2013 European Control Conference (ECC)
July 17-19, 2013, Zürich, Switzerland.

Stabilizing Linear Model Predictive Control: On the Enlargement of the Terminal Set

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Abstract— **It is well known that a large terminal set leads to a large region where the MPC problem is feasible without the need for a long prediction horizon. This paper proposes a new method for the enlargement of the terminal set. Contrary to existing approaches, the method uses the convex hull of a trajectory as the basis for the construction. This trajectory may be any feasible trajectory of the system terminating in an invariant set around the origin and is not restricted to consist only of equilibrium points. The method is applied in an MPC scheme where the terminal set is calculated at initialization.**

I. Introduction

The terminal set is a crucial design parameter in stabilizing MPC. Usually, it is an invariant set for the closed loop consisting of the system and a linear controller. Satisfaction of the terminal constraint guarantees that the predicted trajectory can be extended by a convergent trajectory. However, the inclusion of a terminal constraint limits the region where the MPC problem is feasible, and, in turn, the region of attraction of the closed loop system. The reason for this is that the terminal constraint requires that the terminal set is reached in a number of steps equal to the prediction horizon. Clearly, a larger terminal set leads to a larger region of attraction for a fixed horizon length.

There exist many approaches to the problem of enlarging the region of attraction for terminal set MPC. In [1], a contractive time-varying terminal constraint is proposed, where the terminal set for a given sampling instant is not necessarily invariant itself, but is a subset of the one-step controllable set of the terminal set at the next sampling instant. The authors of [2] propose an approach where the unstable states of the system are constrained to the set of states which can be controlled to the terminal set in a fixed number of steps. The authors of [3] propose to constrain the terminal state to a set of states which are controlled to the terminal set in a fixed number of steps by a local controller.

A simple method of achieving a large terminal set is to use a detuned local control law with the drawback of loosing local optimality [4]. Interpolation based MPC is a way to tackle the dilemma of having a large terminal region with local sub-optimality on the one hand and a small terminal region with guaranteed local optimality on the other hand. Instead of using only one terminal set based on one linear feedback law, the terminal set in interpolation based MPC is defined as the convex hull of multiple terminal sets based on different controllers. Some of these controllers may be tuned for local optimality, others for a large region of attraction. See, e.g., [4]–[6]. The resulting local controller is a convex interpolation of multiple linear feedback laws and is consequently nonlinear. These approaches extend the region of attraction while maintaining some level of local optimality.

In [7] it was shown that the convex hull of states on a trajectory that terminates at the origin is control invariant. The control law guaranteeing invariance is a convex combination of the inputs associated with the states on the trajectory.

Based on this insight, we propose an enlarged terminal set defined as the convex hull of multiple sets which are scaled and translated instances of one invariant kernel set. The translations are defined by the states on a trajectory terminating in a non-translated set around the origin. Refer to Fig. 1 for an illustration.

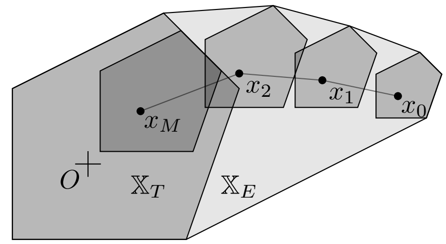

Fig. 1. The enlarged terminal set $\mathbb{X}_E$ is defined as the convex hull of several translated and scaled instances of some invariant set $\mathbb{X}_T$, where the translations are defined by a trajectory of the system.

The terminal cost function on this set is defined as an interpolation between the values of a cost function defined on the translated sets. Similarly, the local controller is defined as an interpolation of controllers defined on the translated sets.

In contrast to the interpolation approaches in [4]–[6], the individual sets between which the interpolation takes place are not required to contain the origin and are not required to be invariant under the individual feedback laws.

Note that a similar approach was pursued in [8]. Therein, a control strategy is presented where the control input is calculated by finding an interpolation between precomputed trajectories that minimizes the resulting interpolated cost. However, a potentially very large number of trajectories has to be computed offline in order to guarantee that the measured state can be expressed as a combination of the starting points of these trajectories. Furthermore, the method to compute the cost proposed in [8] requires a number of calculations that grows exponentially with the number of

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precomputed trajectories.

Another approach related to the one in this work was presented in [9], where the steady state is an additional decision variable and the terminal state is required to be contained in an invariant set around this steady state, effectively making the union of infinitely many invariant sets around points in a set of equilibria the terminal set. This concept was extended for example in [10]–[12] to nonlinear systems. In contrast, the terminal set proposed in this paper is constructed from states on trajectories which are not necessarily equilibria.

The enlarged terminal set presented in this paper shares some properties with the tubes employed in tube MPC and especially with the tubes in [13] which are based on homothety. That is, similar to tube MPC, the local controller ensures that the system state remains in a tube around a predicted trajectory. Compare also [14] and [15] which use similar concepts in the context of linear parameter varying systems and ellipsoidal sets.

As in the interpolation approaches in [4]–[6], we expect little loss of local optimality for the MPC scheme as the invariant set based on the linear (optimal) controller is contained in the terminal set. A rigorous sub-optimality estimate is, however, beyond the scope of this work.

Based on this new type of terminal set, we propose an MPC algorithm where the terminal set is calculated online at initialization. While this calculation requires itself the solution of an MPC problem, the horizon of the MPC problem solved at every time step may have a far shorter horizon, greatly reducing the computational burden thereafter.

The remainder of the paper is organized as follows. The introductory section concludes with a few notes on notation. In Section II the problem setup and basic assumptions are presented. In Section III the construction of the terminal set and the terminal cost function is described. Further it is shown that this terminal set and terminal cost function can be used to construct a stabilizing MPC scheme. In Section IV the MPC scheme with online calculation of the terminal constraint is presented. Section V contains some notes on convexification. In Section VI the complexity and optimality of the algorithm are discussed. In Section VII the effectiveness of the approach is demonstrated in an example. Section VIII concludes the paper.

Notation: For sets $\mathbb{Y}, \mathbb{Z} \subset \mathbb{R}^n$, a scalar $a \in \mathbb{R}$ and a matrix $A \in \mathbb{R}^{m\times n}$ define $a\mathbb{Y} = \{ay | y \in \mathbb{Y}\}$, $A\mathbb{Y} = \{Ay | y \in \mathbb{Y}\}$, and the Minkowski set addition $\mathbb{Y} \oplus \mathbb{Z} = \{y + z | y \in \mathbb{Y}, y \in \mathbb{Y}\}$. Let $\operatorname{convh}(\mathbb{Y})$ denote the convex hull of $\mathbb{Y}$. Let $\mathbf{1}$ denote a column vector of ones of a dimension that is clear from the context.

II. Setup and Definitions

We consider discrete-time linear systems of the form

$$x_{k+1} = Ax_k + Bu_k \tag{1}$$

subject to the constraints

$$x_k \in \mathbb{X} \subset \mathbb{R}^n, \quad u_k \in \mathbb{U} \subset \mathbb{R}^m, \tag{2}$$

for all $k \geq 0$. Define the kernel set $\mathbb{X}_T \subset \mathbb{R}^n$, the kernel controller $\kappa_T(x) := Kx$ with $K \in \mathbb{R}^{m\times n}$, the kernel cost function $V_T(\cdot) : \mathbb{R}^n \to \mathbb{R}$ and the stage cost function $l(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$. The goal is to asymptotically stabilize the origin of (1) while satisfying (2).

For the remainder of this work the following assumptions are required to hold:

Assumption 1: The sets $\mathbb{X}$, $\mathbb{X}_T$ and $\mathbb{U}$ are convex, compact, have a non-empty interior and contain the origin in their interior.

Assumption 2: It holds that $\mathbb{X}_T \subseteq \mathbb{X}$ and $K\mathbb{X}_T \subseteq \mathbb{U}$. Further, it holds that the set $\mathbb{X}_T$ is positively invariant for the dynamics $(A + BK)$, that is $(A + BK)\mathbb{X}_T \subseteq \mathbb{X}_T$.

Assumption 3: The stage cost $l(\cdot,\cdot)$ is positive definite and convex, that is, for all $x, y \in \mathbb{R}^n$, all $u, v \in \mathbb{R}^m$ and all $\lambda \in [0, 1]$ it holds that $l\big(\lambda x + (1 - \lambda)y, \lambda u + (1 - \lambda)v\big) \leq \lambda l(x, u) + (1 - \lambda)l(y, v)$.

Assumption 4: The kernel cost function $V_T(\cdot)$ is positive definite and convex. Further, for all $x \in \mathbb{X}_T$ it holds that $V_T\big((A + BK)x\big) \leq V_T(x) - l(x, Kx)$.

Assumption 5: There exists a real number $r \geq 1$ such that for all scalars $c \geq 0$, all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$ it holds that $V_T(cx) \leq c^r V_T(x)$ and $l(cx, cu) \leq c^r l(x, u)$.

Remark 1: Assumption 5 is trivially satisfied for positively homogeneous functions of degree r, e.g., for functions defined by vector norms with $r = 1$ and for quadratic functions with $r = 2$.

Remark 2: From Assumptions 3, 4 and 5 it follows that for all scalars $c \geq 0$, all $x, y \in \mathbb{R}^n$ and all $u, v \in \mathbb{R}^m$ it holds that

$$\begin{aligned} V_T(x + cy) &= V_T\left((1 + c)\left(\frac{1}{1 + c}x + \frac{c}{1 + c}y\right)\right) \\ &\leq (1 + c)^r V_T\left(\frac{1}{1 + c}x + \frac{c}{1 + c}y\right) \\ &\leq (1 + c)^{r-1}\big(V_T(x) + cV_T(y)\big), \end{aligned} \tag{3}$$

where in the last line the convexity of $V_T(\cdot)$ was exploited. Similarly, it holds that

$$l\big(x + cy, u + cv\big) \leq (1 + c)^{r-1}\big(l(x, u) + cl(y, v)\big). \tag{4}$$

III. Construction of the Enlarged Terminal Set and the Terminal Cost

The enlarged terminal set $\mathbb{X}_E$ is defined as the convex hull of instances of the set $\mathbb{X}_T$ which are translated by the states on the trajectory $(\mathbf{x}^\star_M, \mathbf{u}^\star_{M-1}) = ((x^\star_0, \dots, x^\star_M), (u^\star_0, \dots, u^\star_{M-1}))$ of system (1) and scaled by the factors $\mathbf{c}^\star_M := (c^\star_0, \dots, c^\star_M)$. See Fig. 1 for an example of such a terminal set.

The set of valid trajectories is given by

$$\begin{aligned} \mathcal{D}^\star_M := \big\{(\mathbf{x}_M, \mathbf{u}_{M-1})\big| x_M \in \mathbb{X}_T,\ \forall i = 0, \dots, M - 1 :& \\ (x_i, u_i) \in \mathbb{X} \times \mathbb{U},\ x_{i+1} = Ax_i + Bu_i\big\}.& \end{aligned} \tag{5}$$

In order to guarantee $\mathbb{X}_E \subseteq \mathbb{X}$, satisfaction of the input constraints for the local controller and invariance of $\mathbb{X}_E$, the constraints

$$
\begin{aligned}
c_i &\geq 0 && i = 0, \ldots, M && \text{(6a)}\\
c_i &\leq c_{i+1} && i = 0, \ldots, M-1 && \text{(6b)}\\
\{x_i^\star\} \oplus c_i \mathbb{X}_T &\subseteq \mathbb{X} && i = 0, \ldots, M-1 && \text{(6c)}\\
\{u_i^\star\} \oplus c_i K \mathbb{X}_T &\subseteq \mathbb{U} && i = 0, \ldots, M-1 && \text{(6d)}\\
\{x_M^\star\} \oplus c_M \mathbb{X}_T &\subseteq \mathbb{X}_T && && \text{(6e)}
\end{aligned}
$$

are imposed on the scaling factors. Summarizing, the constraint set for the scaling factors is defined as

$$
\mathcal{D}_M^S(\mathbf{x}_M^\star, \mathbf{u}_{M-1}^\star) := \big\{\mathbf{c}_M \big| \text{ (6a) to (6e) hold}\big\}. \tag{7}
$$

Remark 3: By the definition of $\mathcal{D}_M^\star$ and the assumptions on the involved sets, there always exists a $\mathbf{c}_M$ such that inequalities (6a) to (6e) are satisfied (that is $c_i = 0$ for all i).

For $(\mathbf{x}_M^\star, \mathbf{u}_{M-1}^\star) \in \mathcal{D}_M^\star$ and $\mathbf{c}_M^\star \in \mathcal{D}_M^S(\mathbf{x}_M^\star, \mathbf{u}_{M-1}^\star)$ the enlarged terminal set $\mathbb{X}_E$ is defined as

$$
\begin{aligned}
\mathbb{X}_E &:= \operatorname{convh}\left(\bigcup_{i=0}^{M-1} \big\{\{x_i^\star\} \oplus c_i^\star \mathbb{X}_T\big\} \cup \mathbb{X}_T \right)\\
&= \Bigg\{ x \in \mathbb{R}^n \Bigg| \exists \lambda_i, \bar{\lambda} \geq 0, \sum_{i=0}^{M-1} \lambda_i + \bar{\lambda} = 1,\\
&\qquad \exists \hat{x}_i, \bar{x} \in \mathbb{X}_T \colon\ x = \sum_{i=0}^{M-1} \lambda_i \left(x_i^\star + c_i^\star \hat{x}_i\right) + \bar{\lambda}\bar{x} \Bigg\}, \qquad (8)
\end{aligned}
$$

which is based on the definition of the convex hull as the union of all convex combinations of states contained in the argument of $\operatorname{convh}(\cdot)$. The following lemma allows the reduction of the numbers of variables in the description of $\mathbb{X}_E$ by aggregating the variables $\hat{x}_i$ into one single variable $\hat{x}$.

Lemma 1: It holds that

$$
\begin{aligned}
\mathbb{X}_E = \Bigg\{ x \in \mathbb{R}^n \Bigg| &\exists \lambda_i, \bar{\lambda} \geq 0, \sum_{i=0}^{M-1} \lambda_i + \bar{\lambda} = 1,\\
&\exists \hat{x}, \bar{x} \in \mathbb{X}_T \colon\ x = \sum_{i=0}^{M-1} \lambda_i \left(x_i^\star + c_i^\star \hat{x}\right) + \bar{\lambda}\bar{x} \Bigg\}. \qquad (9)
\end{aligned}
$$

Proof: Let $\lambda_i \geq 0$ and $\hat{x}_i \in \mathbb{X}_T$ be given. For the case that $\sum_{i=0}^{M-1} \lambda_i c_i^\star = 0$, choose $\hat{x} = 0$. Otherwise, choose

$$
\hat{x} = \frac{\sum_{i=0}^{M-1} \lambda_i c_i^\star \hat{x}_i}{\sum_{i=0}^{M-1} \lambda_i c_i^\star}. \tag{10}
$$

The fact that $\mathbb{X}_T$ is convex and contains the origin implies that in both cases it holds that $\hat{x} \in \mathbb{X}_T$. Furthermore, it holds that $\sum_{i=0}^{M-1} \lambda_i c_i^\star \hat{x}_i = \sum_{i=0}^{M-1} \lambda_i c_i^\star \hat{x}$. It follows that every state contained in the set (8) is also contained in the set (9). On the other hand, if λ_i and $\hat{x} \in \mathbb{X}_T$ are given, choose $\hat{x}_i = \hat{x}$, $i = 0, \ldots, M-1$ to obtain $\sum_{i=0}^{M-1} \lambda_i c_i^\star \hat{x}_i = \sum_{i=0}^{M-1} \lambda_i c_i^\star \hat{x}$. Hence, every state contained in the set (9) is also contained in the set (8).

Therefore, the sets (8) and (9) are identical, which completes the proof. ∎

The λ_i, $\bar{\lambda}$, $\hat{x}$ and $\bar{x}$ are not necessarily unique for a given $x \in \mathbb{X}_E$. Define the set that contains all possible parameterizations $p := (\boldsymbol{\lambda}, \bar{\lambda}, \hat{x}, \bar{x}) = \big((\lambda_0, \ldots, \lambda_{M-1}), \bar{\lambda}, \hat{x}, \bar{x}\big)$ as

$$
\begin{aligned}
\Gamma_M(x) := \Bigg\{ &(\boldsymbol{\lambda}, \bar{\lambda}, \hat{x}, \bar{x}) \in \mathbb{R}^M \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \Bigg| \lambda_i, \bar{\lambda} \geq 0,\\
&\sum_{i=0}^{M-1} \lambda_i + \bar{\lambda} = 1,\ \hat{x}, \bar{x} \in \mathbb{X}_T, x = \sum_{i=0}^{M-1} \lambda_i \left(x_i^\star + c_i^\star \hat{x}\right) + \bar{\lambda}\bar{x} \Bigg\}.
\end{aligned} \tag{11}
$$

It holds that $\mathbb{X}_E = \{x \in \mathbb{R}^n \,|\Gamma_M(x) \neq \emptyset\}$. From the definition of $\mathbb{X}_E$ and $\mathbf{c}_M^\star$ it follows that $\mathbb{X}_E \subseteq \mathbb{X}$.

For given $(\mathbf{x}_M^\star, \mathbf{u}_{M-1}^\star) \in \mathcal{D}_M^\star$ and $\mathbf{c}_M^\star \in \mathcal{D}_M^S(\mathbf{x}_M^\star, \mathbf{u}_{M-1}^\star)$, any $x \in \mathbb{X}_E$ and a $p(x) \in \Gamma_M(x)$, the terminal controller is defined as

$$
\kappa_E(x) := \sum_{i=0}^{M-1} \lambda_i \left(u_i^\star + c_i^\star K \hat{x}\right) + \bar{\lambda} K \bar{x}. \tag{12}
$$

As $\kappa_E(x)$ is a convex combination of inputs contained in $\mathbb{U}$, it holds that $\kappa_E(x) \in \mathbb{U}$ for any $p(x) \in \Gamma_M(x)$. Finally, for any $p(x) \in \Gamma_M(x)$, the terminal cost is defined as

$$
\begin{aligned}
V_E(x) := (1 + c_{M-1}^\star)^{r-1} \sum_{i=0}^{M-1} \lambda_i \Bigg(&\sum_{j=i}^{M-1} l(x_j^\star, u_j^\star)\\
&+ V_T(x_M^\star) + c_i^\star V_T(\hat{x}) \Bigg) + \bar{\lambda} V_T(\bar{x}).
\end{aligned} \tag{13}
$$

Lemma 2: The function $V_E(x)$ is positive definite in the sense that for all $x \in \mathbb{X}_E$ and any $p(x) \in \Gamma_M(x)$ it holds that $V_E(x) \geq 0$ and $V_E(x) = 0$ implies $x = 0$.

Proof: The first statement follows from the positive definiteness of $V_T(\cdot)$ and $l(\cdot, \cdot)$. For the second statement, consider that $V_E(x) = 0$ implies by the positive definiteness of $V_T(\cdot)$ and $l(\cdot, \cdot)$ that $\lambda_i(x_i^\star + c_i^\star \hat{x}) = 0$ for all $i = 0, \ldots, M-1$ and $\bar{\lambda}\bar{x} = 0$. Hence, $V_E(x) = 0$ implies $x = \sum_{i=1}^{M-1} \lambda_i \left(x_i^\star + c_i^\star \hat{x}\right) + \bar{\lambda}\bar{x} = 0$. ∎

For simplicity, we omit the implied dependence of $\kappa_E(x)$ and $V_E(x)$ on $p(x)$. We are not interested in the specific choice of $p(x)$, but only in the existence of a $p(x)$ such that the enlarged terminal set $\mathbb{X}_E$ and the terminal cost function $V_E(\cdot)$ satisfy the standard assumptions used to establish asymptotic stability in terminal set MPC. That is, the terminal set is positively invariant for the closed loop system with the terminal controller and the terminal cost is a Lyapunov function defined on the terminal set. This is expressed in the following lemma.

Lemma 3: For an $x \in \mathbb{X}_E$ let $\kappa_E(x)$ and $V_E(x)$ be given and denote $x^+ = Ax + B\kappa_E(x)$. There always exists a $p(x^+) \in \Gamma_M(x^+)$ such that $V_E(x^+) - V_E(x) \leq -l\big(x, \kappa_E(x)\big)$.

The interested reader is referred to [16] for a proof of this statement.

IV. Online Calculation of the Terminal Set and Model Predictive Control Scheme

In this section we present a control scheme where the enlarged terminal set is computed online at initialization. This terminal set, defined by $\{x \in \mathbb{R}^n \,|\Gamma_M(x) \neq \emptyset\}$, is then used for all sampling instants in the MPC problem $\mathcal{P}_N^0(x)$ with an horizon length N which yields the control input.

For given $(\mathbf{x}_M^\star, \mathbf{u}_{M-1}^\star) \in \mathcal{D}_M^\star$ and $\mathbf{c}_M^\star \in \mathcal{D}_M^S(\mathbf{x}_M^\star, \mathbf{u}_{M-1}^\star)$, the decision variables for the MPC problem are $(\mathbf{x}_N, \mathbf{u}_{N-1}, p) = \big((x_0, \ldots, x_N), (u_0, \ldots, u_{N-1}), (\lambda_0, \ldots, \lambda_{M-1}), \bar{\lambda}, \hat{x}, \bar{x}\big)$, taken from the constraint set

$$\mathcal{D}_N^0(x) := \big\{(\mathbf{x}_N, \mathbf{u}_{N-1}, p)\big| x_0 = x,\ p \in \Gamma_M(x_N) \quad (14)$$
$$\forall i = 0, \ldots, N-1 : (x_i, u_i) \in \mathbb{X} \times \mathbb{U}, x_{i+1} = Ax_i + Bu_i\big\}.$$

The MPC problem $\mathcal{P}_N^0(x)$ takes the form

$$\begin{aligned} &\underset{(\mathbf{x}_N, \mathbf{u}_{N-1}, p)}{\text{minimize}} && \sum_{i=0}^{N-1} l(x_i, u_i) + V_E(x_N) \\ &\text{subject to} && (\mathbf{x}_N, \mathbf{u}_{N-1}, p) \in \mathcal{D}_N^0(x), \end{aligned} \qquad (15)$$

with the solution denoted by $(\mathbf{x}_N^0, \mathbf{u}_{N-1}^0, p^0)$. The control input resulting from the solution of the MPC problem is $\kappa_0(x) = u_0^0(x)$ and the closed loop system is defined by $x_{k+1} = Ax_k + B\kappa_0(x_k)$.

Lemma 4: Let Assumptions 1 to 5 be satisfied. Let further problem $\mathcal{P}_N^0(x)$ be feasible for a given $x \in \mathbb{X}$. Then $\mathcal{P}_N^0(x^+)$ is feasible for $x^+ = Ax + B\kappa_0(x)$. Furthermore, the origin of the closed loop system is asymptotically stable.

Proof Sketch: The proof follows from Lemma 3 and standard arguments in MPC, see for example [17]. Due to space limitations, only a sketch of the proof is given here. The interested reader is referred to [16] for a detailed proof.

Given a solution of $\mathcal{P}_N^0(x)$, use the local controller to extend the solution trajectory by the input $\kappa_E(x_N^0)$ and the state $Ax_N^0 + B\kappa_E(x_N^0)$. By Lemma 3, this state is contained in the terminal set $\mathbb{X}_E$ and there exists a $p\big(Ax_N^0 + B\kappa_E(x_N^0)\big) \in \Gamma_M\big(Ax_N^0 + B\kappa_E(x_N^0)\big)$ such that $V_E\big(Ax_N^0 + B\kappa_E(x_N^0)\big) - V_E\big(x_N^0\big) \leq -l\big(x_N^0, \kappa_E(x_N^0)\big)$. This implies the claimed recursive feasibility property and a decrease of the optimal cost, which in turn implies asymptotic stability of the origin. ■

The question remains how to compute $(\mathbf{x}_M^\star, \mathbf{u}_{M-1}^\star)$ and $\mathbf{c}_M^\star$ which are needed to define $\Gamma_M(x)$. In the following, one possible approach is shown.

The trajectory $(\mathbf{x}_M^\star, \mathbf{u}_{M-1}^\star)$ is chosen as the end piece of the solution of an initial MPC problem $\tilde{\mathcal{P}}_L(x)$ with horizon length L, which will be defined below. The parameter M is chosen such that $M \leq L$. Define the constraint set

$$\tilde{\mathcal{D}}_L(x) := \big\{(\mathbf{x}_L, \mathbf{u}_{L-1})\big| x_0 = x,\ \forall i = 0, \ldots, L-1 :$$
$$(x_i, u_i) \in \mathbb{X} \times \mathbb{U},\ x_{i+1} = Ax_i + Bu_i,\ x_L \in \mathbb{X}_T\big\}. \quad (16)$$

For all $x \in \mathbb{R}^n$ it holds that $\tilde{\mathcal{D}}_L(x) \subseteq \mathcal{D}_L^\star$. Define the optimization problem $\tilde{\mathcal{P}}_L(x)$ by

$$\begin{aligned} &\underset{(\mathbf{x}_L, \mathbf{u}_{L-1})}{\text{minimize}} && \sum_{i=0}^{L-1} l(x_i, u_i) + V_T(x_L) \\ &\text{subject to} && (\mathbf{x}_L, \mathbf{u}_{L-1}) \in \tilde{\mathcal{D}}_L(x). \end{aligned} \qquad (17)$$

Given a solution $(\tilde{\mathbf{x}}_L, \tilde{\mathbf{u}}_{L-1})$ of $\tilde{\mathcal{P}}_L(x)$, choose

$$(x_{M-i}^\star, u_{M-i}^\star) = (\tilde{x}_{L-i}, \tilde{u}_{L-i}),\ i = 1, \ldots, M \qquad (18)$$

and $x_M^\star = \tilde{x}_L$. It holds that $(\mathbf{x}_M^\star, \mathbf{u}_{M-1}^\star) \in \mathcal{D}_M^\star$.

The scaling factors $\mathbf{c}_M^\star$ are calculated by solving the optimization problem $\mathcal{P}_M^S\big((\mathbf{x}_M^\star, \mathbf{u}_{M-1}^\star)\big)$ defined by

$$\begin{aligned} &\underset{\mathbf{c}_M}{\text{minimize}} && -\sum_{i=0}^{M} c_i \\ &\text{subject to} && \mathbf{c}_M \in \mathcal{D}_M^S(\mathbf{x}_M^\star, \mathbf{u}_{M-1}^\star). \end{aligned} \qquad (19)$$

Summarizing, the algorithm reads as follows.

Algorithm 1 MPC with Online Terminal Set Calculation at Initialization

1: obtain initial state x
2: solve problem $\tilde{\mathcal{P}}_L(x)$ and obtain $(\tilde{\mathbf{x}}_L, \tilde{\mathbf{u}}_{L-1})$
3: calculate $(\mathbf{x}_M^\star, \mathbf{u}_{M-1}^\star)$
4: solve problem $\mathcal{P}_M^S\big((\mathbf{x}_M^\star, \mathbf{u}_{M-1}^\star)\big)$ and obtain the scaling factors $\mathbf{c}_M^\star$
5: solve problem $\mathcal{P}_N^0(x)$ and obtain $\mathbf{u}_{N-1}^0$
6: apply $u = u_0^0$ to the system
7: wait for next time instant
8: obtain current state x
9: go to 5.

The following lemma gives a relation between the different trajectory lengths L, M and N that guarantee the initial feasibility of $\mathcal{P}_N^0(x)$.

Lemma 5: If $N \geq L - M$ and $\tilde{\mathcal{P}}_L(x)$ is feasible, then $\mathcal{P}_N^0(x)$ is feasible.

Proof: For $N \leq L$ define the candidate trajectory $\big(\mathbf{x}_N^c, \mathbf{u}_N^c\big) := \big((x_0^c, \ldots, x_N^c), (u_0^c, \ldots, u_{N-1}^c)\big)$ and choose $(x_i^c, u_i^c) = (\tilde{x}_i, \tilde{u}_i)$, $i = 0, \ldots, N-1$ and $x_N^c = \tilde{x}_N$. By assumption it holds that $M - L + N \in [0, M]$ and by (18) it holds that $x_N^c = \tilde{x}_N = x_{M-L+N}^\star \in \mathbb{X}_E$ and, hence, $\mathcal{P}_N^0(x)$ is feasible. For $N > L$ define the candidate trajectory

$$\begin{aligned} &\big(\mathbf{x}_N^c, \mathbf{u}_N^c\big) \qquad (20) \\ &:= \big((\tilde{x}_0, \ldots, \tilde{x}_L, (A+BK)\tilde{x}_L, \ldots, (A+BK)^{N-L}\tilde{x}_L), \\ &\quad (\tilde{u}_0, \ldots, \tilde{u}_L, K\tilde{x}_L, K(A+BK)\tilde{x}_L, \ldots \\ &\qquad \ldots, K(A+BK)^{N-L-1}\tilde{x}_L)\big). \end{aligned}$$

As it holds that $(A + BK)^i \tilde{x}_L \in \mathbb{X}_T$ for all $i \geq 0$, it holds that $\mathcal{P}_N^0(x)$ is feasible. ■

Remark 4: There are many alternative ways to obtain $(\mathbf{x}_M^\star, \mathbf{u}_{M-1}^\star)$. For example, one could drop the constraint $\tilde{x}_0 = x$ in $\tilde{\mathcal{P}}_L(x)$ and replace it by an additional term in the cost that penalizes the distance between $\tilde{x}_0$ and x. This would allow a reduction of the number of decision variables in $\tilde{\mathcal{P}}_L(x)$, as the states at the beginning of the trajectory

$(\tilde{\mathbf{x}}_L, \tilde{\mathbf{u}}_{L-1})$ are discarded anyway. Another option would be to extend the trajectory $(\tilde{\mathbf{x}}_L, \tilde{\mathbf{u}}_{L-1})$ by states and inputs $\big((A+BK)^{i+1}\tilde{x}_L, K(A+BK)^i\tilde{x}_L\big)$, $i = 0, \dots, \hat{L}$, for some $\hat{L} \geq 0$, which are contained in $\mathbb{X}_T$ and $K\mathbb{X}_T$, respectively. This would allow for non-zero $c_i^\star$ in the case that $\tilde{x}_L$ lies on the boundary of $\mathbb{X}_T$.

V. Convexification

In this section we briefly discuss how the the constraints and cost functions involved in the optimization problems $\mathcal{P}_M^S\big((\mathbf{x}_M^\star, \mathbf{u}_{M-1}^\star)\big)$ and $\mathcal{P}_N^0(x)$ can be convexified under the assumption that the sets $\mathbb{X}, \mathbb{U}$ and $\mathbb{X}_T$ are polyhedral. Let

$$\mathbb{X} = \{x \in \mathbb{R}^n | H_x x \leq \mathbf{1}\} \tag{21a}$$

$$\mathbb{U} = \{u \in \mathbb{R}^m | H_u x \leq \mathbf{1}\} \tag{21b}$$

$$\mathbb{X}_T = \{x \in \mathbb{R}^n | H_T x \leq \mathbf{1}\} = \operatorname{convh}\left(\bigcup_{j=1}^{h_V} v_j\right), \tag{21c}$$

with matrices $H_x \in \mathbb{R}^{h_x \times n}$, $H_u \in \mathbb{R}^{h_u \times m}$, $H_T \in \mathbb{R}^{h_T \times n}$ and where v_j are the h_V extreme points of $\mathbb{X}_T$. Note that the constraints (6c) to (6e) are, by the convexity of $\mathbb{X}_T$, equivalent to

$$H_x(x_i^\star + c_i v_j) \leq \mathbf{1} \tag{22a}$$

$$H_u(u_i^\star + c_i K v_j) \leq \mathbf{1} \tag{22b}$$

$$H_T(x_M^\star + c_M v_j) \leq \mathbf{1} \tag{22c}$$

$$i = 0, \dots, M-1, \quad j = 1, \dots, h_V,$$

which are linear inequalities in $\mathbf{c}_M$.

The constraint $x = \sum_{i=1}^{M-1} \lambda_i\,(x_i^\star + c_i^\star \hat{x}) + \bar{\lambda}\bar{x}$ is, however, non-convex. By replacing $p = (\boldsymbol{\lambda}, \bar{\lambda}, \hat{x}, \bar{x})$ and $\Gamma_M(x)$ in $\mathcal{P}_N^0(x)$ with $p' = (\boldsymbol{\lambda}, \bar{\lambda}, \hat{x}', \bar{x}')$ and

$$\Gamma'_M(x) := \Bigg\{ (\boldsymbol{\lambda}, \bar{\lambda}, \hat{x}', \bar{x}') \in \mathbb{R}^M \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \,\Bigg|\, \lambda_i, \bar{\lambda} \geq 0, \sum_{i=0}^{M-1} \lambda_i + \bar{\lambda} = 1,\ \hat{x}' \in \sum_{i=0}^{M-1} \lambda_i c_i^\star \mathbb{X}_T,\ \bar{x}' \in \bar{\lambda}\mathbb{X}_T,\ x = \sum_{i=0}^{M-1} \lambda_i x_i^\star + \hat{x}' + \bar{x}' \Bigg\}, \tag{23}$$

the constraint is made convex.

Lemma 6: For any fixed x, the sets $\Gamma'_M(x)$ and $\Gamma_M(x)$ are equivalent in the sense that

$$\big\{x \in \mathbb{R}^n \big| \Gamma'_M(x) \neq \emptyset\big\} = \big\{x \in \mathbb{R}^n \big| \Gamma_M(x) \neq \emptyset\big\}. \tag{24}$$

The interested reader is referred to [16] for a proof of this statement.

Note that $\hat{x}' \in \sum_{i=0}^{M-1} \lambda_i c_i^\star \mathbb{X}_T$ can be written as $H_T\hat{x}' \leq \sum_{i=0}^{M-1} \lambda_i c_i^\star \mathbf{1}$, which is a linear inequality. The cost function of the MPC problem $\mathcal{P}_N^0(x)$ contains terms of the form $\bar{\lambda}V_T(\bar{x})$ which are non-convex. Consider now two cases.

Case a): The functions $l(\cdot,\cdot)$ and $V_T(\cdot)$ are infinity norms, that is $l(x,u) = \|P_l(x^T, u^T)^T\|_\infty$ and $V_T(x) := \|Px\|_\infty$ for matrices $P_l \in \mathbb{R}^{p_l \times (n+m)}$ and $P \in \mathbb{R}^{p \times n}$ with full column rank. This implies that $l(\cdot,\cdot)$ and $V_T(\cdot)$ are positively homogeneous of degree 1 such that Assumption 5 holds with equality and $r = 1$. In this case, problem $\mathcal{P}_N^0(x)$ is redefined as $\mathcal{P}_N^a(x)$ with $\Gamma'_M(x)$ instead of $\Gamma_M(x)$ and the redefined terminal cost function

$$V_E^a(x) := \sum_{i=0}^{M-1} \lambda_i \left(\sum_{j=i}^{M-1} l(x_j^\star, u_j^\star) + V_T(x_M^\star) \right) + V_T(\hat{x}') + V_T(\bar{x}') \tag{25}$$

instead of $V_E(x)$.

Case b): The functions $l(\cdot,\cdot)$ and $V_T(\cdot)$ are quadratic, that is $l(x,u) = x^TQx + u^TRu$ and $V_T(x) = x^TPx$ for positive definite matrices Q, R, and P. This implies that $l(\cdot,\cdot)$ and $V_T(\cdot)$ are positively homogeneous of degree 2 such that Assumption 5 holds with equality and $r = 2$. In this case problem $\mathcal{P}_N^0(x)$ is redefined as $\mathcal{P}_N^b(x)$ with $\Gamma'_M(x)$ instead of $\Gamma_M(x)$ and the redefined terminal cost function

$$V_E^b(x) := \eta \tag{26}$$

instead of $V_E(x)$, subject to the additional constraint

$$\begin{bmatrix} \eta - d\sum_{i=0}^{M-1} \lambda_i J_i & \hat{x}'^T & \bar{x}'^T \\ \hat{x}' & \frac{1}{d}\sum_{i=0}^{M-1} \lambda_i c_i^\star P^{-1} & 0 \\ \bar{x}' & 0 & \bar{\lambda}P^{-1} \end{bmatrix} \geq 0, \tag{27}$$

where $J_i := \sum_{j=i}^{M-1} l(x_j^\star, u_j^\star) + V_T(x_M^\star)$ and $d := (1 + c_{M-1}^\star)$. The decision variables for $\mathcal{P}_N^b(x)$ are $(\mathbf{x}_N, \mathbf{u}_{N-1}, p', \eta)$.

Both $\mathcal{P}_N^a(x)$ and $\mathcal{P}_N^b(x)$ are convex optimization problems and can be solved efficiently.

Lemma 7: The redefined problems $\mathcal{P}_N^a(x)$ and $\mathcal{P}_N^b(x)$ with $\Gamma'_M(x)$ instead of $\Gamma_M(x)$ and $V_E^a(\cdot)$ or $V_E^b(\cdot)$ respectively instead of $V_E(x)$ are equivalent to problem $\mathcal{P}_N^0(x)$ in the sense that if a trajectory is optimal for one of the problems it is also optimal for the other problems.

Proof Sketch: Due to space limitations, only a sketch of the proof is given here. The interested reader is referred to [16] for a detailed proof.

By Lemma 6, for a given x_N and a $p' \in \Gamma'(x_N)$ there always exists a $p \in \Gamma(x_N)$ and vice versa. This implies that a trajectory that is feasible for one of the problems must also be feasible for the other problems. The equivalence of the cost functions can be checked by evaluating the cost function of one problem for the trajectory that is a solution of one of the other problems. As these values are identical it follows that none of the problems can have a solution with a lower objective function that the other problems and, hence, a trajectory that is optimal for one problem is also optimal for the other problems. ■

VI. Complexity and Optimality

The main advantage of the proposed MPC algorithm is the enlargement of the feasible region without an increase in length of the prediction horizon. On the other hand, additional decision variables have to be included in the optimization problem to describe the enlarged terminal set.

We consider here the case when only the predicted inputs, and not the predicted states, are decision variables, that is, the predicted states are expressed as functions of the initial state and the predicted inputs.

In Table I, the numbers of decision variables and constraints for the MPC problem $\mathcal{P}_N^a(x)$ are compared to those of a "standard" MPC problem $\tilde{\mathcal{P}}_L(x)$, which, by Lemma 5, has the same feasible region as Algorithm 1 if $N \geq L - M$. Here, L is the prediction horizon of the initial MPC scheme, M is the length of the trajectory used to define $\mathbb{X}_E$ and N is the prediction horizon of the MPC scheme used to calculate the control input. As the cost is positively homogeneous of degree 1, only linear inequality and equality constraints appear.

In Table II, a similar comparison for the case of a quadratic cost function is shown, which involves an LMI constraint and an additional decision variable η as indicated in (26) and (27).

TABLE I

NUMBER OF DECISION VARIABLES AND CONSTRAINTS FOR A STANDARD MPC PROBLEM $\tilde{\mathcal{P}}_L(x)$ AND THE PROPOSED MPC PROBLEM $\mathcal{P}_N^a(x)$ FOR INFINITY NORM COST FUNCTIONS

	$\tilde{\mathcal{P}}_L(x)$	$\mathcal{P}_N^a(x)$
number of decision variables	Lm	$Nm + M + 1 + 2n$
number of inequality constraints	$Lh_u + (L-1)h_x + h_T$	$Nh_u + (N-1)h_x + M + 1 + 2h_T$
number of equality constraints	0	$1 + n$

TABLE II

NUMBER OF DECISION VARIABLES AND CONSTRAINTS FOR A STANDARD MPC PROBLEM $\tilde{\mathcal{P}}_L(x)$ AND THE PROPOSED MPC PROBLEM $\mathcal{P}_N^b(x)$ FOR QUADRATIC COST FUNCTIONS

	$\tilde{\mathcal{P}}_L(x)$	$\mathcal{P}_N^b(x)$
number of decision variables	Lm	$Nm + M + 2 + 2n$
number of inequality constraints	$Lh_u + (L-1)h_x + h_T$	$Nh_u + (N-1)h_x + M + 1 + 2h_T$
number of equality constraints	0	$1 + n$
number of decision variables in the LMI constraint	0	$2n + M + 2$
dimension of the LMI constraint	0	$(2n+1) \times (2n+1)$

From Table I it can be seen that the MPC scheme $\mathcal{P}_N^a(x)$ has less decision variables and constraints than the standard MPC scheme $\tilde{\mathcal{P}}_L(x)$ if m and h_u are large and L is large compared to N. The reason for this is that the inputs $\mathbf{u}_{M-1}^\star$ are not used in the definition of the terminal set $\mathbb{X}_T$ and only the scalars $\lambda_i, \bar{\lambda}$ and the variables $\hat{x}$ and $\bar{x}$ are added as decision variables.

The initialization procedure in Algorithm 1 consist of solving $\tilde{\mathcal{P}}_L(x)$ and $\mathcal{P}_M^S\big((\mathbf{x}_M^\star, \mathbf{u}_{M-1}^\star)\big)$ once. For linear constraints $\mathcal{P}_M^S\big((\mathbf{x}_M^\star, \mathbf{u}_{M-1}^\star)\big)$ is a linear program with $M+1$ variables and $h_V(M(h_x + h_u) + h_T)$ constraints. Note that the convex hull which defines the enlarged terminal set $\mathbb{X}_E$ is never computed explicitly, i.e., neither the extreme points nor the hyperplanes defining $\mathbb{X}_E$ have to be computed.

We expect that our proposed MPC scheme maintains local optimality, as for any state $x \in \mathbb{X}_T$ the parameter $p(x)$ can be chosen such that $V_E(x) = V_T(x)$. The factor $(1 + c_{M-1}^\star)^{r-1}$ distorts the cost. However, it can be chosen arbitrarily close to 1 by choosing a small $c_{M-1}^\star$. For a positively homogeneous cost of degree $r = 1$ it also reduces to 1. We expect that the performance of the trajectory $(\mathbf{x}_M^\star, \mathbf{u}_{M-1}^\star)$ affects the performance of the entire MPC algorithm. A rigorous examination of the sub-optimality of the approach is, however, beyond the scope of this work.

Solving the initial problems offline in a parametric fashion or acquiring $(\mathbf{x}_M^\star, \mathbf{u}_{M-1}^\star)$ and $\mathbf{c}_M^\star$ in an other efficient way would further increase the efficiency of the approach.

VII. SIMULATION RESULTS

To illustrate the approach, the proposed method is applied to a double integrator. Let the system matrices be given by

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \tag{28}$$

and the state and input constraint sets by $\mathbb{U} = [-1, 1]$ and $\mathbb{X} = [-20, 20] \times [-20, 20]$, such that $n = 2, m = 1, h_u = 2$ and $h_x = 4$. The feedback matrix K defining the kernel controller is chosen as the solution of the LQR problem $(A + BK)^T P(A + BK) - P = -(Q + K^T RK)$ with $Q = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $R = 1$.

The kernel set $\mathbb{X}_T$ is determined as the maximal admissible set, calculated with the algorithm in [18], for the dynamics $(A + BK)$. After simplification (using, e.g., the Multi-Parametric Toolbox [19]) and rounding to two decimals, the kernel set $\mathbb{X}_T$ can be represented by the matrix

$$H_T = \begin{bmatrix} 0.43 & 1.03 \\ -0.43 & -1.03 \\ -0.11 & 0.18 \\ 0.11 & -0.18 \end{bmatrix}, \tag{29}$$

such that $h_T = 4$.

We define the stage cost as $l(x, u) = x^T Qx + u^T Ru$ and the kernel cost as $V_T(x) = x^T Px$. For these functions, Assumption 5 holds with $r = 2$.

The horizon lengths are chosen as $L = 10$, $M = 9$, $N = 1$ and the initial state is $x_0 = (20, 0)^T$ at the boundary of the state constraint set. In Fig. 2 the region $\tilde{\mathcal{X}}_L$ is shown, where $\tilde{\mathcal{P}}_L(x)$ and in turn the whole algorithm is feasible. Additionally, the sets $\mathbb{X}_T$ and $\mathbb{X}_E$ and the trajectory $\mathbf{x}_M^\star$ are depicted. The scaling factors become $\mathbf{c}_M^\star = (0.00, \ldots, 0.00, 0.74, 0.93)$ after rounding to two decimals.

Remark 5: For one particular set $\mathbb{X}_E$, problem $\mathcal{P}_N^0(x)$ is feasible for all $x \in \tilde{\mathcal{X}}_L$ for any $N \geq L$. However, $\mathcal{P}_N^0(x)$ might be feasible for much lower N, depending on the particular initial condition. As stated in Lemma 5, for the initial condition at which $\mathbb{X}_E$ was constructed, feasibility is ensured for any $N \geq L - M$.

In Fig. 3 trajectories of the closed loop system for the control law $u = \kappa_0(x)$, resulting from $\mathcal{P}_N^0(x)$ are shown for different initial conditions and the respective minimal values for N such that $\mathcal{P}_N^0(x)$ is feasible, but with the same terminal

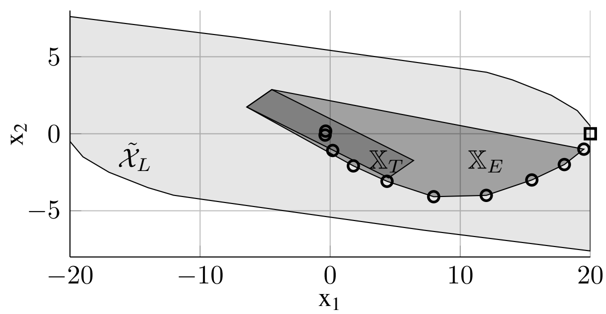

Fig. 2. Feasible region $\tilde{\mathcal{X}}_L$ for the problem $\tilde{\mathcal{P}}_L(x)$, sets $\mathbb{X}_T$ and $\mathbb{X}_E$, and the trajectory $\mathbf{x}^\star_M$ (circles).

set $\mathbb{X}_E$ calculated at $x_0 = (20, 0)^T$. For comparison, the trajectory of the closed loop system for a controller resulting from the standard MPC problem $\tilde{\mathcal{P}}_L(x)$ is shown for the initial condition $x_0 = (20, 0)^T$. This trajectory is almost identical to the trajectory generated by $\mathcal{P}^0_N(x)$ for the same initial condition and $N = 1$.

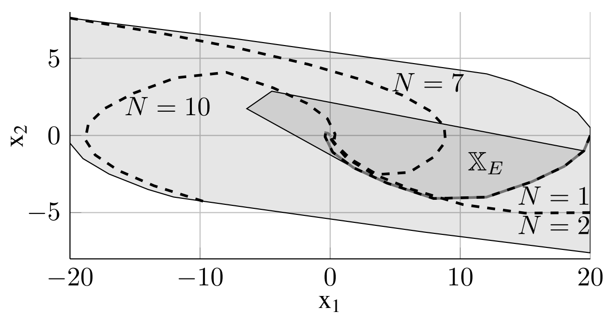

Fig. 3. Trajectories of the closed loop system for the controller resulting from $\tilde{\mathcal{P}}_L(x)$ (gray, solid) and from $\mathcal{P}^0_N(x)$ (black, dashed).

The terminal set can be further enlarged by applying the procedure to obtain $\mathbb{X}_E$ for multiple trajectories with different initial conditions and then taking the convex hull of all resulting enlarged terminal sets. The formal details of this approach are under development.

VIII. Conclusion and Outlook

In this paper a method to construct a terminal set from a given trajectory was proposed. This makes it possible to achieve a large terminal set based on a nonlinear, and possibly optimal, feedback law. This in turn makes it possible to have a large region of attraction of the closed loop system while keeping the number of decision variables and constraints low. As shown in the example, there is very little loss of performance when compared to a standard MPC scheme.

Further research will address the offline parametric solution of the terminal set construction problem. This will make solving the problems $\tilde{\mathcal{P}}_L(x)$ and $\mathcal{P}^S_M\big((\mathbf{x}^\star_M, \mathbf{u}^\star_{M-1})\big)$ online unnecessary, further reducing the computational burden. Another topic to be addressed is the estimation of the sub-optimality of the MPC algorithm.

Acknowledgment

The authors would like to thank Saša V. Raković for the fruitful discussions we had on the topic.

The authors would like to thank the German Research Foundation (DFG) for financial support within the Cluster of Excellence SimTech (EXC 310/1) and within the collaborative research project "Optimization Based Control of Uncertain Systems".

The Multi-Parametric Toolbox [19] and YALMIP [20] were used for the simulations.

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