

On an approximation with prescribed zeros of SISO abstract boundary control systems

O. V. Iftime and T. C. Ionescu

Abstract—Finite-dimensional approximations of partial differential equations are used not only for simulation, but also for controller design. Modal truncation and numerical approximation are common practical methods for approximating distributed parameter systems. The modal approximation preserves the exact, low-order poles of the original system. However, the zeros of modal approximations may differ significantly from those of the original distributed parameter system. In particular, right half-plane zeros, which are not present in the original infinite-dimensional model, may appear in modal truncations. In this paper we consider a boundary control system and propose a moment matching based approximation which preserves a prescribed set of zeros. To illustrate the advantages of the method, we consider its application to the heat equation with Neumann boundary control at the right end (HENBCR). Although the modal approximation provides good error bounds for the HENBCR, it contains non-minimum phase zeros which lead to erroneous predictions. The moment matching approach sketched in this paper yields an approximation of the HENBCR with minimum phase zeros only. We consider that the numerical example is very interesting and convincing for the reader. Due to space limitations, further theoretical analysis will be addressed in the full paper.

I. INTRODUCTION

Systems modelled by partial differential equations with boundary control arise in many applications, for instance, heating processes, active noise control and flexible structures. A finite-dimensional approximation of partial differential equations is customary used not only for simulation, but also for designing controllers. Modal truncation and numerical approximation (such as Finite Element Methods) are common practical methods used for approximating distributed parameter systems.

The modal approximation preserves the exact (low-order) poles of the given system. However, the zeros of modal approximations may differ significantly from those of the original distributed parameter system. It has been shown in the literature that modal truncations of a controlled beam yield inaccurate system zeros. In particular, in specific modal truncations, right half-plane zeros, which are not present in the original infinite-dimensional model, appear. It has been also found that Finite Element Methods exhibit similar problems.

The zeros of the transfer function of a system are significant for controller design. For instance, the presence of right

half-plane zeros restricts the achievable sensitivity reduction. The presence of right half-plane zeros also renders the use of an adaptive controller impractical, in most situations. Hence, it is important to obtain left half-plane zeros when approximating distributed parameter systems (see also [2]).

In this paper we consider boundary control systems as defined in [16] and also discussed in [12]. A representation of the transfer function of a boundary control system is defined in terms of an abstract elliptic problem associated with the boundary control system. A moment matching problem is formulated based on an operator Sylvester equation. Then one can approximate the boundary control system by moment matching with preservation of zeros. The outcome is a class of finite-dimensional, reduced order models that match a set of prescribed moments of the given system and preserve some of the poles and zeros of the original distributed parameter system.

For numerical computation, we consider the one-dimensional heat equation with Neumann boundary control at the right end. The zeros of a specific modal approximation are easily calculated, illustrating the presence of right half-plane zeros. The Finite Element Methods also generate right half-plane zeros [12]. Using moment matching and properly selecting the free parameters of the resulting reduced order model we avoid right half-plane zeros, i.e., only left half-plane zeros are present. We also point out that in the class of finite-dimensional models approximating the one-dimensional heat equation with Neumann boundary control at the right end there are models which yield better infinity norm of the error than the modal truncation counterparts.

The paper is structured as follows. In Section II we review the required notations. In Section III we describe the basics of moment matching. A class of finite-dimensional, parameterized, reduced order models that achieve moment matching is proposed. Furthermore, the free parameters are selected such that pole placement and preservation of zeros occurs. In Section IV we illustrate the theory with the example of approximating the one-dimensional heat equation with Neumann boundary control at the right end. We compare our approximant with the one obtained by modal truncation. Placing not only a pole, but also a set of zeros, we obtain a fourth order model of the original system which has the infinity norm of the approximation error smaller than the one corresponding to the fourth order modal truncation. The paper is completed by a conclusions section.

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II. PRELIMINARIES

Let \mathcal{Z} , Z , \mathcal{U} and \mathcal{Y} be Hilbert spaces endowed with the corresponding inner products, where \mathcal{Z} is a dense subspace of Z with the continuous, injective embedding $i_{\mathcal{Z}}$. A boundary control system (Δ, Γ, C) is defined formally by

$$\begin{aligned} \frac{d}{dt}z(t) &= \Delta z(t), \\ z(0) &= z_0, \\ \Gamma z(t) &= u(t), \quad t \geq 0 \\ y(t) &= Cz(t), \end{aligned} \quad (1)$$

where $\Delta \in \mathcal{L}(\mathcal{Z}, Z)$, $\Gamma \in \mathcal{L}(\mathcal{Z}, \mathcal{U})$ and $C \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ (see [1], [12] and the references therein). In this paper we consider only SISO systems and we will assume that the boundary control system satisfies the same assumptions as in [12]. Our goal is to approximate the infinite-dimensional state linear system (1) by a finite-dimensional, reduced order model such that poles and zeros of the original system are preserved. The reduced order model is obtained by projection onto a particular finite-dimensional subspace of the state-space Z and using moment matching techniques.

Any boundary control system (Δ, Γ, C) can be put into a state-space form (A, B, C)

$$\begin{aligned} \dot{z}(t) &= Az(t) + Bu(t), \quad t \geq 0, \quad z(0) = z_0, \\ y(t) &= Cz(t). \end{aligned} \quad (2)$$

The operators A , B and C are defined as in [16]. Note that, in general, the operators B and C are unbounded on the state space. The operator A is $A = \Delta i$ with i the canonical injection from $\mathcal{W} = \{z \in \mathcal{Z} \mid \Gamma z = 0\}$ to Z , where \mathcal{W} is the completion of $D(A)$ in the graph norm of A . Then A generates a C_0 -semigroup on Z and $B \in \mathcal{L}(\mathcal{U}, [D(A^*)]')$ where $[D(A^*)]'$ is the dual of $[D(A^*)]$. For the well-posedness of boundary control systems we refer to [1]. In the sequel, we assume that $\mathcal{U} = \mathcal{Y} = \mathbb{C}$.

Let $\sigma(A)$ and $\rho(A)$ denote the spectrum of A and the resolvent set of A , respectively. The transfer function K of the state linear system (A, B, C) is given by

$$K(s) = C(sI - A)^{-1}B \in \mathbb{C}, \quad s \in \rho_{\infty}(A), \quad (3)$$

where $\rho_{\infty}(A)$ is the component of $\rho(A)$ that contains an interval $[r, \infty)$. We make throughout the paper the following assumption.

Assumption 1. A is a Riesz-spectral operator.

III. MOMENT MATCHING WITH PRESCRIBED ZEROS

First, we recall the notion of moment. Consider the boundary control system (Δ, Γ, C) and its state-space form (A, B, C) as in (2), with the transfer function $K(s)$. The moments of (A, B, C) at a given point s^* in $\rho_{\infty}(A)$ are the coefficients of the Taylor expansion of the transfer function $K(s)$ about s^* , similar to the finite-dimensional case (see, e.g. [10], [11]).

Definition 1. Let $s^* \in \rho_{\infty}(A)$ and $k \in \mathbb{N}$. The k -moment of (2) at s^* is $\eta_k(s^*) \in \mathbb{C}$ defined by

$$\eta_k(s^*) = \frac{(-1)^k}{k!} \frac{d^k K}{ds^k}(s^*). \quad (4)$$

Let $s^* \in \rho_{\infty}(A)$ ($s^* \neq \infty$), and consider $L_{\nu} : \mathbb{C}^{\nu} \rightarrow \mathbb{C}$ and $\Sigma_{\nu} : \mathbb{C}^{\nu} \rightarrow \mathbb{C}^{\nu}$ given by $L_{\nu} = [1 \ 0 \ \dots \ 0]$ and

$$\Sigma_{\nu} = \begin{bmatrix} s^* & 1 & 0 & \dots & 0 \\ 0 & s^* & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & s^* & 1 \\ 0 & \dots & \dots & 0 & s^* \end{bmatrix}.$$

One can write the operator Sylvester equation

$$\tilde{\Pi}\Sigma_{\nu} - A\tilde{\Pi} = BL_{\nu}, \quad (5)$$

where $\tilde{\Pi} : \mathbb{C}^{\nu} \rightarrow D(A) \subset Z$.

Note that the operator B can be unbounded in general.

We make the following major standing assumption, throughout the rest of the paper. Due to space limitations, we shall analyze this assumption in the full paper.

Assumption 2. Equation (5) has a unique solution $\tilde{\Pi}$ and $\dim(\text{ran}(\tilde{\Pi})) = \nu$.

The following result (which is an extension of a similar result obtained for the finite-dimensional case [11]) provides a relation of the k -moments of (2), at $s^* \in \rho_{\infty}(A)$, with $\tilde{\Pi}$.

Lemma 1. Consider the boundary control system (Δ, Γ, C) and its corresponding state-space representation (A, B, C) . Suppose that Assumption 2 holds and let $\tilde{\Pi}$ be the unique solution of the OSE (5). Then the first ν moments of system (2) at $\{s^*, \dots, s^*\} = \sigma(\tilde{\Sigma}_{\nu})$ satisfy

$$[\eta_0(s^*) \ \dots \ \eta_{\nu-1}(s^*)] = C\tilde{\Pi} \text{diag}\{1, -1, \dots, (-1)^{\nu-1}\}. \quad \square$$

Proof: Using Assumption 2 and Assumption 1, $\tilde{\Pi}$ can be written using the linearly independent vectors $\{v_k\}_k$, and the equality follows. \blacksquare

Remark 1. Note that, in Lemma 1, Σ_{ν} and L_{ν} have a particular form.

Consider the boundary control system (Δ, Γ, C) , with the corresponding state-space form (A, B, C) in (2). Let $(\tilde{F}, \tilde{G}, \tilde{H})$ be a finite-dimensional reduced order model associated to (A, B, C) , given by equations of the form

$$\begin{aligned} \dot{\xi}(t) &= \tilde{F}\xi(t) + \tilde{G}u(t), \quad t \geq 0, \quad \xi(0) = \xi_0, \\ \psi(t) &= \tilde{H}\xi(t), \end{aligned} \quad (6)$$

where $\xi_0 \in \mathbb{C}^{\nu}$, $\xi(t) \in \mathbb{C}^{\nu}$, $\tilde{F} : \mathbb{C}^{\nu} \rightarrow \mathbb{C}^{\nu}$, $\tilde{G} : \mathbb{C} \rightarrow \mathbb{C}^{\nu}$, and $\tilde{H} : \mathbb{C}^{\nu} \rightarrow \mathbb{C}$.

Let $s_i \in \rho_{\infty}(A) \cap \rho(\tilde{F})$, $i = 0, \dots, l$, $l \geq 0$. Take $j_i \geq 0$ such that

$$\sum_{i=0}^l (j_i - 1) = \mu. \quad (7)$$

For each i , let $\eta_0(s_i), \dots, \eta_{j_i}(s_i)$ denote the first $j_i + 1$ moments of the system (A, B, C) at the given points s_i and let $\hat{\eta}_0(s_i), \dots, \hat{\eta}_{j_i}(s_i)$ denote the first $j_i + 1$ moments of the system $(\tilde{F}, \tilde{G}, \tilde{H})$ at s_i (see Definition 1, [10], [11]).

Definition 2 (Moment matching). A system $(\tilde{F}, \tilde{G}, \tilde{H})$ matches μ moments of a given system (A, B, C) at $\{s_0, \dots, s_l\}$, if

$$\eta_k(s_i) = \hat{\eta}_k(s_i), \quad (8)$$

for all $k = 0, \dots, j_i$, $i = 0, \dots, l$, with μ satisfying (7). \square

We are now ready to define the class of reduced order models $(\tilde{F}, \tilde{G}, \tilde{H})$ of the given system (A, B, C) , that achieve moment matching.

Consider $\tilde{S} : \mathbb{C}^\nu \rightarrow \mathbb{C}^\nu$, with $\sigma(\tilde{S}) = \{s_0, \dots, s_l\}$, where each s_i has multiplicity $j_i + 1$, such that

$$\sigma(\tilde{S}) \cap \sigma(A) = \emptyset. \quad (9)$$

Let $\tilde{L} : \mathbb{C}^\nu \rightarrow \mathbb{C}$ be such that the pair (\tilde{L}, \tilde{S}) is observable.

A system $(\tilde{F}, \tilde{G}, \tilde{H})$ is a *reduced (μ -th) order model* of (A, B, C) at $\sigma(\tilde{S})$ if the following two conditions are satisfied simultaneously

$$\sigma(\tilde{F}) \cap \sigma(\tilde{S}) = \emptyset, \quad (10a)$$

$$C\tilde{\Pi} = \tilde{H}, \quad (10b)$$

where $P : \mathbb{C}^\nu \rightarrow \mathbb{C}^\nu$ is the unique solution of the OSE

$$P\tilde{S} - \tilde{F}P = \tilde{G}\tilde{L}, \quad (11)$$

and $\tilde{\Pi} : \mathbb{C}^\nu \rightarrow \mathbb{C}^\nu$ is the unique solution of the OSE

$$\tilde{\Pi}\tilde{S} - A\tilde{\Pi} = B\tilde{L}.$$

Note that $\tilde{G} : \mathbb{C} \rightarrow \mathbb{C}^\nu$ is a free parameter. For each \tilde{G} , solving the Sylvester equation (11) yields a unique P . Hence, let

$$(\tilde{F}, \tilde{G}, \tilde{H})_{\tilde{G}}$$

denote a class of reduced order models (6) parameterized in \tilde{G} . The following result states that the class of reduced order models $(\tilde{F}, \tilde{G}, \tilde{H})_{\tilde{G}}$ at $\sigma(\tilde{S})$ achieves moment matching.

Theorem 1. Consider the state-space representation (A, B, C) of a boundary control system (Δ, Γ, C) . Let \tilde{S} be such that relation (9) holds and let \tilde{L} be such that the pair (\tilde{L}, \tilde{S}) is observable. If Assumption 2 holds, then there exists $\{(\tilde{F}, \tilde{G}, \tilde{H})_{\tilde{G}}\}$, a class of reduced order models of (A, B, C) at $\sigma(\tilde{S})$, parameterized in \tilde{G} such that, for all $(\tilde{F}, \tilde{G}, \tilde{H}) \in \{(\tilde{F}, \tilde{G}, \tilde{H})_{\tilde{G}}\}$, the equalities (8) are satisfied for $\mu = \nu$, i.e., moment matching is achieved. \square

Based on Theorem 1, we characterize $\{(\tilde{F}, \tilde{G}, \tilde{H})_{\tilde{G}}\}$, the class of finite-dimensional models on \mathbb{C}^ν that approximate (A, B, C) and match its moments at $\sigma(\tilde{S})$.

Corollary 1. Consider a boundary control system (Δ, Γ, C) . Let \tilde{S} be such that relation (9) holds and let \tilde{L} be such that the pair (\tilde{L}, \tilde{S}) is observable. Then, if Assumption 2 holds, the class of finite-dimensional reduced order models $\{(\tilde{F}, \tilde{G}, \tilde{H})_{\tilde{G}}\}$ that match ν moments of (A, B, C) at $\sigma(\tilde{S})$

is described by equations of the form

$$\begin{aligned} \dot{\xi} &= (\tilde{S} - \tilde{G}\tilde{L})\xi + \tilde{G}u, \quad t \geq 0, \quad \xi(0) = \xi_0, \\ \psi &= C\tilde{\Pi}\xi, \end{aligned} \quad (12)$$

with $\xi_0 \in \mathbb{C}^\nu$, $\xi(t) \in \mathbb{C}^\nu$ and $\psi(t) \in \mathbb{C}$. \square

By properly selecting \tilde{G} , it is possible to place the zeros of the reduced order model, at the zeros of the system to be approximated.

Proposition 1. Consider a boundary control system (Δ, Γ, C) . Furthermore, consider the class of reduced order models

$$\{(\tilde{S} - \tilde{G}\tilde{L}, \tilde{G}, C\tilde{\Pi})_{\tilde{G}}\}$$

that approximate (Δ, Γ, C) such that the assumptions in Theorem 1 are satisfied. Let $z_1, \dots, z_l \in \mathbb{C}$, $l \leq \nu$. Then the following statements are equivalent.

i. There exists a subset of models from the class $\{(\tilde{S} - \tilde{G}\tilde{L}, \tilde{G}, C\tilde{\Pi})_{\tilde{G}}\}$, with the property that the set of zeros of each model contains z_1, \dots, z_l .

ii. There exist \tilde{G} such

$$\det \begin{bmatrix} sI - \tilde{S} & \tilde{G} \\ C\tilde{\Pi} & 0 \end{bmatrix} = 0, \quad (13)$$

for $s = z_i$, $i = 1, \dots, l$. \square

Note that the proposed moment matching approach follows the arguments of [6], taking into account the particularities related to the boundary control systems.

IV. EXAMPLE

Consider the one-dimensional heat equation with Neumann boundary conditions at the right end ([12]), described by the equations

$$\begin{aligned} \frac{\partial z}{\partial t}(x, t) &= \frac{\partial^2 z}{\partial x^2}(x, t), \quad x \in [0, 1], \\ z(x, 0) &= 0, \quad x \in [0, 1], \\ \frac{\partial z}{\partial x}(0, t) &= 0, \quad t > 0, \\ \frac{\partial z}{\partial x}(1, t) &= u(t), \quad t > 0, \\ y(t) &= z(x_1, t), \end{aligned} \quad (14)$$

where the output is the temperature measured at $x_1 \in [0, 1]$. In this example one has that

$$\mathcal{Z} := \{z \in H^2(0, 1) \mid z'(0) = 0\}, \text{ and } \mathcal{U} = \mathcal{Y} = \mathbb{C}.$$

The system (14) can be written in the form (A, B, C) as in (2).

The operator A defined as $Ah = \frac{d^2 h}{dx^2}$, with

$$D(A) = \left\{ h \in Z \mid h, \frac{dh}{dx} \text{ absolutely continuous,} \right.$$

$$\left. \frac{d^2 h}{dx^2} \in L_2(0, 1), h(0) = \frac{dh}{dx}(0) = 0 \right\},$$

is a Riesz-spectral, self-adjoint operator ($A = A^*$), hence Assumption 1 holds. A has $\sigma(A) = \{-(n\pi)^2 \mid n \geq 0\}$ and the eigenfunctions $\phi_n(x) = \psi_n(x) = \cos(n\pi x)$, $n \geq 0$ as a basis, where ϕ_n are the eigenfunctions of A and ψ_n are the eigenfunctions of A^* .

The operator operator B is such that $Bu \in D(A)$. Furthermore, its adjoint is $B^* \in L(D(A), \mathbb{R})$ such that

$$B^* \psi(x) = \psi(1), \quad (\forall) \psi \in D(A).$$

The operator C is such that $Cz \in \mathbb{C}$, $Cz(x, t) = z(x_1, t)$, with $x_1 \in [0, 1]$, fixed. Note that the input and output spaces are \mathbb{C} . The transfer function of the system (14) is

$$K(s) = \frac{\cosh(x_1 \sqrt{s})}{\sqrt{s} \sinh(\sqrt{s})}.$$

Modal approximation of order N : According to [5] the modal truncation transfer function of order N is

$$\begin{aligned} K_{\text{mod}}^N(s) &= \sum_{n=0}^{N-1} \frac{1}{s - \lambda_n} C \phi_n(x) B^* \psi_n(x) \\ &= \sum_{n=0}^{N-1} \frac{1}{s - \lambda_n} C \phi_n(x) B^* \phi_n(x) \\ &= \frac{1}{s} + 2 \sum_{n=1}^{N-1} \frac{(-1)^n \cos(n\pi x_1)}{s - \lambda_n}, \end{aligned} \quad (15)$$

where ϕ_n and ψ_n are from the normalized basis.

Approximation that matches four prescribed moments: Let $x_1 = 1/3$ and let the interpolation points be $s_1 = 3$, $s_2 = 4.4$, $s_3 = 7.3$ and $s_4 = 10j$. Note that $s_i \in \rho_\infty(A)$, $i = 1, \dots, 4$. We compute the class of reduced order models $\{(\tilde{S} - \tilde{G}\tilde{L}, \tilde{G}, \tilde{H})\}_{\tilde{G}}$, described by equations (12) that match the moments $\eta_0(s_i)$, $i = 1, \dots, 4$ of the system (14). Select

$$\tilde{L} = [1 \ 1 \ 1 \ 1],$$

$$\tilde{S} = \text{diag}\{s_1, s_2, s_3, s_4\}.$$

Note that

$$\sigma(\tilde{S}) \cap \sigma(A) = \emptyset,$$

and it can be shown that the assumptions about the solvability of the Sylvester equation are satisfied. Furthermore, (\tilde{L}, \tilde{S}) is observable. In particular, the OSE (5) has a unique solution $\tilde{\Pi} = [v_1 \ v_2 \ v_3 \ v_4]$ of finite rank, with $v_i \in D(A)$, $i = 1, \dots, 4$ (note that $Bu \in D(A)$). Further,

$$\dim(\text{ran}(\tilde{\Pi})) = 4,$$

hence Assumption 2 holds. According to Lemma 1, we have that $[\eta_0(2) \ \eta_0(3.3) \ \eta_0(5) \ \eta_0(6)] = C[v_1 \ v_2 \ v_3 \ v_4]$, yielding

$$\begin{aligned} \eta_0(3) &= 0.2470, \\ \eta_0(4.4) &= 0.1491, \\ \eta_0(7.3) &= 0.0715, \\ \eta_0(10j) &= -0.0585 - 0.0454j. \end{aligned} \quad (16)$$

By (12) we have

$$\begin{aligned} \tilde{F} &= \begin{bmatrix} 3 - g_1 & -g_1 & -g_1 & -g_1 \\ -g_2 & 4.4 - g_2 & -g_2 & -g_2 \\ -g_3 & -g_3 & 7.3 - g_3 & -g_3 \\ -g_4 & -g_4 & -g_4 & 10j - g_4 \end{bmatrix}, \\ \tilde{G} &= [g_1 \ g_2 \ g_3 \ g_4]^T, \\ \tilde{H} &= [0.2470 \ 0.1491 \ 0.0715 \ -0.0585 - 0.0454j]. \end{aligned} \quad (17)$$

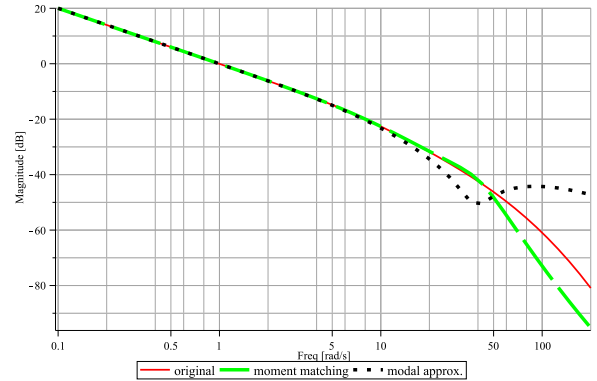
\tilde{G} must be chosen such that condition (10a) is satisfied, which is equivalent to $\tilde{G} \in \mathbb{C}^{2*}$. Hence, a class of fourth order models of (14) that match the moments from (16) is

$$\{(\tilde{F}, \tilde{G}, \tilde{H})\}_{\mathbb{C}^{2*}} = \left\{ (\tilde{F}, \tilde{G}, \tilde{H}) \text{ as in (17)} \mid \tilde{G} \in \mathbb{C}^{2*} \right\}.$$

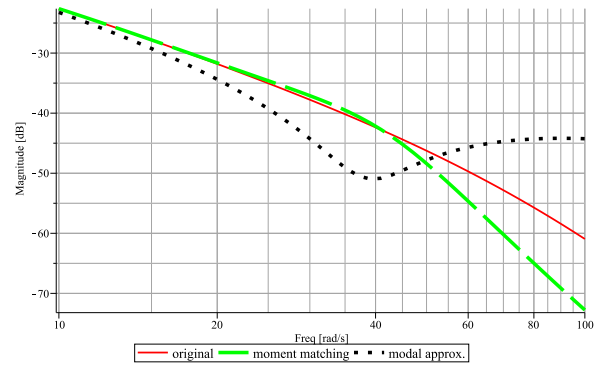
We compute the parameters \tilde{G} that place a pole of the reduced order model at 0 and three zeros at prescribed values, e.g., (some of) the zeros of system (14). For example, let

$$z_1 = -\frac{75\pi^2}{4}, \quad z_2 = -\frac{363\pi^2}{4} \text{ and } z_3 = -\frac{507\pi^2}{4}.$$

Hence, a reduced order model of (A, B, C) with the pole $\lambda = 0$ and the zeros z_1, z_2 and z_3 is $(\tilde{F}_z, \tilde{G}_z, \tilde{H}) \in \{(\tilde{F}, \tilde{G}, \tilde{H})\}_{\mathbb{C}^{2*}}$ described by equation (18).



(a) Magnitude plot.



(b) Magnitude plot zoom in.

Fig. 1. Magnitude plots of the original system, fourth order model that matches the moments in (16), preserving the pole $\lambda = 0$ and the zeros $z_1 = -\frac{75\pi^2}{4}$, $z_2 = -\frac{363\pi^2}{4}$ and $z_3 = -\frac{507\pi^2}{4}$, and fourth order modal truncation.

The magnitude plots in Fig. 1 show that the fourth order model $(\tilde{F}_z, \tilde{G}_z, \tilde{H})$ approximates (14) as accurately as the fourth order modal truncation K_{mod}^4 in (15), with $N = 4$, i.e.,

$$K_{\text{mod}}^4(s) = \frac{1}{s} - \frac{1}{s + \pi^2} - \frac{1}{s + 4\pi^2} + \frac{2}{s + 9\pi^2}.$$

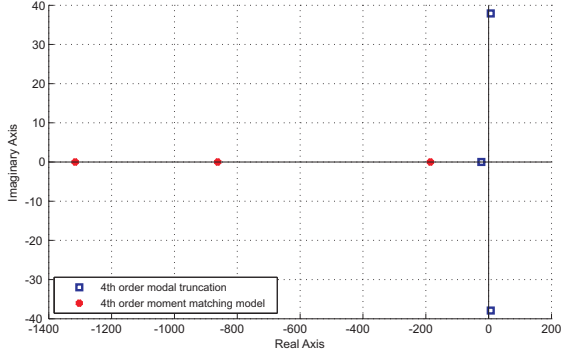


Fig. 2. Stable zeros of the fourth order model that matches the moments in (16) vs. the unstable zeros of the fourth order modal truncation.

Fig. 2 shows that in the modal truncation, right half-plane zeros are present, whereas the moment matching based model has all the zeros placed in the left half-plane, e.g., at some of the zeros of the given system, by a proper selection of the free parameters \tilde{G} . In terms of the L_∞ norm of the approximation error, $(\tilde{F}_z, \tilde{G}_z, \tilde{H})$ yields $2.2 \cdot 10^{-3}$, better than the fourth order modal approximation, which yields $6.975 \cdot 10^{-3}$.

A model with real coefficients that accurately approximates (14) is yielded by the selection of interpolation points at $s_1 = 2$, $s_2 = 3.3$, $s_3 = 5$, $s_4 = 6$. We obtain a fourth order model $(\tilde{F}_z, \tilde{G}_z, \tilde{H})$ as in (19), that approximates (14) with an H_∞ norm of the error of $2.406 \cdot 10^{-3}$ lower than the fourth order modal approximation counterpart. Note that the real model does not have any zeros in the right half-plane.

| Reduction method | Reduced order model | L_∞ error norm $\cdot 10^{-3}$ |
|-------------------------------------|----------------------------------|---------------------------------------|
| Moment matching with zero placement | K_{red} in (18), ord. 4 | 2.200 |
| Moment matching with zero placement | K_{red} in (19), ord. 4 | 2.406 |
| Modal truncation | K_{mod}^4 | 6.975 |

TABLE I

L_∞ NORM OF THE APPROXIMATION ERROR: FOURTH ORDER MOMENT MATCHING BASED MODEL VS. FOURTH ORDER MODAL TRUNCATION.

Increasing the number of matched moments by one, and properly selecting the interpolation points and the zeros to be placed yields a low order, more accurate approximation of (14), i.e., the infinity norm of the error is much lower than the 16th order modal truncation counterpart. To this end, let $s_1 = 3$, $s_2 = 5.4$, $s_3 = 7.3$, $s_4 = j$ and $s_5 = 34j$. Then the fifth order model that matches the moments of (14) at s_i , $i = 1, \dots, 5$ and has four zeros at $-(3/4) \cdot 25\pi^2$, $-(3/4) \cdot 49\pi^2$, $-(3/4) \cdot 121\pi^2$ and $-(3/4) \cdot 169\pi^2$ and a pole

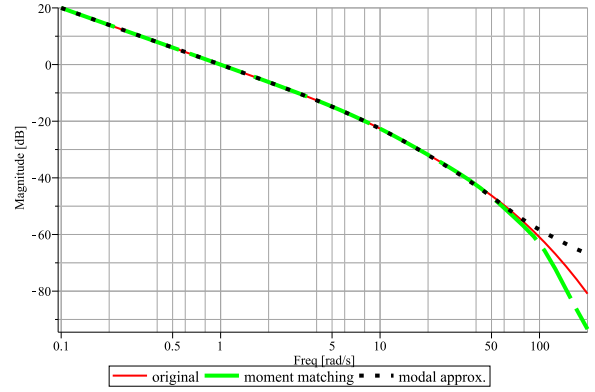


Fig. 3. Magnitude plot of the original system, fifth order model that matches the prescribed moments, preserving the pole $\lambda = 0$ and the zeros $z_1 = -\frac{75\pi^2}{4}$, $z_2 = -\frac{147\pi^2}{4}$, $z_3 = -\frac{363\pi^2}{4}$ and $z_4 = -\frac{507\pi^2}{4}$, and 16th order modal truncation.

at 0 is given by $(\tilde{F}_z, \tilde{G}_z, \tilde{H})$ as in (20) (see Fig. 3). The infinity norm of the approximation error is $3.415 \cdot 10^{-4} < 4.106 \cdot 10^{-4}$, the approximation error yielded by the 16th order modal truncation.

| Reduction method | Reduced order model | L_∞ error norm $\cdot 10^{-3}$ |
|-------------------------------------|----------------------------------|---------------------------------------|
| Moment matching with zero placement | K_{red} in (20), ord. 5 | 0.342 |
| Modal truncation | K_{mod}^{16} | 0.411 |

TABLE II

L_∞ NORM OF THE APPROXIMATION ERROR: FIFTH ORDER MOMENT MATCHING BASED MODEL VS. 16TH ORDER MODAL TRUNCATION.

V. CONCLUSIONS

In this paper we have addressed the problem of moment matching for a class of single-input, single-output, boundary control systems, based on the unique solution of an operator Sylvester equation. The result is a class of parameterized, finite-dimensional, reduced order models that match a set of prescribed moments of the given system. The moments can be obtained by any efficient numerical procedure, e.g., Krylov methods. Further theoretical analysis will be addressed in the full paper.

We consider the same example as in [12]. In [12] (see also [2]) it has been shown that the modal approximation of a boundary control system, unfortunately, may have zeros in the right-half plane. It has also been found that Finite Element Methods exhibit the same problem. We have shown that by placing not only some of the poles of the system, but also zeros, one can obtain a low order approximation that yields an infinity norm of the error, lower than high order modal truncation. Thus, contrary to modal truncation or finite-element methods, where the zeros of the approximation may widely differ from the original zeros, the proposed method allows for the zeros to be placed at the original values. We consider that the numerical example is very interesting and convincing.

$$\tilde{F}_z = \begin{bmatrix} -327.2739770 - 1149.348889j & -330.2739770 - 1149.348889j & -330.2739770 - 1149.348889j & -330.2739770 - 1149.348889j \\ 1112.005919 + 2609.250553j & 1116.405919 + 2609.250553j & 1112.005919 + 2609.250553j & 1112.005919 + 2609.250553j \\ -1006.133641 - 1412.496738j & -1006.133641 - 1412.496738j & -998.833641 - 1412.496738j & -1006.133641 - 1412.496738j \\ 164.0248901 - 58.10766043j & 164.0248901 - 58.10766043j & 164.0248901 - 58.10766043j & 164.0248901 - 58.10766043j \end{bmatrix},$$

$$\tilde{G}_z = \begin{bmatrix} 330.2739770 + 1149.348889j \\ -1112.005919 - 2609.250553j \\ 1006.133641 + 1412.496738j \\ -164.0248901 + 58.10766043j \end{bmatrix}, \quad \tilde{H} = [0.2470 \ 0.1491 \ 0.0715 \ -0.0585 - 0.0454j], \quad (18)$$

$$K_{\text{red}}(s) \approx \frac{8.762 \cdot 10^{-4} s^3 + (0.204 + 0.0024j)s^2 + (132.987 + 1.556j)s + 181.551 + 212.59j}{s^4 + (45.677 + 0.703j)s^3 + (2148.403 + 24.223j)s^2 + (18168.551 + 212.59j)s}$$

$$\tilde{F}_z = \begin{bmatrix} 2810.461757 & 2808.461757 & 2808.461757 & 2808.461757 \\ -13791.92164 & -13788.62164 & -13791.92164 & -13791.92164 \\ 28606.76226 & 28606.76226 & 28611.76226 & 28606.76226 \\ -17683.27881 & -17683.27881 & -17683.27881 & -17677.27881 \end{bmatrix}, \quad \tilde{G}_z = \begin{bmatrix} -2808.461757 \\ 13791.92164 \\ -28606.76226 \\ 17683.27881 \end{bmatrix}, \quad \tilde{H} = [0.4068 \ 0.2186 \ 0.1248 \ 0.096], \quad (19)$$

$$K_{\text{red}}(s) = \frac{8.471 \cdot 10^{-4} s^3 + 0.1975s^2 + 128.5638s + 17564.1679}{s^4 + 43.6764s^3 + 2079.5636s^2 + 17564.1679s}$$

$$\tilde{F}_z = \begin{bmatrix} -3851.6512 - 6820.0181j & -3854.6512 - 6820.0181j & -3854.6512 - 6820.0181j & -3854.6512 - 6820.0181j & -3854.6512 - 6820.0181j \\ 15132.7243 + 17164.3748j & 15138.1243 + 17164.3748j & 15132.7243 + 17164.3748j & 15132.7243 + 17164.3748j & 15132.7243 + 17164.3748j \\ -11020.9048 - 10210.8055j & -11020.9048 - 10210.8055j & -11013.6048 - 10210.8055j & -11020.9048 - 10210.8055j & -11020.9048 - 10210.8055j \\ -497.5651 - 6.8116j & -497.5651 - 6.8116j & -497.5651 - 6.8116j & -497.5651 - 6.8116j & -497.5651 - 6.8116j \\ 138.4881 - 66.2294j & 138.4881 - 66.2294j & 138.4881 - 66.2294j & 138.4881 - 66.2294j & 138.4881 - 66.2294j \end{bmatrix},$$

$$\tilde{G}_z = \begin{bmatrix} 3854.6512 + 6820.0181j \\ -15132.7243 - 17164.3748j \\ 11020.9048 + 10210.8055j \\ 497.5651 + 6.8116j \\ -138.4881 + 66.2294j \end{bmatrix}, \quad \tilde{H} = [0.24702 \ 0.1119 \ 0.0715 \ -0.11 - 0.9894j - 0.0094 + 0.0042j], \quad (20)$$

$$K_{\text{red}}(s) = \frac{(2.0385 + 1.892j) \cdot 10^{-5} s^4 - 0.5(1+j)s^3 - (48.1801 - 44.7174j)s^2 - (15448.878 - 14338.5602j)s - (1.5331 - 1.4229j) \cdot 10^6}{s^5 + (86.2088 - 95.51j)s^4 + (4366.9356 - 3967.022j)s^3 + (1.8568 - 1.7253j) \cdot 10^5 s^2 + (1.5331 - 1.4229j) \cdot 10^6 s}$$

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