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FREE TIME FRACTIONAL OPTIMAL CONTROL PROBLEMS*

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***Abstract*— We present necessary optimality conditions for a class of optimal control problems. The dynamic constraints involve fractional-order and integer-order derivatives and the final time is free. Optimality conditions are obtained using variational principles and some typical problems are solved by approximating the fractional derivatives in terms of integer ones.**

I. INTRODUCTION

An optimal control problem consists in the finding of control signals that make a system satisfy certain constraints while an objective functional is optimized. In a fractional optimal control problem, at least one fractional order derivative or integral is present in the formulation of the problem.

There is a growing interest to the modeling of physical phenomena in terms of fractional operators [1], [2], [8], [11], [15]. This gives us the insight that, sooner or later, such problems will appear in our real-world life. This work is an effort to contribute with some solution methods to a class of fractional optimal control problems and has a potential to be followed by similar works to improve these methods or to try new techniques. For the readers' convenience, we introduce some basic concepts on fractional calculus. We may say that fractional calculus is the integral and differential calculus of real order. Three of the most common fractional operators considered are the Riemann-Liouville fractional integral (RLFI) and derivative (RLFD), and Caputo fractional derivative (CFD), which are defined by

$$
\begin{aligned}
{}_aI_t^{\alpha}x(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}x(\tau)d\tau && \text{(left RLFI)},\\
{}_tI_b^{\alpha}x(t) &= \frac{1}{\Gamma(\alpha)}\int_t^b (\tau-t)^{\alpha-1}x(\tau)d\tau && \text{(right RLFI)},\\
{}_aD_t^{\alpha}x(t) &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{x(\tau)d\tau}{(t-\tau)^{1+\alpha-n}} && \text{(left RLFD)},\\
{}_tD_b^{\alpha}x(t) &= \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_t^b \frac{x(\tau)d\tau}{(\tau-t)^{1+\alpha-n}} && \text{(right RLFD)},\\
{}_a^CD_t^{\alpha}x(t) &= \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{x^{(n)}(\tau)d\tau}{(t-\tau)^{1+\alpha-n}} && \text{(left CFD)},\\
{}_t^CD_b^{\alpha}x(t) &= \frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^b \frac{x^{(n)}(\tau)d\tau}{(\tau-t)^{1+\alpha-n}} && \text{(right CFD)},
\end{aligned}
$$

where n is the smallest integer larger than α, and Γ is the gamma function, that is,

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}\,dt, \quad z>0.$$

When the order is a negative integer, we recover an n-fold integral, and a derivative of order n when the order is a positive integer. If x and $x^{(i)}$, for $i=1,\ldots,n-1$, vanish at $t=a$, then ${}_aD_t^{\alpha}x(t) = {}_a^CD_t^{\alpha}x(t)$, and if they vanish at $t=b$, then ${}_tD_b^{\alpha}x(t) = {}_t^CD_b^{\alpha}x(t)$.

For numerical purposes, it is sometimes useful to approximate these fractional operators using integer order derivatives. In this way one can benefit of all advantages of the classical optimal control theory. Recently, in [4], [13], a new approximation formula has been obtained, with the advantage that we only need the first derivative. The formula is the following:

$$
\begin{aligned}
{}_aD_t^{\alpha}x(t) \simeq{}& A(\alpha,N)(t-a)^{-\alpha}x(t) + B(\alpha,N)(t-a)^{1-\alpha}\dot{x}(t)\\
&-\sum_{p=2}^{N} C(\alpha,p)(t-a)^{1-p-\alpha}V_p(t),
\end{aligned} \tag{1}
$$

where $V_p(t)$ is defined as the solution of the system

$$
\begin{cases}
\dot{V}_p(t) = (1-p)(t-a)^{p-2}x(t)\\
V_p(a) = 0
\end{cases}
$$

for $p=2,3,\ldots,N$, and the coefficients $A=A(\alpha,N), B=B(\alpha,N)$ and $C_p = C(\alpha,p)$ are given by the formulas

$$
\begin{aligned}
A(\alpha,N) &= \frac{1}{\Gamma(1-\alpha)}\left[1+\sum_{p=2}^{N}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha)(p-1)!}\right],\\
B(\alpha,N) &= \frac{1}{\Gamma(2-\alpha)}\left[1+\sum_{p=1}^{N}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!}\right],\\
C(\alpha,p) &= \frac{1}{\Gamma(2-\alpha)\Gamma(\alpha-1)}\frac{\Gamma(p-1+\alpha)}{(p-1)!}.
\end{aligned}
$$

Assuming $L_2 = \max_{\tau\in[a,t]}|\ddot{x}(\tau)|$, this approximation gives a truncation error

$$|E_{tr}(t)| \le L_2 \frac{e^{(1-\alpha)^2+1-\alpha}}{\Gamma(2-\alpha)(1-\alpha)N^{1-\alpha}}(t-a)^{2-\alpha}$$

(see [13]). For the right Riemann-Liouville fractional derivative, we have

$$
\begin{aligned}
{}_tD_b^{\alpha}x(t) \simeq{}& A(\alpha,N)(b-t)^{-\alpha}x(t) + B(\alpha,N)(b-t)^{1-\alpha}\dot{x}(t)\\
&-\sum_{p=2}^{N} C(\alpha,p)(b-t)^{1-p-\alpha}W_p(t),
\end{aligned} \tag{2}
$$

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where W_p is the solution of the differential equation

$$\begin{cases} \dot{W}_p(t) = -(1-p)(b-t)^{p-2}x(t), \\ W_p(b) = 0. \end{cases}$$

To approximate the Caputo fractional derivative, we may use the formulas that establish a relation between the two fractional operators,

$${}_a^C D_t^\alpha x(t) = {}_a D_t^\alpha x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(a)(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)}$$

and

$${}_t^C D_b^\alpha x(t) = {}_t D_b^\alpha x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(b)(b-t)^{k-\alpha}}{\Gamma(k-\alpha+1)},$$

and the relation given in Eq. (1).

II. NECESSARY CONDITIONS FOR OPTIMALITY

Let $\alpha \in (0,1)$, $a \in \mathbb{R}$, L and f be two differentiable functions with domain $[a,+\infty[\times\mathbb{R}^2$ and ϕ be a differentiable function with domain $[a,+\infty[\times\mathbb{R}$. The fundamental problem is stated in the following way:

$$\min \quad J(x,u,T) = \int_a^T L(t,x(t),u(t))\,dt + \phi(T,x(T)) \quad (3)$$

subject to

$$M\dot{x}(t) + N\,{}_a^C D_t^\alpha x(t) = f(t,x(t),u(t)) \quad (4)$$

and

$$x(a) = x_a, \quad (5)$$

with $(M,N) \neq (0,0)$ and x_a a fixed real number (later we consider the cases T and/or $x(T)$ fixed). Here, T is a variable number with $a < T < \infty$. We assume that the state variable x is differentiable and that the control u is continuous. We remark that we obtain the classical case when $N = 0$; the case $M = 0$ and fixed T has already been studied for different types of fractional derivatives (see, e.g., [1], [2], [3], [7], [8], [14], [15]). In [9] a special type of the proposed problem is studied, for a fixed T and without dependence of M. Here we are interested not only on optimal trajectories x and control functions u, but also on the time T for which the functional attains its minimum value.

A. FRACTIONAL NECESSARY CONDITIONS

To deduce the necessary conditions that an optimal triplet (x,u,T) must satisfy, we will use a Lagrange multiplier to adjoin the constraints (4) and (5) to the performance functional (3). To start, we define the Hamiltonian function H by the formula

$$H(t,x,u,\lambda) = L(t,x,u) + \lambda f(t,x,u),$$

where λ is the Lagrange multiplier, and so we can rewrite the initial problem using the new function:

$$\mathscr{J}(x,u,T,\lambda) = \int_a^T H - \lambda[M\dot{x} + N\,{}_a^C D_t^\alpha x]\,dt + \phi(T,x(T)).$$

Next, consider the variation functions

$$x + \delta x,\; u + \delta u,\; T + \delta T \text{ and } \lambda + \delta\lambda,$$

with $\delta x(a) = 0$ by the imposed boundary condition on $t = a$. We use the well-known fact that the first variation of $\mathscr{J}$ must vanish when it is evaluated at the optimal control. We also define a new variable

$$\delta x_T = [x + \delta x](T + \delta T) - x(T).$$

Further, for convenience, we will assume from now on that $\dot{\delta x}(T) = 0$. By Taylor's theorem,

$$[x+\delta x](T+\delta T) - [x+\delta x](T) = \dot{x}(T)\delta T + O(\delta T^2),$$

and so

$$\delta x(T) = \delta x_T - \dot{x}(T)\delta T + O(\delta T^2).$$

By the arbitrariness of the variation functions, and the assumptions above, the following theorem is proven.

Theorem 2.1: If (x,u,T) is a minimizer of (3), under the dynamic constraint (4) and the initial condition (5), then there exists a function λ for which the triplet (x,u,λ) satisfies the next conditions:

The Hamiltonian system

$$\begin{aligned} &\frac{\partial H}{\partial x}(t,x,u,\lambda) = -M\dot{\lambda}(t) + N\,{}_t D_T^\alpha \lambda(t), \\ &\frac{\partial H}{\partial \lambda}(t,x,u,\lambda) = M\dot{x}(t) + N\,{}_a^C D_t^\alpha x(t), \end{aligned} \quad (6)$$

for all $t \in [a,T]$;

The stationary condition

$$\frac{\partial H}{\partial u}(t,x,u,\lambda) = 0, \quad \text{for all } t \in [a,T]; \quad (7)$$

The transversality conditions

$$\left[H - N\lambda\,{}_a^C D_t^\alpha x + N\dot{x}_t I_T^{1-\alpha}\lambda + \frac{\partial \phi}{\partial t}\right]_{t=T} = 0,$$
$$\left[M\lambda + N\,{}_t I_T^{1-\alpha}\lambda - \frac{\partial \phi}{\partial x}\right]_{t=T} = 0.$$

Let us study some particular cases, namely when restrictions are imposed on the end time T or on $x(T)$.

Corollary 2.2: Under assumptions of Theorem 2.1,

1) If T is fixed and $x(T)$ is free, then the transversality condition is replaced by
$$\left[M\lambda + N\,{}_t I_T^{1-\alpha}\lambda - \frac{\partial \phi}{\partial x}\right]_{t=T} = 0.$$
2) If $x(T)$ is fixed and T is free, then the transversality condition is replaced by
$$\left[H - N\lambda\,{}_a^C D_t^\alpha x + N\dot{x}_t I_T^{1-\alpha}\lambda + \frac{\partial \phi}{\partial t}\right]_{t=T} = 0.$$
3) If T and $x(T)$ are both fixed, then we have no transversality condition.
4) If the terminal point $x(T)$ belongs to a fixed curve, i.e., if there exists a differentiable curve γ such that $x(T) = \gamma(T)$, then the transversality condition is replaced by
$$\left[H - N\lambda\,{}_a^C D_t^\alpha x + N\dot{x}_t I_T^{1-\alpha}\lambda + \frac{\partial \phi}{\partial t} - \dot{\gamma}\left(M\lambda + N\,{}_t I_T^{1-\alpha}\lambda - \frac{\partial \phi}{\partial x}\right)\right]_{t=T} = 0.$$

5) If T is fixed and $x(T) \geq K$, for some fixed $K \in \mathbb{R}$, then the transversality condition is replaced by

$$\left[M\lambda + N\,{}_tI_T^{1-\alpha}\lambda - \frac{\partial \phi}{\partial x}\right]_{t=T} \leq 0$$

and

$$(x(T)-K)\left[M\lambda + N\,{}_tI_T^{1-\alpha}\lambda - \frac{\partial \phi}{\partial x}\right]_{t=T} = 0.$$

6) If $x(T)$ is fixed and $T \leq K$, for some fixed $K \in \mathbb{R}$, then the transversality condition is replaced by

$$\left[H - N\lambda\,{}_a^CD_t^\alpha x + N\dot{x}_t I_T^{1-\alpha}\lambda + \frac{\partial \phi}{\partial t}\right]_{t=T} \geq 0$$

and

$$(T-K)\left[H - N\lambda\,{}_a^CD_t^\alpha x + N\dot{x}_t I_T^{1-\alpha}\lambda + \frac{\partial \phi}{\partial t}\right]_{t=T} = 0.$$

Proof: The first three conditions are obvious. The fourth follows from

$$\delta x_T = \gamma(T+\delta T) - \gamma(T) = \dot{\gamma}(T)\delta T + O(\delta T^2).$$

To prove *5*, observe that we have two possible cases. If $x(T) > K$, then δx_T may take negative and positive values, and so we get

$$\left[M\lambda(t) + N\,{}_tI_T^{1-\alpha}\lambda(t) - \frac{\partial \phi}{\partial x}(t,x(t))\right]_{t=T} = 0.$$

On the other hand, if $x(T) = K$, then $\delta x_T \geq 0$ and so

$$-\delta x_T\left[M\lambda(t) + N\,{}_tI_T^{1-\alpha}\lambda(t) - \frac{\partial \phi}{\partial x}(t,x(t))\right]_{t=T} \geq 0.$$

The proof of the last condition is similar. ■

A few remarks: case *1* of Corollary 2.2 was proven in [7] for $(M,N) = (0,1)$ and $\phi \equiv 0$. Also, if we allow $\alpha = 1$, we obtain the necessary conditions for the standard case (see e.g. [6]):

$$\begin{cases} \dfrac{\partial H}{\partial x}(t,x,u,\lambda) = -\dot{\lambda}(t), \\ \dfrac{\partial H}{\partial \lambda}(t,x,u,\lambda) = \dot{x}(t), \end{cases} \qquad \begin{cases} \dfrac{\partial H}{\partial u}(t,x,u,\lambda) = 0, \\ \lambda(T) = 0. \end{cases}$$

B. APPROXIMATED INTEGER ORDER NECESSARY CONDITIONS

Using approximation (1), up to order K, we can transform the original problem (3)–(5) to the following classical problem:

$$\min \quad \tilde{J}(x,u,T) = \int_a^T L(t,x(t),u(t))\,dt + \phi(T,x(T))$$

subject to

$$\begin{cases} \dot{x}(t) = \left(f - \frac{NAx(t)}{(t-a)^\alpha} + \sum_{p=2}^K \frac{NC_pV_p(t)}{(t-a)^{p-1+\alpha}} + \frac{Nx(a)(t-a)^{-\alpha}}{\Gamma(1-\alpha)}\right)\xi(t) \\ \dot{V}_p(t) = (1-p)(t-a)^{p-2}x(t), \quad p = 2,\ldots,K, \end{cases}$$

and

$$\begin{cases} x(a) = x_a, \\ V_p(a) = 0, \quad p = 2,3,\ldots,K, \end{cases} \tag{8}$$

where $A = A(\alpha,K)$, $B = B(\alpha,K)$ and $C_p = C(\alpha,p)$ are the coefficients in approximation (1) and

$$\xi(t) = \frac{1}{M + NB(t-a)^{1-\alpha}}.$$

Now that we are dealing with an integer order problem, we can follow a classical procedure, see [10], by defining a Hamiltonian

$$H = \lambda_1\left(f - \frac{NAx(t)}{(t-a)^\alpha} + \sum_{p=2}^K \frac{NC_pV_p(t)}{(t-a)^{p-1+\alpha}} + \frac{Nx(a)(t-a)^{-\alpha}}{\Gamma(1-\alpha)}\right)\xi$$
$$+ \sum_{p=2}^K (1-p)(t-a)^{p-2}x(t)\lambda_p(t) + L(t,x,u)$$

where $\boldsymbol{\lambda} = (\lambda_1,\lambda_2,\ldots,\lambda_K)$ is the Lagrange multiplier vector. Assuming $\mathbf{x} = (x,V_2,...,V_K)$, the necessary conditions of optimality,

$$\begin{cases} \dot{\mathbf{x}} = \frac{\partial H}{\partial \boldsymbol{\lambda}}, \\ \dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \mathbf{x}}, \\ \frac{\partial H}{\partial u} = 0, \end{cases}$$

result in a two point boundary value problem. In addition to the boundary conditions (8), assuming that $(T^*,\mathbf{x}^*,\boldsymbol{u}^*)$ is the optimal triplet, the necessary conditions of optimality imply

$$\left[\frac{\partial \phi}{\partial \mathbf{x}}\right]^{tr}\delta\mathbf{x}_T + \left[H + \frac{\partial \phi}{\partial t}\right]\delta T = 0.$$

In this formula $[\cdot]^{tr}$ stands for the transpose of $[\cdot]$.

Our problem benefits the fact that V_p, $p = 2,\ldots,K$, are auxiliary variables that we are not concerned with their values at the final time. In other words, $V_p(T)$, $p = 2,\ldots,K$, are free and, therefore,

$$\lambda_p(T) = \frac{\partial \phi}{\partial V_p}\Big|_{t=T}, \quad p = 2,\ldots,K.$$

The value of $\lambda_1(T)$ is determined by the value of $x(T)$. If it is free, then $\lambda_1(T) = \frac{\partial \phi}{\partial x}|_{t=T}$. Whenever the final time is free, a transversality condition of the form

$$\left[H(t,\mathbf{x},\boldsymbol{u},\boldsymbol{\lambda}) - \frac{\partial \phi}{\partial t}\right]_{t=T} = 0$$

completes the set of required boundary conditions.

III. NUMERICAL TREATMENT AND EXAMPLES

In this section we try to apply the necessary conditions introduced in Section II to solve test problems. Solving an optimal control problem, analytically, is an optimistic goal and is unreachable unless for simple cases. Therefore, we apply numerical methods and softwares to solve our problems. In each case we try to solve the problem either by applying fractional necessary conditions or by approximating the problem and solving the resulting classical one. In order to achieve a good sense of confidence, we first solve a problem with fixed final time. The exact solution, the optimal control and trajectory, are known for this example and hence we can compare them with our numerical solutions.

A. FIXED FINAL TIME

Example 3.1: Consider the optimal control problem:

$$J(x,u) = \int_0^1 (tu - (\alpha+2)x)^2 \, dt \quad \rightarrow \quad minimize$$

under the constraints

$$\begin{cases} \dot{x}(t) + {}_0^C D_t^\alpha x(t) = u(t) + t^2 \\ x(0) = 0 \\ x(1) = \frac{2}{\Gamma(3+\alpha)}. \end{cases}$$

The optimal solution is given by

$$(\overline{x},\overline{u}) = \left(\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}, \frac{2t^{\alpha+1}}{\Gamma(\alpha+2)} \right),$$

since $\overline{x}(0) = 0$, $\overline{x}(1) = \frac{2}{\Gamma(3+\alpha)}$, ${}_0^C D_t^\alpha \overline{x}(t) = t^2$, $J(x,u) \geq 0$, for all pair (x,u), and $J(\overline{x},\overline{u}) = 0$.

Let us see that $(\overline{x},\overline{u})$ satisfies the fractional necessary conditions of optimality. In this case, the Hamiltonian is defined by

$$H(t,x,u,\lambda) = (tu - (\alpha+2)x)^2 + \lambda(u + t^2)$$

and

1) $\dfrac{\partial H}{\partial u}(t,\overline{x},\overline{u},\lambda) = 2(t\overline{u} - (\alpha+2)\overline{x})t + \lambda = \lambda$ and so $\lambda(t) = 0$, for all $t \in [0,1]$;
2) $\dfrac{\partial H}{\partial x}(t,\overline{x},\overline{u},0) = -2(\alpha+2)(t\overline{u} - (\alpha+2)\overline{x}) = 0$ is verified;
3) $\dfrac{\partial H}{\partial \lambda}(t,\overline{x},\overline{u},0) = \overline{u} + t^2$ is verified;
4) the transversality condition becomes (cf. Corollary 2.2)

$$\lambda(t) + {}_t I_T^{1-\alpha} \lambda(t) = 0$$

at $t = T$, which is also verified.

First, we apply the fractional necessary conditions. The Hamiltonian is

$$H = (tu - (\alpha+2)x)^2 + \lambda u + \lambda t^2.$$

The stationary condition (7) implies that $u(t) = \frac{\alpha+2}{t}x(t) - \frac{\lambda(t)}{2t^2}$ and hence

$$H = -\frac{\lambda^2}{4t^2} + \frac{\alpha+2}{t} x\lambda + t^2\lambda. \tag{9}$$

Finally, (6) is simplified to

$$\begin{cases} \dot{x}(t) + {}_0^C D_t^\alpha x(t) = -\frac{\lambda}{2t^2} + \frac{\alpha+2}{t}x + t^2 \\ -\dot{\lambda} + {}_t D_1^\alpha \lambda(t) = \frac{\alpha+2}{t}\lambda \\ x(0) = 0 \\ x(1) = \frac{2}{\Gamma(3+\alpha)}. \end{cases}$$

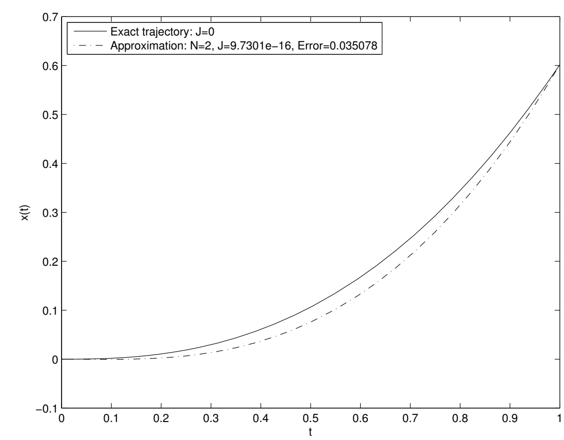
At this point we encounter a fractional boundary value problem that needs to be solved to reach optimal solutions. A handful of methods can be found in the literature to solve this problem. Nevertheless, we use approximations (1) and (2), up to order N, that have been introduced in [4] and used in [9], [13]. With our choice of approximation, the fractional problem is transformed into an integer order boundary value problem:

$$\begin{cases} \dot{x} = \frac{1}{1+Bt^{1-\alpha}} \left(\left(\frac{\alpha+2}{t} - At^{-\alpha} \right) x + \sum_{p=2}^N \frac{C_p V_p}{t^{p+\alpha-1}} - \frac{\lambda}{2t^2} + t^2 \right) \\ \dot{V}_p = (1-p)t^{p-2}x, \quad p = 2,\ldots,N \\ \dot{\lambda} = \frac{1}{1+B(1-t)^{1-\alpha}} \left(\left(\frac{A}{(1-t)^\alpha} - \frac{\alpha+2}{t} \right) \lambda - \sum_{p=2}^N \frac{C_p W_p}{(1-t)^{p+\alpha-1}} \right) \\ \dot{W}_p = -(1-p)(1-t)^{p-2}\lambda, \quad p = 2,\ldots,N \end{cases}$$

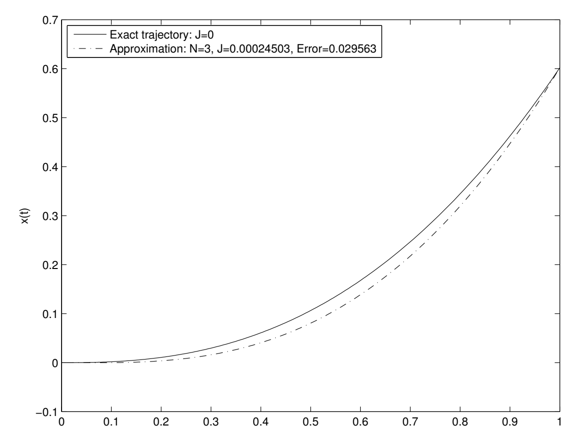
subject to the boundary conditions

$$\begin{cases} x(0) = 0, \quad x(1) = \frac{2}{\Gamma(3+\alpha)} \\ V_p(0) = 0, \quad p = 2,\ldots,N \\ W_p(1) = 0, \quad p = 2,\ldots,N. \end{cases}$$

For $N = 2$, $N = 3$ and $\alpha = 0.5$, the trajectories are depicted in Figure 1. Since the exact solution for this problem is known, for each N, we compute an error of approximation using the maximum norm. Assume that $\overline{x}(t_i)$ is the approximated values on the discrete time horizon $a = t_0, t_1, \ldots, t_n$. Then the error is given by $E = \max_i(|x(t_i) - \overline{x}(t_i)|)$.



(a) $x(t)$, $N = 2$, $E = 0.035$



(b) $x(t)$, $N = 3$, $E = 0.029$

Fig. 1. Exact (solid lines) versus numerical approximations of Example 3.1 using fractional necessary conditions.

Another approach is to approximate the original problem by substituting (1) for fractional derivative. Following the procedure discussed in Section II, Example 3.1 becomes

$$\tilde{J}(x,u) = \int_0^1 (tu - (\alpha+2)x)^2\, dt \quad \to \quad minimize$$

under the constraints

$$\begin{cases} \dot{x} = \frac{1}{1+Bt^{1-\alpha}}\left(u + t^2 - At^{-\alpha}x + \sum_{p=2}^{N} C_p t^{1-p-\alpha} V_p\right) \\ \dot{V}_p = (1-p)t^{p-2}x, \quad p = 2,\ldots,N \\ x(0) = 0 \\ x(1) = \frac{2}{\Gamma(3+\alpha)} \\ V_p(0) = 0, \quad p = 2,\ldots,N. \end{cases}$$

The Hamiltonian system for this classical problem is

$$H = [tu - (\alpha+2)x]^2 + \sum_{p=2}^{N}(1-p)t^{p-2}\lambda_p x + \frac{\lambda_1}{1+Bt^{1-\alpha}}\left(-At^{-\alpha}x + \sum_{p=2}^{N} C_p t^{1-p-\alpha} V_p + u + t^2\right).$$

Using the fact that the control is not constrained for this example, we have $\frac{\partial H}{\partial u} = 0$ along the optimal control and

$$u(t) = \frac{\alpha+2}{t}x(t) - \frac{\lambda_1(t)}{2t^2(1+Bt^{1-\alpha})}.$$

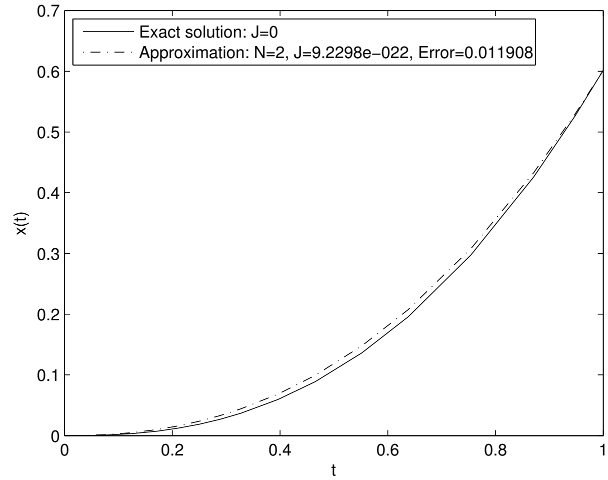
Finally, the Hamiltonian becomes

$$H = \phi_0\lambda_1^2 + \phi_1 x\lambda_1 + \sum_{p=2}^{N}\phi_p V_p\lambda_1 + \phi_{N+1}\lambda_1 + \sum_{p=2}^{N}(1-p)t^{p-2}x\lambda_p, \tag{10}$$

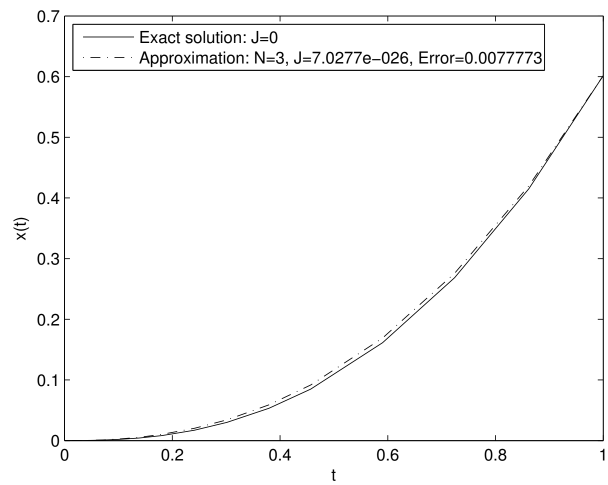
where $\phi_0(t) = \frac{-1}{4t^2(1+Bt^{1-\alpha})^2}$, $\phi_1(t) = \frac{\alpha+2-At^{1-\alpha}}{t(1+Bt^{1-\alpha})}$, $\phi_p(t) = \frac{C(\alpha,p)t^{1-p-\alpha}}{1+Bt^{1-\alpha}}$ and $\phi_{N+1}(t) = \frac{t^2}{1+Bt^{1-\alpha}}$. The necessary conditions of optimality

$$\begin{cases} \dot{\mathbf{x}} = \frac{\partial H}{\partial \lambda} \\ \dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \mathbf{x}} \end{cases}$$

then gives a two point boundary value problem that is solved using Matlab's `bvp4c` built in function for $N=2$ and $N=3$. The results are depicted in Figure 2.



(a) $x(t)$, $N=2$, $E=0.012$



(b) $x(t)$, $N=3$, $E=0.008$

Fig. 2. Exact (solid lines) versus numerical approximations of Example 3.1 using approximated integer order necessary conditions.

B. FREE FINAL TIME

Now that we are equipped with a numerical method to solve fractional optimal control problems, let the final time be free in previous example. More precisely, consider the following problem.

Example 3.2: Find an optimal triplet $(x(\cdot),u(\cdot),T)$ that minimizes

$$J(x,u) = \int_0^T (tu - (\alpha+2)x)^2\, dt$$

under the constraints

$$\begin{cases} \dot{x}(t) + {}_0^C D_t^{\alpha} x(t) = u(t) + t^2 \\ x(0) = 0 \\ x(T) = 1. \end{cases}$$

The solution procedure is exactly the same as the previous example with fixed final time. We use the fractional necessary conditions and, after, we approximate the fractional terms. The only difference is that there is here an extra unknown: the terminal time T. The boundary condition for this new unknown is chosen appropriately from the variants of transversality conditions discussed in Corollary 2.2, i.e.,

$$\left[H(t,x,u,\lambda) - \lambda(t){}_a^C D_t^{\alpha} x(t) + \dot{x}(t){}_t I_T^{1-\alpha}\lambda(t)\right]_{t=T} = 0.$$

Since we require λ to be continuous, ${}_t I_T^{1-\alpha}\lambda(t)\big|_{t=T} = 0$ (cf. [12, pag. 46]) and so $\lambda(T) = 0$ satisfies (9).

One possible way to solve this problem consist in translating it to the interval $[0,1]$ by the change of variable $t = Ts$ [5]. In this setting, either we add T to the problem as a new variable with the dynamics $\dot{T}(s) = 0$, or we treat it as a parameter. The following form is a parametric boundary value problem for $p=2,\dots,N$:

$$\begin{cases} \dot{x}(s) = \frac{T}{1+B(Ts)^{1-\alpha}}\left(\left(\frac{\alpha+2}{Ts}-\frac{A}{(Ts)^{\alpha}}\right)x+(Ts)^2\right. \\ \qquad \left. +\sum_{p=2}^{N} C_p (Ts)^{1-p-\alpha}V_p - \frac{\lambda(s)}{2(Ts)^2}\right) \\ \dot{V}_p(s) = T(1-p)(Ts)^{p-2}x \\ \dot{\lambda}(s) = \frac{T}{1+B(1-Ts)^{1-\alpha}}\left(\left(A(1-Ts)^{-\alpha}-\frac{\alpha+2}{Ts}\right)\lambda\right. \\ \qquad \left. -\sum_{p=2}^{N} C_p (1-Ts)^{1-p-\alpha}W_p\right) \\ \dot{W}_p(s) = -T(1-p)(1-Ts)^{p-2}\lambda \end{cases}$$

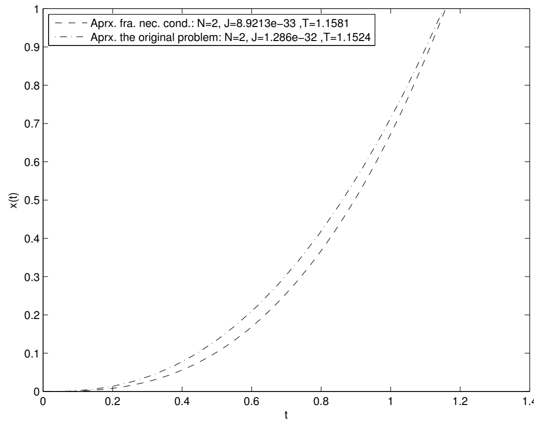
subject to the boundary conditions

$$\begin{cases} x(0)=0 \\ V_p(0)=0 \\ W_p(1)=0, \end{cases} \qquad \begin{cases} x(1)=1 \\ \lambda(1)=0. \end{cases}$$

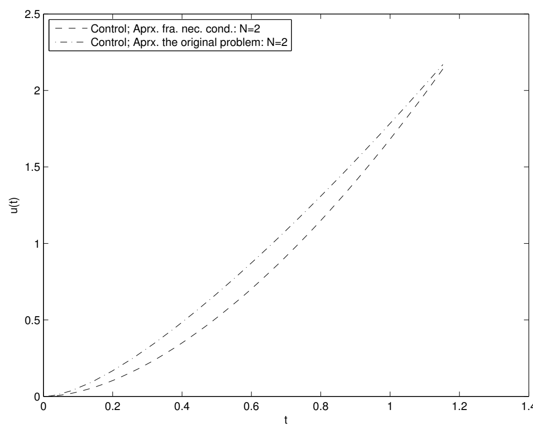
This parametric boundary value problem is solved for $N=2$ and $\alpha=0.5$ by Matlab's `bvp4c` function. The outcome is shown in Figure 3.

We also solve Example 3.2 by directly transforming it to a classical problem with free final time. As it is well known in the classical theory of optimal control, when the final time is free, the Hamiltonian must vanish at terminal point, i.e., we have $H|_{t=T}=0$ provided H is given by (10) [10]. For $N=2$, the necessary conditions of optimality give the following two point boundary value problem:

$$\begin{cases} \dot{x}(t) = 2\phi_0(t)\lambda_1(t)+\phi_1(t)x(t)+\phi_2(t)V_2(t)+\phi_3(t) \\ \dot{V_2}(t) = -x(t) \\ \dot{\lambda_1}(t) = -\phi_1(t)\lambda_1(t)+x(t) \\ \dot{\lambda_2}(t) = -\phi_2(t)\lambda_1(t). \end{cases}$$



(a) $x(t)$, $N=2$



(b) $u(t)$, $N=2$

Fig. 3. Numerical solutions to Example 3.2 using fractional necessary conditions (dashed lines) and approximation to an integer order problem (dash-dotted lines).

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