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Observer-Based Stabilization of Uncertain Linear Systems with Recycle: An LMI Approach

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Abstract

This paper deals with the control of processes having recycle streams. These processes can be modeled as state delayed systems. In this work the problem of robust design of an observed-based controller for these systems is presented. The system is assumed to have norm-bounded uncertainties which are independent in every matrix involved in its state space realization. To ensure asymptotic stability of the closed loop system, a Lyapunov functional is used to obtain delay-independent design conditions. The controller design is accomplished by means of a convex optimization procedure formulated using linear matrix inequalities (LMIs). Numerical examples are provided to illustrate the main characteristics of the proposed design method.

I. INTRODUCTION

Industrial processes with recycle are usually found in the process industries, some examples are very common in chemical plants. The integration of different operational units into one recycle system for economical and environmental benefits forces the simultaneous study of all the units [1]. More recently, [8] has approached similar problems with the transfer function methodology. It is stated there that recycling processes can lead to instability when the feedback gain is larger than unity, as for example the recycle of the energy developed by an exothermic reaction in an adiabatic plug flow reactor for feed preheating.

As previous works had shown, the general view of controlling each unit alone to ensure performance for the complete processes may not be satisfactory in all cases [3]. Moreover, the classical transfer function approach has been used to tackle the design of controllers for this type of systems [2], [1]. However, the design of robust controllers might become a hard task if uncertainties are introduced in both forward and recycle subsystems. Therefore, an alternative to the classical approach might be an useful tool. These processes can also be modeled as a state delayed linear system. Considering all this aspects, the design of controllers for this kind of systems is not a trivial task. The existence of mixed uncertainties together with the presence of time delays represent powerful obstacles for the classical transfer function approach. Even more, it is a common case not to have all the state variables available for feedback. Therefore, design strategies for observed-based controllers represent a research area with a large field of possible applications in the industrial context.

The design of observer based controller has been studied by Ibrir [4], where design conditions are derivated in terms of LMIs for systems having no time delay. In this work we will extend these results, looking forward to solve the problem of designing controllers for systems with recycles. To achieve that, we will consider a single-delay state model that can be used to represent a recycle process. Moreover, in order to make the model more realistic, it will be considered that the system is affected by norm-bounded uncertainties which can represent, as for instance, measurement errors or time varying parametric uncertainties. As a result of this research, new systematic tools will be available for the design of this kind of controllers. Furthermore, as the main result of this work is a set of convex optimization constrains, they can be solved through any available LMI solver as the SeDuMi [5] interfaced by the parser YALMIP [6].

This work is organized as follows: Section 1 states the problem. Section 2 provides some preliminary results, useful to prove the main result of this work. Section 3 presents the observer based controllers. The performance of this controller is illustrated by means of an example in Section 4. Finally some conclusion and future research issues are drawn in Section 5.

Through this paper, matrix inequalities such as $A<0$ $(A>0)$ are used to indicate that matrix A is symmetric negative (or positive) definite. The notation "$\star$" is used to indicate symmetric block within a matrix inequality. Identity matrix and null matrix are represented respectively by $\mathbf{I}$ and $\mathbf{0}$.

II. PROBLEM STATEMENT

Let us consider the recycle system in Fig. 1. State space representations of the subsystems H_1 and H_2 may be:

$$H_1: \quad \begin{aligned} \dot{x}_1(t) &= A_1(\cdot)x_1(t) + B_1(\cdot)u_1(t-\theta_1) \\ y(t) &= C_1 x_1(t) \end{aligned}$$

$$H_2: \quad \begin{aligned} \dot{x}_2(t) &= A_2(\cdot)x_2(t) + B_2(\cdot)u_2(t-\theta_2) \\ y_2(t) &= C_2 x_2(t) \end{aligned}$$

H_1 represents the main process to be observed. H_2 is a recycle stream that takes the output of system H_1 and adds it dynamically to its input.

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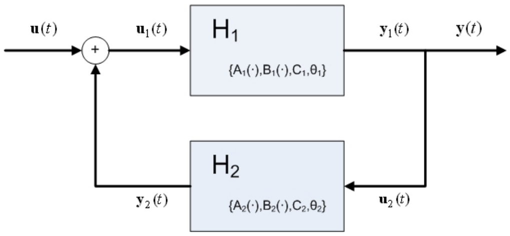

Fig. 1. Recycle Plant

The notation $M(\cdot)$ is for uncertain matrices. This uncertainties may be due, for example, measurement errors of the parameters. In that case an arbitrary uncertain matrix may be represented as $M(\cdot) = M + \Delta M(t)$. With M a precisely known matrix and $\Delta M(t)$ the time varying uncertain component.

In this paper, a particular case of the recycle system will be considered. For simplicity of the results, only one time delay will be considered. This is $\theta_1 = 0$ and $\theta_2 = \tau$. This assumption leads to an overall transfer function between input $u(s)$ and output $y(s)$ with an exponential term in the denominator, a case difficult to treat with conventional control design procedures.

With $x(t) = [x_1(t)x_2(t)]'$, and taking into account that $u_1(t) = y_2(t) + u(t)$ and $u_2(t) = y(t)$, the whole system may be represented as:

$$\begin{aligned}\dot{x}(t) =& \begin{bmatrix} A_1 & B_1C_2 \\ \mathbf{0} & A_2 \end{bmatrix} x(t) + \begin{bmatrix} \Delta A_1(t) & \Delta B_1(t)C_2 \\ \mathbf{0} & \Delta A_2(t) \end{bmatrix} x(t) \\ &+ \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ B_2C_1 & \mathbf{0} \end{bmatrix} x(t-\tau) + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \Delta B_2(t)C_1 & \mathbf{0} \end{bmatrix} x(t-\tau) \\ &+ \begin{bmatrix} B_1 \\ \mathbf{0} \end{bmatrix} u(t) + \begin{bmatrix} \Delta B_1(t) \\ \mathbf{0} \end{bmatrix} u(t) \\ y(t) =& \begin{bmatrix} C_1 & \mathbf{0} \end{bmatrix} x(t)\end{aligned}$$

This is indeed a particular case of the system described in equation (1).

$$\begin{aligned}\dot{x}(t) =& (A + \Delta A(t))x(t) + (D + \Delta D(t))x(t-\tau) \\ &+ (B + \Delta B(t))u(t) \\ y(t) =& Cx(t)\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is a state vector, $u(t) \in \mathbb{R}^m$ is a control input and τ is an unknown time delay of the system. The matrices $A \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ are constant known matrices. The uncertain matrices may be defined as follow

$$\begin{aligned}\Delta A(t) = M_A F_A(t) N_A: &\quad F_A(t)'F_A(t) \leq \gamma_A{}^2\mathbf{I} \\ \Delta B(t) = M_B F_B(t) N_B: &\quad F_B(t)'F_B(t) \leq \gamma_B{}^2\mathbf{I} \\ \Delta D(t) = M_D F_D(t) N_D: &\quad F_D(t)'F_D(t) \leq \gamma_D{}^2\mathbf{I}\end{aligned}$$

Where M_A, M_B, M_D, N_A, N_B and N_D, are matrices of proper dimensions and $\gamma_A \in \mathbb{R}$, $\gamma_B \in \mathbb{R}$ and $\gamma_D \in \mathbb{R}$. Here on, for simplicity, the time dependence indicator (t) next to the F-matrices will be omitted.

There are no uncertainties in the output matrix C, because it is a common case that the output is just a linear combination of a subset of the state variables. Further treatment of the case where a matrix ΔC is included, can be found in [4] and [9]. On the contrary, uncertainties are considered around the B matrix. This can be regarded as a more realistic case from the system point of view.

For this system, the main objective is to find a stabilizing controller $u(t) = K\hat{x}(t)$ such that system (1) is asymptotically stable. The variable $\hat{x}(t)$ is the state vector of the following observer

$$\begin{aligned}\dot{\hat{x}}(t) =& A\hat{x}(t) + D\hat{x}(t-\tau) + Bu(t) \\ &+ Ly(t) - LC\hat{x}(t)\end{aligned} \tag{2}$$

The error between the state vector $x(t)$ and the observed state vector $\hat{x}(t)$ can be defined as $\tilde{x}(t) = x(t) - \hat{x}(t)$. Therefore, system (1) in closed loop and the dynamics of the error can be described by the following expressions

$$\begin{aligned}\dot{x}(t) =& (A + \Delta A + BK + \Delta BK)x(t) \\ &+ (D + \Delta D)x(t-\tau) - (BK + \Delta BK)\tilde{x}(t)\end{aligned} \tag{3}$$

$$\begin{aligned}\dot{\tilde{x}}(t) =& (\Delta A + \Delta BK)x(t) + \Delta Dx(t-\tau) \\ &+ (A - LC - \Delta BK)\tilde{x}(t) + D\tilde{x}(t-\tau)\end{aligned} \tag{4}$$

The problem to be solved is to find matrices $K \in \mathbb{R}^{m\times n}$ and $L \in \mathbb{R}^{n\times p}$ such that the system described by expressions (3) and (4) is asymptotically stable.

III. PRELIMINARY RESULTS

In order to prove the main result of this work, several results and theorems are needed. First, a well known fact is given by:

Proposition 1: For given vectors $v \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, a constant matrix M of appropriate dimensions and for any $\mu > 0$, the following statement is true

$$\left(\frac{1}{\sqrt{\mu}}v - \sqrt{\mu}Mw\right)'\left(\frac{1}{\sqrt{\mu}}v - \sqrt{\mu}Mw\right) \geq 0$$

By factorization we obtain:

$$v'Mw + w'M'v \leq \frac{1}{\mu}v'v + \mu w'M'Mw$$

From this, the following lemmas can be easily obtain:

Lemma 1: For arbitrary matrices M_X, F_X and N_X of proper dimensions, $P > 0$ and vectors $y, z \in \mathbb{R}^n$; if $F_X'F_X \leq \gamma_x{}^2\mathbf{I}$, then

$$\begin{aligned}&y'N_X'F_X'M_X'Pz + z'PM_XF_XN_Xy \leq \\ &\qquad \gamma_x{}^2\frac{1}{\mu_X}y'N_X'N_Xy + \mu_X z'PM_XM_X'Pz\end{aligned} \tag{5}$$

Proof: Identifying $v = F_XN_Xy$, $M = M_X'P$ and $w = z$, with the help of *Proposition 1* the following can be written

$$\begin{aligned}&y'N_X'F_X'M_X'Pz + z'PM_XF_XN_Xy \leq \\ &\qquad \frac{1}{\mu_X}y'N_X'F_X'F_XN_Xy + \mu_X z'PM_XM_X'Pz\end{aligned}$$

Taking into account that $F_X'F_X \leq \gamma_x{}^2\mathbf{I}$, we obtain easily the expression (5).

Lemma 2: For arbitrary matrices K, M_X, F_X and N_X of proper dimensions, $P > 0$ and vectors $y, z \in \mathbb{R}^n$; if $F_X'F_X \leq \gamma_x{}^2\mathbf{I}$, then

$$\begin{aligned}&y'K'N_X'F_X'M_X'Pz + z'PM_XF_XN_XKy \leq \\ &\qquad \gamma_x{}^2\frac{1}{\mu_X}y'K'N_X'N_XKy + \mu_X z'PM_XM_X'Pz\end{aligned}$$

Proof: Similar to preceding proof, but identifying $v = F_XN_XKy$, $M = M_X'P$ and $w = z$.

Lemma 3: For given vectors $y,z \in \mathbb{R}^n$ and constant matrices $P > 0$, B and K of appropriate dimensions we have

$$-y'PBKz - z'K'B'Py \leq \frac{1}{\mu_K} z'K'Kz + \mu_K y'PBB'Py, \mu_K > 0$$

Proof: Identifying $v = -Kz$, $M = B'P$ and $w = y$ and using *Proposition 1*.

Lemma 4: For any $\varepsilon > 0$ and for $P > 0$, such that $P - \varepsilon\mathbf{I}$ is a full rank matrix,

$$\varepsilon^2 P^{-1} > -P + 2\varepsilon\mathbf{I}$$

Proof: Proof can be found in [4].

IV. OBSERVED-BASED CONTROLLER

The main result of this work can be summarized in the following theorem.

Theorem 1: If there exist real $n \times n$ symmetric matrices $X > 0$, $Q' > 0$, $X_1' > 0$, $Q_1' > 0$, two real matrices $Z \in \mathbb{R}^{m\times n}$ and $W \in \mathbb{R}^{n\times p}$, and strictly positive constants μ_{PA}, μ_{PB}, μ_{PD}, μ_{QA}, μ_{QB}, μ_{QD}, μ_K, α and β such that linear matrix inequalities (6), (7), (8) and (9) hold. Then, the recycling system (1) and observer (2) are stabilized by the feedback controller $u(t) = K\hat{x}(t)$, and $L = Q^{-1}W$, $K = ZP$, with $P = X^{-1}$, regardless of the value of the delay.

$$\begin{bmatrix} -X & \mathbf{I} \\ \star & -(2\beta - \alpha)\mathbf{I} \end{bmatrix} < 0 \tag{6}$$

$$\begin{bmatrix} -\frac{\mu_{PB}}{\gamma_B{}^2}\mathbf{I} & \mathbf{0} & \mathbf{0} & N_B Z \\ \star & -\frac{\mu_{QB}}{\gamma_B{}^2}\mathbf{I} & \mathbf{0} & N_B Z \\ \star & \star & -\mu_K\mathbf{I} & Z \\ \star & \star & \star & -\alpha\mathbf{I} \end{bmatrix} < 0 \tag{7}$$

$$\begin{bmatrix} \mathscr{Q}_{11} & QD & QM_A & QM_B & QM_D & \beta\mathbf{I} & \mathbf{0} \\ \star & -Q_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \beta\mathbf{I} \\ \star & \star & -(2-\mu_{QA})\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & -\frac{2-\mu_{QB}}{2}\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & -(2-\mu_{QD})\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & \star & -X & \mathbf{0} \\ \star & \star & \star & \star & \star & \star & -X \end{bmatrix} < 0 \tag{8}$$

$$\mathscr{Q}_{11} = A'Q + QA - C'W' - WC + Q_1$$

$$\begin{bmatrix} \mathscr{P}_{11} & DX & XN_A' & XN_A' & Z'N_B' & Z'N_B' & \mathbf{0} & \mathbf{0} \\ \star & -X_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & XN_D' & XN_D' \\ \star & \star & -\frac{\mu_{PA}}{\gamma_A{}^2}\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & -\frac{\mu_{QA}}{\gamma_A{}^2}\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & -\frac{\mu_{PB}}{\gamma_B{}^2}\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & \star & -\frac{\mu_{QB}}{\gamma_B{}^2}\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & \star & \star & -\frac{\mu_{PD}}{\gamma_D{}^2}\mathbf{I} & \mathbf{0} \\ \star & \star & \star & \star & \star & \star & \star & -\frac{\mu_{QD}}{\gamma_D{}^2}\mathbf{I} \end{bmatrix} < 0 \tag{9}$$

$$\mathscr{P}_{11} = XA' + AX + Z'B' + BZ + X_1 + \mu_{PA}M_AM_A' + 2\mu_{PB}M_BM_B' + \mu_{PD}M_DM_D' + \mu_K BB'$$

Proof: A proposed Lyapunov function to prove stability of system in equations (3) and (4) is as follow

$$v(x,\tilde{x},t) = x(t)'Px(t) + \int_0^\tau x(t-s)'P_1x(t-s)ds + \tilde{x}(t)'Q\tilde{x}(t) + \int_0^\tau \tilde{x}(t-s)'Q_1\tilde{x}(t-s)ds$$

with $P > 0, Q > 0, P_1 > 0, Q_1 > 0$. Recall that this functional implies no restriction on the delay value, but merely on its existence. Then

$$\begin{aligned} \dot{v}(x,\tilde{x},t) &= \dot{x}(t)'Px(t) + x(t)'P\dot{x}(t) \\ &+ \dot{\tilde{x}}(t)'Q\tilde{x}(t) + \tilde{x}(t)'Q\dot{\tilde{x}}(t) \\ &+ x(t)'P_1x(t) - x(t-\tau)'P_1x(t-\tau) \\ &+ \tilde{x}(t)'Q_1\tilde{x}(t) - \tilde{x}(t-\tau)'Q_1\tilde{x}(t-\tau) \end{aligned} \tag{10}$$

It is of interest to ensure that $\dot{v}(x,\tilde{x},t) \leq 0$ in order to find sufficient conditions for stability. As there are uncertainties that are only partially known, simply replacing equations (3) and (4) in (10) will not give useful conditions. Approximations to avoid matrices F_A, F_B and F_D must be made. With the help of *Lemma 1* and *Lemma 2*, it can be stated that,

$$\dot{v}(x,\tilde{x},t) \leq w'\mathscr{M}w.$$

Where,

$$\mathscr{M} = \begin{bmatrix} \mathscr{M}_{11} & PD & -PBK & \mathbf{0} \\ \star & \mathscr{M}_{22} & \mathbf{0} & \mathbf{0} \\ \star & \star & \mathscr{M}_{33} & QD \\ \star & \star & \star & -Q_1 \end{bmatrix}$$

$$w = \begin{bmatrix} x(t)' & x(t-\tau)' & \tilde{x}(t)' & \tilde{x}(t-\tau)' \end{bmatrix}'$$

$$\begin{aligned}\mathscr{M}_{11} &= A'P + PA + K'B'P + PBK + P_1 \\ &+ \gamma_A{}^2(\frac{1}{\mu_{PA}} + \frac{1}{\mu_{QA}})N_A'N_A \\ &+ \gamma_B{}^2(\frac{1}{\mu_{PB}} + \frac{1}{\mu_{QB}})K'N_B'N_BK \\ &+ \mu_{PA}PM_AM_A'P + 2\mu_{PB}PM_BM_B'P \\ &+ \mu_{PD}PM_DM_D'P\end{aligned}$$

$$\mathscr{M}_{22} = \gamma_D{}^2(\frac{1}{\mu_{PD}} + \frac{1}{\mu_{QD}})N_D'N_D - P_1$$

$$\begin{aligned}\mathscr{M}_{33} &= A'Q + QA - C'L'Q - QLC + Q_1 \\ &+ \gamma_B{}^2(\frac{1}{\mu_{PB}} + \frac{1}{\mu_{QB}})K'N_B'N_BK \\ &+ \mu_{QA}QM_AM_A'Q + 2\mu_{QB}QM_BM_B'Q \\ &+ \mu_{QD}QM_DM_D'Q\end{aligned}$$

Evidently, $\dot{v}(x,\tilde{x},t) \leq 0$ if $\mathscr{M} < 0$. To achieve this, the biggest problem is the nonlinear term in $\mathscr{M}_{13}$. With the help of *Lemma 3*, we can find a sufficient condition to make matrix $\mathscr{M}$ block-diagonal. This method involves some conservatism, but the inclusion of the new variable μ_K relieves part of this problem. Then we have that if the following inequalities hold, the system will be stable.

$$\begin{bmatrix} \mathscr{M}_{11} + \mu_K PBB'P & PD \\ \star & \mathscr{M}_{22} \end{bmatrix} < 0 \qquad (11)$$

$$\begin{bmatrix} \mathscr{M}_{33} + \frac{1}{\mu_K}K'K & QD \\ \star & -Q_1 \end{bmatrix} < 0 \qquad (12)$$

The first condition is associated with the dynamics of the system and the second one with the dynamics of the observer.

Defining $X = P^{-1}$ and multiplying both sides of condition (11) by

$$T = \begin{bmatrix} X & \mathbf{0} \\ \mathbf{0} & X \end{bmatrix}$$

we can obtain an equivalent condition which is easy to make it linear with respect to its variables. With this transformation and by defining $Z = KX$ and $X_1 = XP_1X$, condition (11) can be rewritten as

$$\begin{bmatrix} \mathscr{R}_{11} & DX \\ \star & \gamma_D{}^2(\frac{1}{\mu_{PD}} + \frac{1}{\mu_{QD}})XN_D'N_DX - X_1 \end{bmatrix} < 0 \qquad (13)$$

$$\begin{aligned}\mathscr{R}_{11} &= XA' + AX + Z'B' + BZ + X_1 \\ &+ \mu_{PA}M_AM_A' + 2\mu_{PB}M_BM_B' + \mu_{PD}M_DM_D' + \mu_K BB' \\ &+ \gamma_A{}^2(\frac{1}{\mu_{PA}} + \frac{1}{\mu_{QA}})XN_A'N_AX \\ &+ \gamma_B{}^2(\frac{1}{\mu_{PB}} + \frac{1}{\mu_{QB}})Z'N_B'N_BZ\end{aligned}$$

Condition (13) is Schur equivalent [11] with condition (9).

To deal with the uncertainties associated with the B matrix, some ingenious steps must be followed. Condition (12) can be Schur expanded as the following (remark that $XP = \mathbf{I}$)

$$\begin{bmatrix} -\frac{\mu_{PB}}{\gamma_B{}^2}\mathbf{I} & \mathbf{0} & \mathbf{0} & N_BK(XP) & \mathbf{0} \\ \star & -\frac{\mu_{QB}}{\gamma_B{}^2}\mathbf{I} & \mathbf{0} & N_BK(XP) & \mathbf{0} \\ \star & \star & -\mu_K\mathbf{I} & K(XP) & \mathbf{0} \\ \star & \star & \star & \mathscr{L}_{44} & QD \\ \star & \star & \star & \star & -Q_1 \end{bmatrix} < 0$$

$$\begin{aligned}\mathscr{L}_{44} &= A'Q + QA - C'L'Q - QLC + Q_1 + \mu_{QA}QM_AM_A'Q \\ &+ 2\mu_{QB}QM_BM_B'Q + \mu_{QD}QM_DM_D'Q\end{aligned}$$

Recalling that $Z = KX$, for any $\alpha > 0$, the last matrix can be rewritten as

$$T_D \begin{bmatrix} -\frac{\mu_{PB}}{\gamma_B{}^2}\mathbf{I} & \mathbf{0} & \mathbf{0} & N_BZ & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & -\frac{\mu_{QB}}{\gamma_B{}^2}\mathbf{I} & \mathbf{0} & N_BZ & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & -\mu_K\mathbf{I} & Z & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & -\alpha\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & -\alpha\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & \star & \mathscr{L}_{44} + \alpha PP & QD \\ \star & \star & \star & \star & \star & \star & \alpha PP - Q_1 \end{bmatrix} T_D' < 0 \qquad (14)$$

With T_D a full ranked transformation as follow,

$$T_D = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & P & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & P & \mathbf{0} & \mathbf{I} \end{bmatrix}$$

As the matrix in (14) is a block-diagonal matrix, and $\alpha > 0$, the inequality holds if and only if two matrix inequalities hold simultaneously. One of them is directly inequality (7) of theorem one. The other condition is a nonlinear inequality given by

$$\begin{bmatrix} \mathscr{L}_{44} + \alpha PP & QD \\ \star & \alpha PP - Q_1 \end{bmatrix} < 0 \qquad (15)$$

To make it linear with respect to its variables, we will make use of the procedure suggested in [4]. Let $\beta > 0$ be a scalar such that

$$X > \frac{\alpha}{\beta^2}\mathbf{I} \qquad (16)$$

By the Schur complement lemma [11], the last expression can be stated as:

$$\begin{bmatrix} -X & \mathbf{I} \\ \star & -\frac{\beta^2}{\alpha}\mathbf{I} \end{bmatrix} < 0$$

Using *Lemma 4*, it can be written that $-\frac{\beta^2}{\alpha}\mathbf{I} \leq -(2\beta-\alpha)\mathbf{I}$. Therefore, a sufficient linear condition to ensure that (16) holds is (6).

From inequality (16), multiplying by PP and by β^2 we obtain,

$$\alpha PP < \beta^2 P$$

Then, a sufficient condition that makes inequality (15) hold is given by

$$\begin{bmatrix} \mathscr{L}_{44}+\beta^2 P & QD \\ \star & \beta^2 P - Q_1 \end{bmatrix} < 0 \quad (17)$$

With the change of variables $W = QL$, the Schur complement of (17) becomes

$$\begin{bmatrix} A'Q+QA-C'W'-WC+Q_1 & QD & QM_A & QM_B & QM_D & \beta\mathbf{I} & \mathbf{0} \\ \star & -Q_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \beta\mathbf{I} \\ \star & \star & -\frac{1}{\mu_{QA}}\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & -\frac{1}{2\mu_{QB}}\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & -\frac{1}{\mu_{QD}}\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & \star & \star & -X & \mathbf{0} \\ \star & \star & \star & \star & \star & \star & -X \end{bmatrix} < 0 \quad (18)$$

The only nonlinearities in (18) are the reciprocals of μ_{QA}, μ_{QB} and μ_{QD}. From *Lemma 4* it follows that $-\mu_{QA}{}^{-1}\mathbf{I} \leq -(2-\mu_{QA})\mathbf{I}$, $-\mu_{QB}{}^{-1}\mathbf{I} \leq -(2-\mu_{QB})\mathbf{I}$ and $-\mu_{QD}{}^{-1}\mathbf{I} \leq -(2-\mu_{QD})\mathbf{I}$. Then, a sufficient condition to fullfill (18) is (8). This ends the proof.

V. EXAMPLE

Consider that H_1 and H_2 are both uncertain first order processes. The delay of H_2, given by τ, is unknown. From this, precisely known matrices that describe this system are:

$$A = \begin{bmatrix} -0.4 & 0.4 \\ 0.0 & -1.0 \end{bmatrix}, B = \begin{bmatrix} 0.4 \\ 0.0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1.0 & 0.0 \end{bmatrix}, D = \begin{bmatrix} 0.0 & 0.0 \\ 0.9 & 0.0 \end{bmatrix}$$

The uncertainties may be represented by the following matrices.

$$M_A = \begin{bmatrix} 0.04 & 0.0 & 0.1 \\ 0.0 & 0.04 & 0.0 \end{bmatrix}, M_B = \begin{bmatrix} 0.04 \\ 0.0 \end{bmatrix}, M_D = \begin{bmatrix} 0.0 \\ 0.09 \end{bmatrix}$$

$$N_A = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \\ 0.0 & 1.0 \end{bmatrix}, N_B = \begin{bmatrix} 1.0 \end{bmatrix}, N_D = \begin{bmatrix} 1.0 & 0.0 \end{bmatrix}$$

$$\gamma_A = \gamma_B = \gamma_D = 1$$

This represents up to a 10% of uncertainty in every element of the precisely known matrices.

A solution to equations (6), (7), (8) and (9) is:

$$X = \begin{bmatrix} 0.2939 & -0.1894 \\ -0.1894 & 0.7485 \end{bmatrix}, X_1 = \begin{bmatrix} 0.3757 & -0.2546 \\ -0.2546 & 0.1892 \end{bmatrix},$$

$$Q = \begin{bmatrix} 10.2244 & -10.1157 \\ -10.1157 & 17.4925 \end{bmatrix}, Q_1 = \begin{bmatrix} 76.2525 & 7.6331 \\ 7.6331 & 11.9956 \end{bmatrix},$$

$$W = \begin{bmatrix} 56.2256 \\ 21.1582 \end{bmatrix}, Z = \begin{bmatrix} -0.37678 & -0.1792 \end{bmatrix},$$

$$\mu_{PA} = 6.0437, \mu_{QA} = 1.5875, \mu_{PB} = 8.5884,$$
$$\mu_{QB} = 1.8417, \mu_{PD} = 3.4593, \mu_{QD} = 1.2750,$$
$$\mu_K = 0.2292, \alpha = 0.9172, \beta = 2.6918$$

From there, we obtain the values of the controller and the observer as:

$$L = \begin{bmatrix} 15.6499 \\ 10.2597 \end{bmatrix}, K = \begin{bmatrix} -1.7156 & -0.6734 \end{bmatrix} \quad (19)$$

To show the performance of this controller we will present some time simulations. The uncertain matrices F_A, F_B and F_D were assumed to be solidarily time varying all through the simulation time. *i.e.* when F_A changes, so does F_B and F_D. Furthermore, the changes of this matrices are, within the bounded norm restriction, completely random all through the simulation time. Fig. 2. shows the evolution of the state variables for not null initial conditions and a null input. On the other hand Fig. 3. shows the same system with the observed-based controller stated in (19).

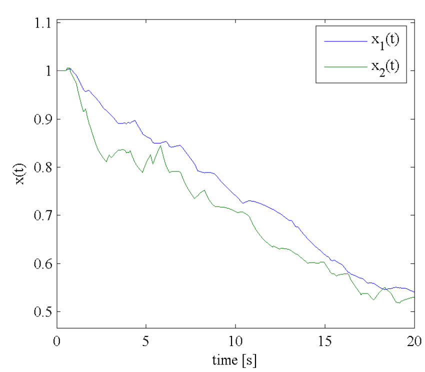

Fig. 2. Open Loop Simulation - State variables

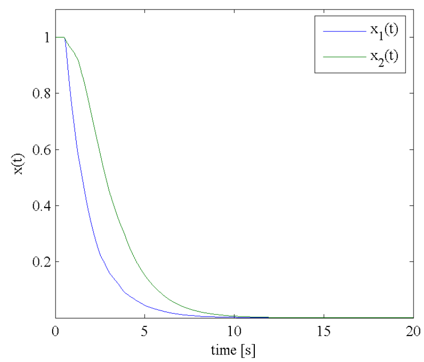

Fig. 3. Closed Loop Simulation - State variables.

Even thought the system behaves as stable for Open Loop, it is clear that the inclusion of the state observed based controller speeds up the performance of the system in a significant way.

If necessary, to ensure a faster speed performance of the controller, adjustments can be made to the A matrix. To obtain a faster closed loop response, A can be replaced in (8) and (9) by $A+\delta\mathbf{I}$, where δ is a positive scalar representing the distance of the poles of the system to the origin. Indeed, considering $\delta = 0.08$, in the study case, leads to:

$$L = \begin{bmatrix} 35.5490 \\ 18.9776 \end{bmatrix}, K = \begin{bmatrix} -2.2176 & -0.8225 \end{bmatrix} \tag{20}$$

The controller stated in (20) makes the system stable without taking into account the value of the delay in the recycle transfer function. This can be seen in Fig. 4, where the step response for the same controller and different delays is shown. In spite of the time changing uncertainties, it can be seen that the systems are all stable but with different dynamics.

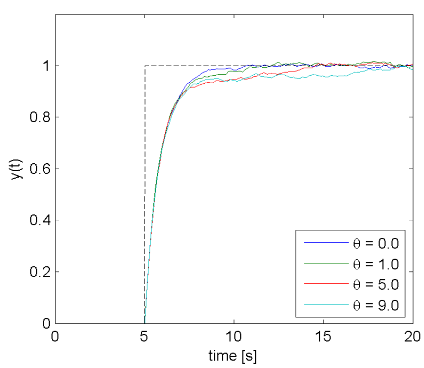

Fig. 4. Closed Loop Simulation - Step response for different delays.

VI. CONCLUSIONS AND FUTURE WORKS

In this work, we have presented a suitable treatment of uncertain systems with delay on the recycle branch. This usually leads to transfer functions that are difficult to treat with other standard controller design procedures. The solution presentend here is in this sence general as it does not deppend on the size of the delay nor on the dimentions of the forward or recycle branchs.

In order to control this kind of systems, convex optimization constrains for the development of observer-based controllers are achieved. In this aspect, the main novelty is the inclusion of uncertainties in input matrix B along with the treatment of time delay systems. The LMI conditions presented here are appropriate to be solved by any standard LMI software. Therefore, they represent a convenient tool for solving the control problem of time-delay uncertain recycle systems in an industrial context.

The authors are focused in developing less restrictive constraints to ensure stability of the systems. Also, research on including multiple delays (namely in the forward and recycle branch) is under investigation. Furthermore, development on using Lyapunov-Krasovskii functionals to find delay-depending constraints to ensure stability is also an interesting field of research.

VII. ACKNOWLEDGMENTS

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