

Combinatorial bounds and scaling laws for noise amplification in networks

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Abstract—Motivated by the problem of designing and analyzing clock synchronization protocols for large-scale networks, we provide combinatorial bounds on the mean square disagreement of synchronization dynamics subject to additive noise. While previous work has used eigenvalues to analyze the mean square disagreement when the synchronization dynamics is governed by a normal matrix, in the non-normal case eigenvalues cease to be an adequate measure. We first show that the so-called 2-norm coefficient of ergodicity used in the study of inhomogeneous Markov chains can be used to bound mean square disagreement even when the underlying dynamics is governed by a non-normal matrix. Our main result then demonstrates that the 2-norm coefficient of ergodicity has a natural combinatorial interpretation as a combined measure of matrix non-normality and graph connectivity. We apply this result to yield new performance guarantees for distributed clock synchronization: we show that in a simple (possibly irregular) random graph model, the natural clock synchronization scheme wherein neighboring nodes average their clocks is order-optimal: it drives the expected clock offsets between any two nodes to at most a constant, independently of the total number of nodes.

I. INTRODUCTION

There has been much recent interest within the control community in the study of distributed clock synchronization schemes [21], [8], [2], [3], [4], [22], [5], [7]. Many sensor network applications require coordination and synchrony from spatially distributed nodes, which often requires the ability to refer to a common notion of time [20], [19]. Consequently, the design of protocols which keep all clocks in the network as close as possible to each other while expending as little energy as possible on clock offset measurements is an important problem in distributed control theory.

In this paper, we are concerned with distributed clock synchronization schemes that come with particularly strong guarantees on performance. Namely, we are interested in the design and analysis of protocols wherein, at every time step, each node measures the offsets between its clock and those of its neighbors, and then adjusts its clock reading; and, as a consequence, the average squared drift between nodes turns out to be a constant independently of the number of nodes n . Such schemes are nearly optimal in the sense that their performance is at most a constant factor away the best one could hope to achieve.

Our interest in this subject stems from its connection to the study of combinatorics of nonnegative dynamic systems.

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Indeed, distributed clock synchronization schemes are naturally modeled as a particular kind of noisy stochastic linear systems. In this paper, our goal is to bound the steady-state of such systems in terms of the combinatorial properties of the network of nodes. We will show that a certain coefficient of ergodicity, which may be thought of as a joint measure of connectivity and non-normality of a graph, provides a way to derive such bounds. Moreover, our bound will allow us to prove that clock synchronization is nearly optimal in the above sense on a general class of random graphs.

Specifically, we will be studying dynamic systems of the form

$$x(t+1) = Ax(t) + w(t), \quad (1)$$

where $A \in \mathbf{R}^{n \times n}$ is a stochastic matrix and $w(t)$ are vectors whose entries are independent with mean zero, with the i 'th entry having variance ν_i^2 . Our main goal is to analyze the quantity E_{ss} defined as

$$E_{ss} = \limsup_{t \rightarrow \infty} \frac{1}{n} E[\|x(t) - \bar{x}(t)\mathbf{1}\|_2^2],$$

where $\mathbf{1}$ is the vector of all ones and

$$\bar{x}(t) = \left(\frac{\sum_{i=1}^n x_i(t)}{n} \right) \mathbf{1}.$$

The study of E_{ss} is motivated by the problem of distributed clock synchronization. Consider n nodes in a network, each maintaining a clock which slowly drifts away from the true time. A distributed clock synchronization protocol uses (noisy) measurements of offsets between neighboring clocks in an effort to control this drift. A common, simple model is to assume that the drift of every clock as consisting of a bias and a random drift term¹ [19]. However, because nodes can identify and correct their bias over time by repeatedly comparing their drift with the drift of neighboring nodes (see [16]), the asymptotic disagreement of the clocks will be determined by the accumulation of the random terms. Since we focus here on asymptotic performance, we assume without loss of generality that the bias terms equal zero. Discretizing time $t = 1, 2, 3, \dots$ and letting $c_i(t)$ be the clock of node i , we have that after correcting for the drift but without implementing any other control scheme, $c_i(t)$ will evolve as

$$c_i(t+1) = c_i(t) + 1 + z_i(t),$$

¹Note, however, that more sophisticated models are available; see in particular [4] which models the internal dynamics of each clock. In this paper, we stick with the simpler linear model.

where $z_i(t)$ is the random drift variable; we assume $z_i(t)$ are independent, have mean zero, and variances μ_i^2 . Among the simplest protocols to control the scheme is a simple averaging scheme, first proposed in [11], wherein every node averages its clock with the clocks of its neighbors, which leads $c_i(t)$ to evolve according to

$$c_i(t+1) = \frac{1}{d(i)} \sum_{j \in N(i)} \hat{c}_{i,j}(t) + 1 + z_i(t),$$

where $N(i)$ is the set of neighbors of node i (which always includes i) and $d(i)$ is cardinality of $N(i)$. Here $\hat{c}_{i,j}$ is the estimate node i has of the clock of node j :

$$\hat{c}_{i,i}(t) = c_i(t), \quad \hat{c}_{i,j}(t) = c_j(t) + w_{ij}(t).$$

The error term $w_{ij}(t)$ arises because it is typically impossible to estimate the clock offset exactly: for example, a possible method is for neighboring nodes to bounce a message back and forth, appending the time they received it after each bounce; after enough bounces, the nodes have a noisy estimate of the round trip time they can use to estimate offsets (see, e.g., [12] for one protocol based on this idea). Since clock offsets are usually estimated this way over bidirectional links, we will be assuming that the graph of interconnections between the nodes is undirected: $i \in N(j)$ if and only if $j \in N(i)$.

Thus if $x_i(t) = c_i(t) - t$ is the difference between the clock reading of node i and the true time, then

$$x_i(t+1) = \frac{1}{d(i)} \sum_{j \in N(i)} x_j(t) + w_i(t),$$

where $w_i(t)$ combines the drift $z_i(t)$ with the estimation error $(1/d(i)) \sum_{j \in N(i)} w_{ij}(t)$. We now have a particular instance of Eq. (1). The quantity E_{ss} then becomes a measure of the success of distributed clock synchronization: it equals the average squared deviation of a node from the average time in the network.

In the remainder of this paper, we will focus our attention on the particular form of Eq. (1) we have just described, i.e., we assume that A is constructed from an undirected graph $G = (V, E)$ by setting $a_{ij} = 1/d(i)$ if $(i, j) \in E$, where $d(i)$ is the out-degree of node i , and $a_{ij} = 0$ otherwise. Since $i \in N(i)$, we will assume that G has a self-loop at every node. Without loss of generality, we will assume that G is connected. We will adopt the terminology that A is the transition matrix of the graph G when A is constructed in this manner.

The quantity E_{ss} may additionally be thought of as a measure of the robustness of the linear system in Eq. (1) to noise. Under some mild additional assumptions on A (namely that it is an irreducible and aperiodic) the iteration $x(t+1) = Ax(t)$ drives the system to consensus, meaning that every $x_i(t)$ approaches the same value; the quantity E_{ss} then measures the steady state expected deviation per node after the addition of the noise term.

Our goal in this paper is to obtain bounds on E_{ss} which can be expressed in terms of combinatorial properties of the

graph G . While it is not hard to derive expressions for E_{ss} in terms of the eigenvalues and eigenvectors of A , or in terms of Lyapunov equations satisfied by A , it is unclear how such expressions scale with network size or topology. By contrast, combinatorial bounds can provide effective performance guarantees for large classes of networks and are consequently useful as design guidelines.

The quantity E_{ss} was previously studied in [24],[25], [26] and is related to the papers [15], [13], [10]. Under the assumption that the matrix A is symmetric, an expression for E_{ss} in terms of the eigenvalues of A was derived in [24]. This was extended in [25], where, in a similar continuous-time model, the assumption of symmetry of A was relaxed to the assumption of normality of a certain related matrix. These expressions are useful because they allow for the derivation of combinatorial bounds on E_{ss} in terms of the underlying graphs, e.g., by applying the Cheeger bound to bound eigenvalues in terms of cuts - see [6] for many examples of this technique.

Unfortunately, these techniques do not appear extendable to the case when A is not symmetric and does not possess any normality properties. In this paper, we provide a technique to bound E_{ss} in many such cases through the 2-ergodicity coefficient, defined next.

Definition 1: The ergodicity coefficient τ of a transition matrix A is defined by

$$\tau = \max_{\sum_{i=1}^n d(i)y(i)=0} \frac{\|Ay\|_2}{\|y\|_2}$$

The quantity τ is one of many ergodicity coefficients defined in the probability literature; we refer the reader to [17], [14] as well as the book [18] and the recent survey [9] as well as the references therein for more information. Our interest in τ comes from the observation that it may be used to bound E_{ss} as follows.

Proposition 2: If $\tau < 1$, then

$$E_{ss} \leq \nu_{\max}^2 \frac{2}{1 - \tau^2}$$

where ν_{\max} is the largest among the ν_i .

This proposition immediately raises the question of whether any combinatorial bounds on τ , particularly those that would imply that $\tau < 1$, are possible. We analyze this question in the present paper. One might guess that, by analogy with the second eigenvalue of A , bounds on τ might follow from various notions of connectivity of the graph G , such as having a small diameter, sharing neighbors, or a having an isoperimetric constant bounded away from zero. Alternatively, one might observe that $\tau < 1$ if the graph G is regular [23], which might lead one to conjecture that graphs which are close to regular have the property that $\tau < 1$. Since, among graphs with self-loops regular graphs correspond to normal matrices (we justify this claim in the following section), this would mean that τ measures non-normality.

The next theorem, which is our main result, shows that while these two guesses are not right on their own, combining them does lead to combinatorial upper bounds on τ .

Theorem 3: 1) For large enough n , the line graph on n nodes (which is a single edge away from being a regular graph) has $\tau > 1$.

- 2) Define $e(S)$ to be the set of edges with exactly one endpoint in S , $|e(S)|$ to be the cardinality of $e(S)$, and

$$|S|_d = \sum_{i \in S} d(i)$$

The isoperimetric constant h is defined through the relations

$$h(S) = \frac{|e(S)|}{|S|_d}$$

and

$$h(G) = \min_{S \subset V, |S|_d \leq |V|_d} h(S).$$

For all large enough n , there exists a graph G on n nodes such that G is neighbor shared (meaning any two nodes share a common neighbor), has diameter 2, with $h(G)$ bounded away from zero (in fact, $h(G) \geq 1/5$), and $\tau = \Omega(\sqrt{n})$.

- 3)

$$\tau < \sqrt{\frac{d_{\max}}{d_{\min}}} (1 - h(G)^2)$$

where d_{\max} is the largest degree of a node in G and d_{\min} is the smallest degree of a node.

Note that the first two items of the theorem are counterexamples to simple connections between τ and either non-normality or connectivity: the first item shows that being combinatorially close to a regular graph (i.e., a normal matrix) does not guarantee that $\tau < 1$, while the second item shows that good connectivity is also insufficient to guarantee that $\tau < 1$. However, the final item suggests that the condition $\tau < 1$ may be thought of as a combined effect of both closeness to normality (due to the $\sqrt{d_{\max}/d_{\min}}$ term) and connectivity (due to the dependence on $h(G)$).

Combining Proposition 2 and Theorem 3 we get the following statement, which is the main conclusion of this paper: if the degrees do not vary too much and the connectivity is not too poor (as measured together by the inequality $\sqrt{\frac{d_{\max}}{d_{\min}}} (1 - h(G)^2) < 1$), then the steady-state variance of the natural clock synchronization scheme is upper bounded by a constant independent of the total number of nodes. We remark again that this is the best possible performance one can hope for.

We illustrate the utility of this with the following application to distributed clock synchronization on random graphs. Specifically, we consider the problem of topology design for distributed clock synchronization under the assumption that every node has enough energy to support a small number of (possibly long-range) links. In lieu of design, an attractive option to simply take a random regular graph. Because random regular graphs have real eigenvalues bounded away from 1, this immediately leads to favorable bounds on E_{ss} : it is easy to see as a consequence of the results in [24] that E_{ss}

is upper bounded by a constant in this case independently of the network size.

However, symmetry and normality of a matrix are very fragile properties: both can be undone simply by changing a single matrix entry. As a result, the performance guarantee obtained in this way is not necessarily valid if only a single edge in the graph is added or removed. By contrast, combinatorial bounds tend to be more considerably robust. We demonstrate the advantage by using Theorem 3 to get favorable on the performance of distributed clock synchronization on an irregular, adversarially perturbed random graph model.

We define a simple model $G_{n,d,c}$ of an irregular, adversarially perturbed random graph as follows: we generate a d -regular random graph on n vertices (for simplicity, let us assume that n is even and the graph is generated by unioning d random disjoint matchings) and we let an adversary arbitrarily delete edges from the last c matchings. We continue to assume that every node in addition has a self-loop. This setup models the situation when a limited number of unreliable communication edges may fail in a way that is difficult to predict in advance.

Corollary 4: For any $c > 0$, if n and d are large enough, then with high probability $G_{n,d,c}$ satisfies

$$E_{ss} \leq 3\nu_{\max}^2.$$

Note that if all the variance ν_i are equal, this bound gives the the best possible performance up to a constant factor.

It is in principle not difficult to obtain effective versions of this corollary bounding E_{ss} for specific n, d, c , and in a later section, we will report on simulations which indicate that performance is good even for small degrees d ; however, while the corollary as stated follows quickly from Theorem 3 and well-known bounds on the isoperimetric constant of random graphs, effective bounds for small n, d, c would require a stronger analysis of the concentration of the isoperimetric constant, which we will not perform here due to space constraints, but which we instead postpone to a possible future work.

The rest of this paper is organized as follows. Section II contains proofs of the all the claims in this section. Section III which compares the theoretical bounds we have derived with a few simulations. Finally, we end with some conclusions and open problems in Section ??.

II. PROOFS OF THE MAIN THEOREMS

Proof: [Proof of Proposition 2] Note that vector π with $\pi_i = d(i) / \sum_{j=1}^n d(j)$ is a left-eigenvector of A corresponding to the eigenvalue 1. Consider the quantity

$$E'_{ss}(t) = E[||x(t) - \pi^T x(t)||_2^2]$$

Since the entries of π are nonnegative and add up to 1,

$$E_{ss} \leq \frac{1}{n} \limsup_t E'_{ss}(t), \quad (2)$$

so that it suffices to bound the latter. Observe that

$$x(t+1) - \pi^T x(t+1)\mathbf{1} = A(x(t) - \pi(t)^T x(t)\mathbf{1}) + w(t) - \pi^T w(t)\mathbf{1}$$

Therefore,

$$E[\|x(t+1) - \pi^T x(t+1)\mathbf{1}\|_2^2] = E[\|A(x(t) - \pi^T x(t)\mathbf{1})\|_2^2] + E[\|w(t) - \pi^T w(t)\mathbf{1}\|_2^2], \quad (3)$$

where we used that every entry of $w(t)$ has mean zero. Now since

$$\pi^T(x(t) - \pi^T x(t)\mathbf{1}) = 0$$

so by definition of τ ,

$$E[\|x(t+1) - \pi^T x(t+1)\mathbf{1}\|_2^2] \leq \tau^2 \|x(t) - \pi^T x(t)\|_2^2 + E[\|w(t) - \pi^T w(t)\mathbf{1}\|_2^2].$$

We can bound the last term in this equation as

$$\begin{aligned} E[\|w(t) - \pi^T w(t)\mathbf{1}\|_2^2] &= \sum_{j=1}^n (n-1)\pi_j^2 E[w_j^2(t)] + \sum_{i=1}^n (1-\pi_i)^2 E[w_i^2(t)] \\ &= \sum_{i=1}^n n\pi_i^2 E[w_i^2(t)] - 2\pi_i E[w_i^2(t)] + E[w_i^2(t)] \\ &= \nu_{\max}^2 \sum_{i=1}^n n\pi_i^2 - 2\pi_i + 1 \\ &\leq 2n\nu_{\max}^2 \end{aligned}$$

Therefore

$$\|x(t+1) - \pi^T x(t+1)\mathbf{1}\|_2^2 \leq \tau^2 \|x(t) - \pi^T x(t)\|_2^2 + 2n\nu_{\max}^2.$$

Since by assumption $\tau < 1$, this implies

$$\limsup_t E'_{ss}(t) = \limsup_t \|x(t+1) - \pi^T x(t+1)\mathbf{1}\|_2^2 \leq \frac{2n\nu_{\max}^2}{1-\tau^2},$$

and now applying Eq. (2), the proof is completed. ■

We next provide a proof of the claim from the previous section that regular graphs correspond to normal matrices.

Proposition 5: Suppose $G = (V, E)$ is an undirected, connected graph with a self-loop at every node, and A is the transition matrix of G . Then A is normal if and only if G is regular.

Proof: In one direction, the regularity of G implies that A is symmetric and therefore normal. Conversely, if A is normal, then $\mathbf{1}^\perp$ would be an invariant subspace of A , which implies that $\mathbf{1}^T A = \mathbf{1}^T$. But if the graph was not regular, then there must exist a node i with the property that its degree is at least as big as the degree of each of its neighbors; and moreover, i has at least one neighbor with strictly smaller degree. Then the i 'th column of A has a sum strictly larger than 1, which is a contradiction. ■

We now provide a proof of our main theorem.

Proof: [Proof of Theorem 3]

- 1) We will show that a line graph on n nodes with $n \geq 163$ has $\tau > 1$. Applying the Courant-Fischer theorem, we argue

$$\tau \geq \inf_{w \in \mathbf{R}^n} \sup_{w^T z = 0} \frac{\|Ax\|_2}{\|x\|_2} = \sigma_2(A) = \sup_{\dim(S)=2} \inf_{x \in S} \frac{\|Ax\|_2}{\|x\|_2}, \quad (4)$$

where the supremum in the last expression is taken over all two dimensional subspaces S of \mathbf{R}^n . Thus it suffices to exhibit a two dimensional subspace S such that every $s \in S$ satisfies $s^T A^T A s \geq (1 + \epsilon)s^T s$ for

some $\epsilon > 0$. We exhibit such an S as $S = \text{span}(v_1, v_2)$ where v_1 is

$$\left(\frac{9}{10}, \frac{11}{10}, 1, 1, \dots, 1, 1, \frac{11}{10}, \frac{9}{10}\right)$$

and v_2 is

$$(-1, -2, -2 + \frac{4}{n-3}, -2 + \frac{8}{n-3}, \dots, -2 + \frac{4(n-3)}{n-3}, 1)$$

Observe that v_1 and v_2 are orthogonal by symmetry; moreover, still, by symmetry, Av_2 is also orthogonal to Av_1 . These two properties immediately imply that for any $s = \alpha v_1 + \beta v_2$, we have

$$\begin{aligned} \frac{s^T A^T A s}{s^T s} &= \frac{\alpha^2 v_1^T A^T A v_1 + 2\alpha\beta v_1^T A^T A v_2 + \beta^2 v_2^T A^T A v_2}{\alpha^2 v_1^T v_1 + \beta^2 v_2^T v_2} \\ &= \frac{\alpha^2 v_1^T A^T A v_1 + \beta^2 v_2^T A^T A v_2}{\alpha^2 v_1^T v_1 + \beta^2 v_2^T v_2}. \end{aligned}$$

Thus we only need to check that $v_1^T A^T A v_1 > v_1^T v_1$ and $v_2^T A^T A v_2 > v_2^T v_2$. The former follows since $(9/10)^2 + (11/10)^2 + 1^2 < 1 + 1 + (31/30)^2$ and as for the latter, observe that:

$$v_2^T v_2 = 1 + 1 + \sum_{i=0}^{n-3} \left(-2 + \frac{4i}{n-3}\right)^2$$

while

$$v_2^T A^T A v_2 = 2\left(\frac{3}{2}\right)^2 + 2\left(\frac{5}{3} - \frac{4}{3(n-3)}\right)^2 + \sum_{i=1}^{n-4} \left(-1 + \frac{4i}{n-3}\right)^2$$

So $v^T A^T A v \geq v^T v$ as long as

$$2\left(\frac{3}{2}\right)^2 + 2\left(\frac{5}{3} - \frac{4}{3(n-3)}\right)^2 \geq 2(4+1).$$

The last statement is true as long as $n \geq 163$.

- 2) We will describe the construction and the proof for odd n ; the argument for even n is similar. We define G to the graph composed of two disjoint stars on $k = (n-1)/2$ nodes (and we assume $k \geq 6$) and an additional node q connected to every other node. Clearly G is neighbor-shared and has diameter 2. We will show that $\tau = \Omega(\sqrt{n})$ and $h(G) \geq 1/5$.

Indeed, from Eq. (4), to show that $\tau = \Omega(\sqrt{n})$ it suffices to find a two-dimensional subspace S such that $\|As\|_2^2 \geq \Omega(n)\|s\|_2^2$. Let v_1 be the vector with 1 at the center of the two stars and 0 everywhere else; and let v_2 be the vector with 1 at the center of one star, -1 at the center of another star, and 0 everywhere else. We claim that $S = \text{span}(v_1, v_2)$ has the property we want. Indeed, observe that v_1 and v_2 are orthogonal; and moreover, by symmetry, so are Av_1 and Av_2 . Consequently, by Eq. (5), it suffices to show that $\|Av_1\|_2^2 \geq \Omega(n)\|v_1\|_2^2$ and $\|Av_2\|_2^2 \geq \Omega(n)\|v_2\|_2^2$. This follows immediately since for both Av_1 and Av_2 , the entries at the leaves of both stars have entries of absolute value $1/4$.

It remains to argue that $h(G) \geq 1/5$. We will prove this on a case by case basis by analyzing several possibilities for the set S in the definition of the isoperimetric constant. Specifically, S either contains

q or it does not, and in either case there are two star-centers in the graph, and S contains 0 or 1 or 2 of them. That is six possibilities total, each of which we examine below. Before we start, we note that because

$$\sum_{i=1}^n d(i) = (k+1) \cdot 2 + 3 \cdot 2(k-1) + (2k+1) \cdot 1 \leq 10k,$$

we have that for any set S in the definition of h , $|S|_d \leq 5k$.

Now suppose S contains q . If S contains neither of the two star centers, then $e(S)$ contains both edges leading from q to the centers, as well as a single edge for every leaf, regardless of whether it is in S ; so $|e(S)| \geq 2k$, which implies that $h \geq (2k)/(5k) = 2/5$ in this case. If S contains q and a single center, then it can't contain more than $(2/3)k$ leaves without violating the condition $|S|_d \leq 5k$, which implies there are always at least $(4/3)k - 2$ leaves it does not contain; in turn, this implies $|e(S)| \geq (4/3)k$ because S contains q , so in this case $h \geq ((4/3)k - 2)/(5k) \geq 1/5$, where the last inequality used that $k \geq 6$. By a similar argument, if S contains q and both centers, then it cannot contain more than $k/3$ leaves; which implies there are $5k/3 - 2$ leaves it does not contain, so $|e(S)| \geq 5k/3 - 2$, and so in this case $h \geq (5k/3 - 2)/(5k) = 1/4$.

Now if S does not contain q , then, once again, S either contains 0, 1, 2 centers. If it contains zero centers, then S must be made up wholly of leaves, and since every leaf has degree three and two outgoing edges, all of which go to non-leaf nodes, we have that in this case $h \geq 2/3$. Now if S contains one center, m_1 leaves in the star corresponding to that center, and m_2 leaves in the other star, then

$$|S|_d = k + 1 + 3m_1 + 3m_2 \leq 4k + 1 + 3m_2$$

while

$$|e(S)| = (k - 1 - m_1) + m_1 + 2m_2 = k - 1 + 2m_2$$

which implies

$$h \geq \frac{k - 1 + 2m_2}{4k + 1 + 3m_2} \geq \min\left(\frac{k - 1}{4k + 1}, \frac{2}{3}\right) \geq \frac{1}{5},$$

where we used $k \geq 6$ in the final inequality. Finally, if S contains two centers and m_1 leaves in one center and m_2 in the other, then

$$\begin{aligned} h(S) &= \frac{(k - 1) - m_1 + m_1 + (k - 1) - m_2 + m_2}{(2(k + 1) + 3m_1 + 3m_2)} \\ &= \frac{2k - 1}{2(k + 1) + 3m_1 + 3m_2} \geq \frac{2k - 1}{8k + 2} \geq 1/5. \end{aligned}$$

In all cases, we conclude that h is at least $1/5$.

- 3) It is standard that the matrix A is self-adjoint in the real inner product

$$\langle x, y \rangle_d = \sum_{i=1}^n x(i)d(i)y(i),$$

since

$$\begin{aligned} \langle Ax, y \rangle_d &= \sum_{i=1}^n \sum_{j \in N(i)} \frac{1}{d(i)} x(j)d(i)y(i) \\ &= 2 \sum_{(i,j) \in E} x(j)y(i) \\ &= \sum_{i=1}^n x(i)d(i) \sum_{j \in N(i)} \frac{1}{d(i)} y(j) \\ &= \langle x, Ay \rangle_d \end{aligned}$$

Since $\mathbf{1}$ is the right-eigenvector of A corresponding to the spectral radius, the Courant-Fischer theorem implies that

$$\lambda_2^2(A) = \max_{\sum_{i=1}^n d(i)z(i)=0} \frac{\langle Az, Az \rangle_d}{\langle z, z \rangle_d}$$

Applying Cheeger's inequality (in the form given in [6], Proposition 6) results in

$$(1 - h(G)^2)^2 \geq \max_{\sum_{i=1}^n d(i)z(i)=0} \frac{\langle Az, Az \rangle_d}{\langle z, z \rangle_d}$$

Since

$$d_{\min} \|y\|_2^2 \leq \langle y, y \rangle_d \leq d_{\max} \|y\|_2^2,$$

this implies that

$$(1 - h(G)^2)^2 \geq \frac{d_{\min}}{d_{\max}} \max_{\sum_{i=1}^n d(i)z(i)=0} \frac{\|Ay\|_2^2}{\|y\|_2^2} \geq \frac{d_{\min}}{d_{\max}} \tau^2$$

and therefore

$$\tau \leq \sqrt{\frac{d_{\max}}{d_{\min}}} (1 - h(G)^2)$$

■

Note that as a consequence of this theorem, well-connected graphs can afford a considerable variation in degrees while still satisfying $\tau < 1$. For example, suppose $h(G) \geq 1/2$. Then, as long as $d_{\max}/d_{\min} \leq 64/49 \approx 1.3$, we will have $\tau < 1$.

Finally, we observe that Corollary 4 is an application of the final part of Theorem 3, coupled with a counting argument; we omit the proof here due to space constraints.

III. SIMULATIONS

We report on two simulations intended to confirm our theoretical results and clarify the (un)importance of symmetry for the performance of clock synchronization. Figure III shows E_{ss} plotted vs n for two classes of graphs. On the left, we plotted a variant of $G_{n,6,2}$; namely, we generated a 4-regular graph and a 2-regular graph, deleted every edge of the 2-regular graph with probability $1/2$, and put the edges together. There is a slight difference between $G_{n,6,2}$ as we defined it earlier and what we generated here, because we did not prune each graph to remove multiple edges. All the random noises were Gaussian with mean zero and unit variance; E_{ss} was estimated by running the iteration $x(t+1) = Ax(t) + w(t)$ a hundred times.

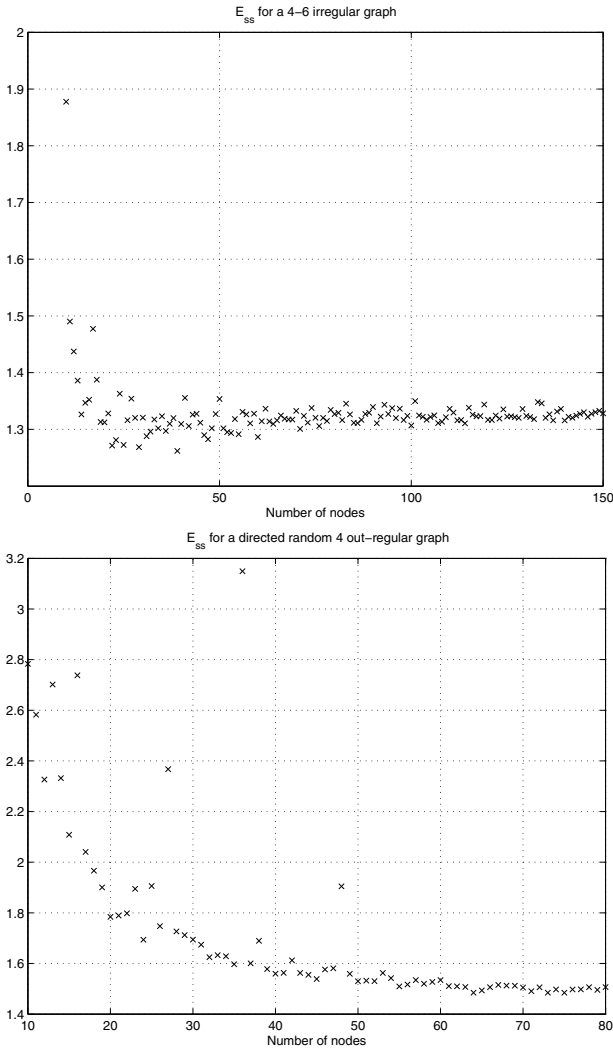


Fig. 1. Two simulations of clock synchronization performance. On the left, a variation on $G_{n,6,2}$ is generated; a 4 regular graph was unioned together with a 2-regular graphs, with every edge of the 2-regular graph being kept with probability $1/2$. Every data point plotted is an average of 600 trials. The number of nodes is on the x-axis while an estimate of E_{ss} is plotted on the y axis. On the right, we plot E_{ss} vs. number of nodes for a directed graph in which every nodes has 4 uniformly distributed out-neighbors. Every data point plotted is an average of 300 trials.

The results appear to confirm our theoretical guarantees: performance appears to be independent of the total number of nodes n . Moreover, they suggest that the bound $E_{ss} \leq 3\nu_{\max}^2$ is considerably loose in this scenario, even for relatively small n and d ; in fact, the performance of clock synchronization appears to be bounded by $1.4\nu_{\max}^2$ here.

On the right of Figure III, we show a simulation for a directed regular graph in which every node has 4 uniformly distributed out-neighbors. The performance here is also independent of the number of nodes. Obtaining a theoretical bound which proves this performance is still an open question, but this simulation demonstrates that even the type-symmetry obtained by basing clock synchronization schemes on undirected graphs is not necessary to achieve excellent performance.

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