j + \delta$  then
5:     compute  $X_i^*, \forall i \in \mathcal{N}$  by (14)
6:     run (15), (16) with initial state  $X_i^*$  and 0,  $\forall i \in \mathcal{N}$ 
7:     compute  $J_i(X_i^*), \forall i \in \mathcal{N}$  by (17)
8:      $\hat{\sigma}_j \leftarrow \arg \min_{i \in \mathcal{N}} J_i(X_i^*)$  by (5)
9:      $\tilde{x}(\hat{t}_j) \leftarrow X_{\hat{\sigma}_j}^*, \hat{x}(\hat{t}_j) \leftarrow X_{\hat{\sigma}_j}^*, s(\hat{t}_j) \leftarrow 0, t_{up} \leftarrow \hat{t}_j$ 
10:    run (18), (19), (20) from  $\hat{t}_j$  to  $\hat{t}_j + \delta$  with
        initial state  $\tilde{x}(\hat{t}_j), \hat{x}(\hat{t}_j), s(\hat{t}_j)$ 
11:   else if  $t > \hat{t}_j + \delta$  then
12:     integrate (18), (19), (20) real time
13:   if (21) holds then
14:      $\tilde{x}(t) \leftarrow \hat{x}(t), t_{up} \leftarrow t$ 
15:   end if
16:   if (23) or (24) holds then
17:      $j \leftarrow j + 1, \hat{t}_j \leftarrow t$ 
18:   end if
19: end if
20: end while

```

VI. CONCLUSION

In this paper, we have considered the estimation problem of switching signal for continuous time switched linear control systems with measurement noise. The main contributions include the following two issues. First, inspired by the work of [8], a mode estimation algorithm is proposed to tackle with the bounded measurement noise and external input. Second, a detection algorithm is presented to detect the mode switching with guaranteed precision, where some update condition and threshold condition are checked all the time. Under the update and threshold conditions, we detect the switching time within a predetermined time interval after the switching occurred. In addition, the precision of the estimation and detection algorithm depends on the maximum norm of the measurement noise and the external input.

Our future work could focus on the following aspects. First, extensions to the system with process disturbance are under development. Second, it is worthwhile to implement the algorithms with better precision and easier conditions to check. Finally, the state estimation will also be considered.

REFERENCES

- [1] A. Alessandri, M. Baglietto, and G. Battistelli, "Minimum-distance receding-horizon state estimation for switching discrete-time linear systems," in *Assessment and Future Directions of Nonlinear Model Predictive Control*, vol. 358 of *Lecture Notes in Control and Information Sciences*, R. Findeisen, F. Allgöwer, and L. T. Biegler, Eds. Springer-Verlag, 2007, pp. 347–358.
- [2] M. Babaali and M. Egerstedt, "Observability of switched linear systems," in *Hybrid Systems: Computation and Control*, vol. 2993 of *Lecture Notes in Computer Science*, R. Alur and G. J. Pappas, Eds. Springer-Verlag, 2004, pp. 48–63.
- [3] M. Babaali and G. J. Pappas, "Observability of switched linear systems in continuous time," in *Hybrid Systems: Computation and Control*, vol. 3414 of *Lecture Notes in Computer Science*, M. Morari and L. Thiele, Eds. Springer-Verlag, 2005, pp. 103–117.
- [4] M. Baglietto, G. Battistelli, and L. Scardovi, "Active mode observability of switching linear systems," *Automatica*, vol. 43, no. 8, pp. 1442–1449, 2007.
- [5] M. Baglietto, G. Battistelli, and L. Scardovi, "Active mode observation of switching systems based on set-valued estimation of the continuous state," *International Journal of Robust and Nonlinear Control*, vol. 19, no. 14, pp. 1521–1540, 2009.
- [6] A. Balluchi, L. Benvenuti, M. D. Di Benedetto, and A. L. Sangiovanni-Vincentelli, "A hybrid observer for the driveline dynamics," in *Proc. of the European Control Conference*, Porto, Portugal, 2001, pp. 618–623.
- [7] A. Balluchi, L. Benvenuti, M. D. Di Benedetto, and A. L. Sangiovanni-Vincentelli, "Design of observers for hybrid systems," in *Hybrid Systems: Computation and Control*, vol. 2289 of *Lecture Notes in Computer Science*, C. J. Tomlin and M. R. Greenstreet, Eds. Springer-Verlag, 2002, pp. 76–89.
- [8] G. Battistelli, "On adaptive stabilization of mode-observable switching linear systems," in *Proc. of 18th IFAC World Congress*, Milano, Italy, 2011, pp. 356–361.
- [9] E. A. Domlan, J. Ragot, and D. Maquin, "Active mode estimation for switching systems," in *Proc. of American Control Conference*, New York, USA, 2007, pp. 1143–1148.
- [10] M. Fliess, C. Join, and W. Perruquetti, "Real-time estimation for switched linear systems," in *Proc. of the 47th IEEE Conference on Decision and Control*, Cancun, Mexico, 2008, pp. 941–946.
- [11] D. Gómez-Gutiérrez, A. Ramírez-Treviño, J. Ruiz-León, and S. Di Gennaro, "On the observability of continuous-time switched linear systems under partially unknown input," *IEEE Transactions on Automatic Control*, vol. 57, no. 3, pp. 732–738, 2012.
- [12] E. A. Medina and D. A. Lawrence, "Reachability and observability of linear impulsive systems," *Automatica*, vol. 44, no. 5, pp. 1304–1309, 2008.
- [13] D. Mincarelli, T. Floquet, and L. Belkoura, "Active mode and switching time estimation for switched linear systems," in *Proc. of 50th IEEE Conference on Decision and Control and European Control Conference*, Orlando, Florida, USA, 2011, pp. 1854–1859.
- [14] H. Shim and A. Tanwani, "Hybrid-type observer design based on a sufficient condition for observability in switched nonlinear systems," To appear in Special Issue on High-gain Observers in Nonlinear Feedback Control at *International Journal of Robust and Nonlinear Control*, <http://dx.doi.org/10.1002/rnc.2901>, 2013.
- [15] A. Tanwani, H. Shim, and D. Liberzon, "Observability for switched linear systems: characterization and observer design," *IEEE Transactions on Automatic Control*, vol. 58, no. 4, pp. 891–904, 2013.
- [16] Y. Tian, T. Floquet, L. Belkoura, and W. Perruquetti, "Switching time estimation for linear switched systems: an algebraic approach," in *Proc. of the 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, Shanghai, P. R. China, 2009, pp. 3909–3913.
- [17] Y. Tian, T. Floquet, L. Belkoura, and W. Perruquetti, "Algebraic switching time identification for a class of linear hybrid systems," *Nonlinear Analysis: Hybrid Systems*, vol. 5, no. 2, pp. 233–241, 2011.
- [18] R. Vidal, A. Chiuso, and S. Soatto, "Observability and identifiability of jump linear systems," in *Proc. of the 41st IEEE Conference on Decision and Control*, Las Vegas, Nevada, USA, 2002, pp. 3614–3619.
- [19] R. Vidal, A. Chiuso, S. Soatto, and S. Sastry, "Observability of linear hybrid systems," in *Hybrid Systems: Computation and Control*, vol. 2623 of *Lecture Notes in Computer Science*, O. Maler and A. Pnueli, Eds. Springer-Verlag, 2003, pp. 526–539.
- [20] G. Xie and L. Wang, "Necessary and sufficient conditions for controllability and observability of switched impulsive control systems," *IEEE Transactions on Automatic Control*, vol. 49, no. 6, pp. 960–966, 2004.
- [21] S. Zhao and J. Sun, "A geometric approach for reachability and observability of linear switched impulsive systems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 11, pp. 4221–4229, 2010.