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Equivalent switching strategy and analog validation of the fractional variable order derivative definition

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Abstract— **The paper presents switching strategy that is equivalent to one type of variable order derivative definitions. The numerical scheme, based on matrix approach, for this type of the definition is also introduced. Using this approach the identity of the switching strategy and considered definition is derived. The switching scheme can be used as an interpretation of this type of definition. Paper presents also numerical examples for introduced methods. Finally, the idea and results of analog (electrical) realization of the switching fractional order integrator (of orders 0.5 and 1) are presented and compared with numerical approach.**

I. Introduction

Fractional calculus is a generalization of traditional integer order integration and differentiation actions onto non-integer order fundamental operator. The idea of such a generalization has been mentioned in 1695 by Leibniz and L'Hospital. In the end of 19th century Liouville and Riemann introduced first definition of fractional derivative. However, only just in late 60' of the 20th century, this idea drew attention of engineers [6]. Fractional calculus was found a very useful tool for modelling behavior of many materials and systems, especially those based on the diffusion processes. One of such devices that can be modelled more efficiently by fractional calculus are ultracapacitors. Models of these electronic storage devices, whose capacity can be even thousands of Farads, based on fractional order models, were presented in [18].

Recently, the case when the order is changing in time, started to be intensively developed. The variable fractional order behavior can be met, for example, in chemistry (when the properties of the system are changing due to chemical reactions), electrochemistry, and other areas. In [10], experimental studies of an electrochemical example of physical fractional variable order system are presented. Papers [10], [12] present methods for numerical realization of fractional variable order integrators or differentiators. The fractional variable order calculus also can be used to obtain variable order fractional noise [13], and to obtain new control algorithms [14]. Some properties of such systems are presented in [15]. In [16], the variable order interpretation of the analog realization of fractional orders integrators, realized as domino ladders, was presented.

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In [19], [20], three general types of variable order derivative definitions can be found, however, these definitions were given without interpretation and derivation. Some other modifications of the mentioned definitions are presented in [11]. In this paper, the switching strategy numerically identical to the 2nd type of fractional derivative definition is introduced, and results of its analog realization are presented. Introduced switching scheme can be used as an interpretation of this definition.

The rest of the paper is organized as follows. Section II presents existing generalizations of Grunwald-Letnikov definition of fractional order derivatives. In Section III, a switching scheme for practical implementation of variable order derivative is given and studied. Section III presents a generalization of the matrix approach for switching order and derivation of identity of the switching scheme and the second type of definition. Section IV presents numerical examples of proposed methods with comparison to the analytical solutions. Finally, Section V presents an analog realization of the switched order integrator and comparison of obtained results to the numerical solutions.

II. Fractional variable order Grunwald-Letnikov type derivatives

As a base of generalization onto variable order derivative the following definition is taken into consideration:

Definition 1: Fractional constant order derivative is defined as follows:

$$ {}_0\mathrm{D}_t^{\alpha} f(t) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{r=0}^{n} (-1)^r \binom{\alpha}{r} f(t-rh), $$

where $n = \lfloor t/h \rfloor$.

For the case of order changing with time (variable order case), three general types of definition can be found in [19], [20]. The first one is obtained by replacement of constant order α by variable order $\alpha(t)$. In this approach all coefficients for past samples are obtained for present value of the order and is given as follows:

Definition 2: The 1st type of fractional variable order derivative is defined as follows:

$$ {}_0\mathrm{D}_t^{\alpha(t)} f(t) = \lim_{h\to 0} \frac{1}{h^{\alpha(t)}} \sum_{r=0}^{n} (-1)^r \binom{\alpha(t)}{r} f(t-rh). $$

The second type of definition assumes that coefficients for past samples are obtained for order that was present for these samples. In this case, the definition has the following form:

Definition 3: The 2nd type of fractional variable order derivative is defined as follows:

$$ {}_0\mathrm{D}_t^{\alpha(t)} f(t) = \lim_{h\to 0} \sum_{r=0}^{n} \frac{(-1)^r}{h^{\alpha(t-rh)}} \binom{\alpha(t-rh)}{r} f(t-rh). $$

The third definition is less intuitive and assumes that coefficients for the newest samples are obtained respectively for the oldest orders. For such a case, the following definition applies:

Definition 4: The 3rd type of fractional variable order derivative is defined as:

$$ {}_0\mathrm{D}_t^{\alpha(t)} f(t) = \lim_{h\to 0} \sum_{r=0}^{n} \frac{(-1)^r}{h^{\alpha(rh)}} \binom{\alpha(rh)}{r} f(t-rh). $$

III. Practical implementation and numerical scheme of switched (variable) order derivative

In this section, the routines and schemes for switching order derivative will be presented. For simplicity, we start with the simplest case of order switching, namely, switching between two real arbitrary constant orders, say α_1 and α_2. Next, this idea will be generalized for a multiple-switching (variable order) case.

A. *Simple-switching order case*

The idea is depicted in Fig. 1, where all the switches S_i, $i = 1, 2$, change their positions depending on an actual value of $\alpha(t)$. If we want to switch from α_1 to α_2, then, before switching time T, we have: $S_1 = b$, $S_2 = a$, and after this time: $S_1 = a$ and $S_2 = b$. At the instant time T, the derivative block of complementary order $\bar{\alpha}_2$ is preconnected on the front of the current derivative block of order α_1, where

$$ \bar{\alpha}_2 = \alpha_2 - \alpha_1. \tag{1} $$

If $\bar{\alpha}_2 < 0$, then ${}_T\mathrm{D}_t^{\bar{\alpha}_2}$ corresponds to integration of $f(t)$; and, if $\bar{\alpha}_2 > 0$, then ${}_T\mathrm{D}_t^{\bar{\alpha}_2}$ corresponds to derivative of $f(t)$, with appropriate order $\bar{\alpha}_2$.

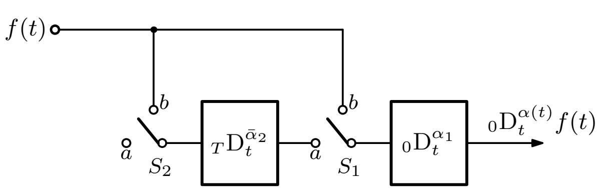

Fig. 1. Structure of simple-switching order derivative (switching from α_1 to α_2)

Now, the numerical scheme corresponding to the above derivative switching structure is introduced. The matrix form of the fractional order derivative is given as follows [5]:

$$ \begin{pmatrix} {}_0\mathrm{D}_0^{\alpha} f(0) \\ {}_0\mathrm{D}_h^{\alpha} f(h) \\ {}_0\mathrm{D}_{2h}^{\alpha} f(2h) \\ \vdots \\ {}_0\mathrm{D}_{kh}^{\alpha} f(kh) \end{pmatrix} = \lim_{h\to 0} W(\alpha, k) \begin{pmatrix} f(0) \\ f(h) \\ f(2h) \\ \vdots \\ f(kh) \end{pmatrix}, $$

where

$$ W(\alpha, k) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ w_{\alpha,1} & 1 & 0 & \dots & 0 \\ w_{\alpha,2} & w_{\alpha,1} & 1 & \dots & 0 \\ w_{\alpha,3} & w_{\alpha,2} & w_{\alpha,1} & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ w_{\alpha,k} & w_{\alpha,k-1} & w_{\alpha,k-2} & \dots & 1 \end{pmatrix}, \tag{2} $$

$W(\alpha, k) \in \mathbb{R}^{(k+1)\times(k+1)}$, $w_{\alpha,i} = \frac{(-1)^i \binom{\alpha}{i}}{h^\alpha}$, and $h = t/k$ is a time step, k is a number of samples.

Lemma 1: For a switching order case, when the switch from order α_1 to order α_2 occurs at time T, the numerical scheme has the following form:

$$ \begin{pmatrix} {}_0\mathrm{D}_0^{\alpha(t)} f(0) \\ {}_0\mathrm{D}_h^{\alpha(t)} f(h) \\ \vdots \\ {}_0\mathrm{D}_{(T-1)h}^{\alpha(t)} f(Th-h) \\ {}_0\mathrm{D}_{Th}^{\alpha(t)} f(Th) \\ \vdots \\ {}_0\mathrm{D}_{kh}^{\alpha(t)} f(kh) \end{pmatrix} = \lim_{h\to 0} W(\alpha_1, k) W(\bar{\alpha}_2, k, T) \begin{pmatrix} f(0) \\ f(h) \\ \vdots \\ f(Th-h) \\ f(Th) \\ \vdots \\ f(kh) \end{pmatrix}, $$

where

$$ W(\bar{\alpha}_2, k, T) = \begin{pmatrix} I_{T,T} & 0_{T,k-T+1} \\ 0_{k-T+1,T} & W(\bar{\alpha}_2, k-T) \end{pmatrix}, $$

and

$$ \alpha(t) = \begin{cases} \alpha_1 & \text{for } t < T, \\ \alpha_1 + \bar{\alpha}_2 & \text{for } t \geq T. \end{cases} $$

The order $\bar{\alpha}_2$, appearing above, is given by relation (1).

Proof: The signal incoming to the block of derivative α_1 can be described as follows

$$ \begin{pmatrix} f(0) \\ f(h) \\ \vdots \\ f(Th-h) \\ {}_{Th}\mathrm{D}_{Th}^{\bar{\alpha}_2} f(Th) \\ \vdots \\ {}_{Th}\mathrm{D}_{kh}^{\bar{\alpha}_2} f(kh) \end{pmatrix} = \lim_{h\to 0} W(\bar{\alpha}_2, k, T) \begin{pmatrix} f(0) \\ f(h) \\ \vdots \\ f(Th-h) \\ f(Th) \\ \vdots \\ f(kh) \end{pmatrix}. $$

Until time T, the input of α_1-block obtains the original function $f(t)$, so, in matrix $W(\bar{\alpha}_2, k, T)$ we have an identity matrix. From time step T, the input signal is passing through the block of derivative $\bar{\alpha}_2$ and we have the sub-matrix $W(\bar{\alpha}_2, k-T)$ that is responsible for starting the $\bar{\alpha}_2$ derivative action from time T. That signal is passing to the α_1-block and has the following matrix form:

$$\begin{pmatrix} {}_0\mathrm{D}_0^{\alpha(t)} f(0) \\ {}_0\mathrm{D}_h^{\alpha(t)} f(h) \\ \vdots \\ {}_0\mathrm{D}_{(T-1)h}^{\alpha(t)} f(Th-h) \\ {}_0\mathrm{D}_{Th}^{\alpha(t)} f(Th) \\ \vdots \\ {}_0\mathrm{D}_{kh}^{\alpha(t)} f(kh) \end{pmatrix} = \lim_{h\to 0} W(\alpha_1, k) \begin{pmatrix} f(0) \\ f(h) \\ \vdots \\ f(Th-h) \\ {}_{Th}\mathrm{D}_{Th}^{\alpha_2} f(Th) \\ \vdots \\ {}_{Th}\mathrm{D}_{kh}^{\alpha_2} f(kh) \end{pmatrix}.$$

■

B. Multiple-switching (variable order) case

In a general case, when there are many switchings between arbitrary orders, we have the following structure presented in Fig. 2. When we switch from the order α_{j-1} to the order α_j at the switch-time instant T_{j-1}, for $j = 2, 3, \ldots$, we have to set:

$$S_i = \begin{cases} a & \text{for } i = 1, \ldots, j-1, \\ b & \text{for } i = j, \end{cases}$$

and the preconnected derivative block (on the front of the previous term) is of the following complementary order:

$$\bar{\alpha}_j = \alpha_j - \alpha_{j-1},$$

where

$$\alpha_{j-1} = \alpha_1 + \sum_{k=1}^{j-2} \bar{\alpha}_{k+1}.$$

The numerical scheme describing the already presented general case of structure allowing to switch between an arbitrary number of orders is given below. What is very important, the numerical scheme of multiple-switching case (when order is switched in each sample time) is equivalent to the 2nd type of variable order derivative.

Theorem 1: Switching order scheme presented in Fig. 2 is equivalent to the 2nd type of variable order derivative (given by Def. 3).

Proof: Let us assume that the order of derivative is changing with every time step, which gives a variable order derivative, and is given as follows:

$$\alpha_k = \sum_{j=0}^{k} \bar{\alpha}_j, \qquad (3)$$

where, in this case, $\alpha_0 = \bar{\alpha}_0$ is a value of initial order. Using Lemma 1, the following numerical scheme is obtained

$$\begin{pmatrix} {}_0\mathrm{D}_0^{\alpha(t)} f(0) \\ \vdots \\ {}_0\mathrm{D}_{kh}^{\alpha(t)} f(kh) \end{pmatrix} = \prod_{j=0}^{k} W(\bar{\alpha}_j, k, jh) \begin{pmatrix} f(0) \\ \vdots \\ f(kh) \end{pmatrix}.$$

The first switching matrices can be described as the following block matrices:

$$\begin{aligned} & W(\bar{\alpha}_0, k, 0) W(\bar{\alpha}_1, k, 1) \\ &= \left(\begin{array}{c|c} 1 & 0_{1,k} \\ \hline R(\bar{\alpha}_0, 1) & W(\bar{\alpha}_0, k-1) \end{array}\right) \left(\begin{array}{c|c} 1 & 0_{1,k} \\ \hline 0 & W(\bar{\alpha}_1, k-1) \end{array}\right) \\ &= \left(\begin{array}{c|c} 1 & 0_{1,k} \\ \hline R(\bar{\alpha}_0, 1) & W(\bar{\alpha}_0 + \bar{\alpha}_1, k-1) \end{array}\right), \end{aligned}$$

where

$$R(\bar{\alpha}, i) = \begin{pmatrix} w_{\bar{\alpha},i} \\ w_{\bar{\alpha},i+1} \\ \vdots \end{pmatrix}$$

is a vector with coefficients given by (2). For a switching in the next sample time we obtain the following numerical scheme:

$$\begin{aligned} & W(\bar{\alpha}_0, k, 0) W(\bar{\alpha}_1, k, 1) W(\bar{\alpha}_2, k, 2) \\ &= \left(\begin{array}{cc|c} 1 & 0 & 0_{1,k-1} \\ w_{\bar{\alpha}_0,1} & 1 & 0_{1,k-1} \\ \hline R(\bar{\alpha}_0, 2) & R(\bar{\alpha}_0 + \bar{\alpha}_1, 1) & W(\bar{\alpha}_1 + \bar{\alpha}_0, k-2) \end{array}\right) \times \\ & \left(\begin{array}{cc|c} 1 & 0 & 0_{1,k-1} \\ 0 & 1 & 0_{1,k-1} \\ \hline 0 & 0 & W(\bar{\alpha}_2, k-1) \end{array}\right) \\ &= \left(\begin{array}{cc|c} 1 & 0 & 0_{1,k-1} \\ w_{\bar{\alpha}_0,1} & 1 & 0_{1,k-1} \\ \hline R(\bar{\alpha}_0, 2) & R(\bar{\alpha}_0 + \bar{\alpha}_1, 1) & W(\bar{\alpha}_0 + \bar{\alpha}_1 + \bar{\alpha}_2, k-2) \end{array}\right). \end{aligned}$$

In the case of $k - th$ switchings of order, we have the following form of the switching matrix $W(\bar{\alpha}_0, k, 0) W(\bar{\alpha}_1, k, 1) \cdots W(\bar{\alpha}_k, k, k)$, i.e.,

$$\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ w_{\bar{\alpha}_0,1} & 1 & 0 & \ldots & 0 \\ w_{\bar{\alpha}_0,2} & w_{\bar{\alpha}_0+\bar{\alpha}_1,1} & 1 & \ldots & 0 \\ w_{\bar{\alpha}_0,3} & w_{\bar{\alpha}_0+\bar{\alpha}_1,2} & w_{\bar{\alpha}_0+\bar{\alpha}_1+\bar{\alpha}_2,1} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ w_{\bar{\alpha}_0,k} & w_{\bar{\alpha}_0+\bar{\alpha}_1,k-1} & w_{\bar{\alpha}_0+\bar{\alpha}_1+\bar{\alpha}_2,k-2} & \ldots & 1 \end{pmatrix}$$

or, shortly, with using relation given by (3)

$$\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ w_{\alpha_0,1} & 1 & 0 & \ldots & 0 & 0 \\ w_{\alpha_0,2} & w_{\alpha_1 1} & 1 & \ldots & 0 & 0 \\ w_{\alpha_0,3} & w_{\alpha_1,2} & w_{\alpha_2,1} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ w_{\alpha_0,k-1} & w_{\alpha_1,k-2} & w_{\alpha_2,k-3} & \ldots & 1 & 0 \\ w_{\alpha_0,k} & w_{\alpha_1,k-1} & w_{\alpha_2,k-2} & \ldots & w_{\alpha_{k-1},1} & 1 \end{pmatrix}.$$

The coefficients of the matrix above are identical with the coefficients given by Def. 3, which completes the proof. ■

IV. Numerical examples

Numerical examples, presented in this section, have been computed in Matlab/Simulink environment, mostly with using dedicated numerical routines, developed by the authors [21]. Number of samples, taken into consideration to

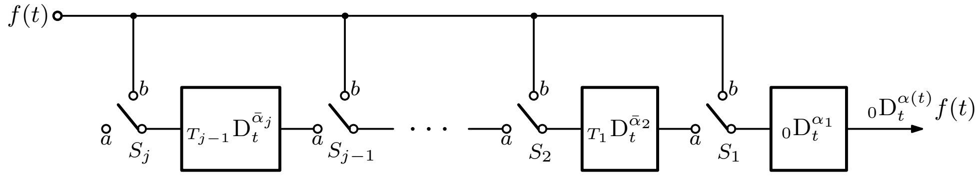

Fig. 2. Structure of multiple-switching order derivatives

numerical evaluation, were chosen in order to cover whole range of time experiment.

Example 1: Let us consider a variable order integration of constant function $f(t) = 1(t)$ with the following sequence of fractional orders $\Lambda = \{-0.4, -1.8, -0.5, -2.5\}$ switched with every one second, i.e.,

$$\alpha(t) = \begin{cases} -0.4 & \text{for } 0 \le t < 1, \\ -1.8 & \text{for } 1 \le t < 2, \\ -0.5 & \text{for } 2 \le t < 3, \\ -2.5 & \text{for } 3 \le t \le 4. \end{cases}$$

The analytical solution of such integration is the following

$${}_0\mathrm{D}_t^{\alpha(t)} f(t) \approx \begin{cases} D_1 & \text{for } 0 \le t < 1, \\ D_2 & \text{for } 1 \le t < 2, \\ D_3 & \text{for } 2 \le t < 3, \\ D_4 & \text{for } 3 \le t \le 4, \end{cases}$$

where

$$\begin{aligned} D_1 &= 1.127 t^{0.4}, \\ D_2 &= 0.597(t-1)^{1.8} + 1.127\left(t^{0.4} - (t-1)^{0.4}\right), \\ D_3 &= 1.128\sqrt{t-2} + 0.597\left((t-1)^{1.8} - (t-2)^{1.8}\right) \\ &\quad + 1.127\left(t^{0.4} - (t-1)^{0.4}\right), \\ D_4 &= 0.3(t-3)^{2.5} + 0.597\left((t-1)^{1.8} - (t-2)^{1.8}\right) \\ &\quad + 1.128\left(\sqrt{t-2} - \sqrt{t-3}\right) + 1.127\left(t^{0.4} - (t-1)^{0.4}\right). \end{aligned}$$

The plot of integration result is presented in Fig. 3.

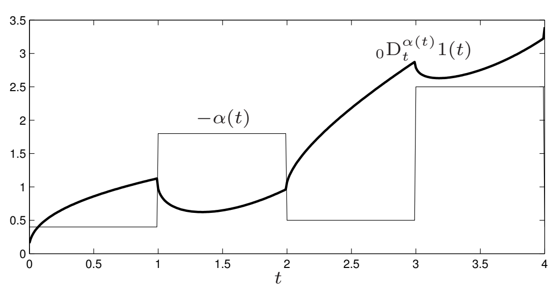

Fig. 3. Plot of the variable order integral of constant function (Ex. 1)

V. Analog realization of the second type of fractional variable order derivative

A. Experimental setup

An analog realization of switching system, directly based on the second type of fractional variable order integral, is presented in Fig. 4. The experimental setup contains:

- dSPACE DS1104 PPC card with a PC,
- operational amplifier TL071,
- analog switches DG303,
- passive elements such as:
  - resistors with the following values: $R = 9.65\mathrm{k}\Omega$, $R_1 = 2\mathrm{k}\Omega$, $R_2 = 8.2\mathrm{k}\Omega$,
  - capacitors with the following value: $C_1 = 470\mathrm{nF}$.

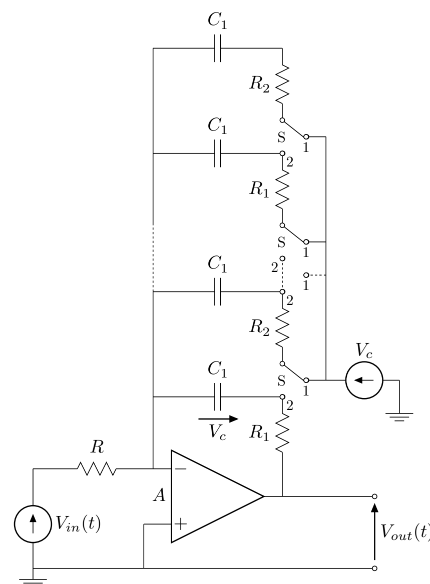

Fig. 4. Analog realization of the 2nd type of fractional variable order integral

Depending on switches position marked as S in Fig. 4 the circuit can be described by fractional order transfer function or traditional transfer function for integral system.

1) For a case, when S-switches are connected to terminals marked as 1, the following transfer function has been obtained:
$$G_1(s) = \frac{1}{T_1 s},$$
where T_1 is a time constant.
2) For a case, when S-switches are connected to terminals marked as 2, the following transfer function has been obtained:
$$G_2(s) = \frac{1}{T_2 s^{0.5}},$$
where T_2 is a time constant of the half order integral; it should be stressed that $T_2 \neq T_1$.

Based on switches position, the system can be switched in two ways:

1) switching from terminals 1 to 2: In this case, the system described by the first order transfer function $G_1(s)$ is switched to system of order half described by transfer function $G_2(s)$. To keep the behavior of the second type of definition it is necessary to maintain a continuous voltage of capacitors in the rest branches, which are connected to terminals marked as 1. The voltages of capacitors are set to the value of voltage on the capacitor in the first order integral system (the first capacitor). The main idea has been shown in Fig. 4. The example of switching matrix is given in the following form:

$$W(-1,5,0)W(0.5,5,3) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0.5 & 1 & 0 \\ 1 & 1 & 1 & 0.375 & 0.5 & 1 \end{pmatrix}.$$

As it could be seen in this switching matrix, before switching we have all the coefficients equal to 1, and after switching coefficients started to have values as the half order integration started in switching time. However, samples for time less than switching time are still taken with coefficients equal to 1 (it is a special case of the first order). It can be interpreted that after the switching time half order integration process starts with the initial conditions equal to the last value of the first order integration process. The initial value of the half order analog model are values of voltage on capacitors in all branches. In the proposed realization of the switching order method, capacitors in branches that are not used in the first order process are charged to the value obtained on the first branch (first order integrator);

2) switching from terminals 2 to 1: In this case the system described by the transfer function $G_2(s)$ (half order) is switched to system described by the transfer function $G_1(s)$ (first order). In this configuration, branches in closed-loop are connected to terminals marked as 2, and after switching, the terminals are changed to 1. The example of switching matrix is given as follows:

$$W(-0.5,5,0)W(-0.5,5,3) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 & 0 & 0 \\ 0.375 & 0.5 & 1 & 0 & 0 & 0 \\ 0.3125 & 0.375 & 0.5 & 1 & 0 & 0 \\ 0.273438 & 0.3125 & 0.375 & 1 & 1 & 0 \\ 0.246094 & 0.273438 & 0.3125 & 1 & 1 & 1 \end{pmatrix}.$$

As one can see, the switching matrix coefficients of samples time for half order integration are changing also after a switching time. This provides that the system has also a part of half order behavior after switching to the first order integration. This effect is temporary because values of half order coefficients decrease with time. After some reasonable time, the value of integration has mainly first order character. In the proposed analog model, the first order integration starts just after switching; this will cause an additional error, however, expected as limited.

B. Experimental results

The parameters of analog models were obtained by identification based on time domain responses, separately for both orders: $\alpha = -1$ and $\alpha = -0.48$, respectively,

$$G_1(s) = \frac{1}{0.00352s}, \quad G_2(s) = \frac{1}{0.071s^{0.48}}.$$

As it was shown in [17], the order of the real analog implementation of the half order integrator is very close, but not equal, to -0.5. It is because of accuracy of used elements. In order to obtain maximum accuracy in time domain, the order of transfer function $G_2(s)$ was chosen as -0.48.

1) Switching between order $\alpha = -1$ and $\alpha = -0.48$: The experimental results of integrator with order switching from $\alpha = -1$ to $\alpha = -0.48$, compared to the numerical results, are presented in Fig. 5. The difference between the 2nd type of derivative definition and its analog implementation is presented in Fig. 6. As it could be seen, the analog realization is very close to the 2nd type of definition (numerical realization). The sample time for all measurements was chosen as 0.01 sec.

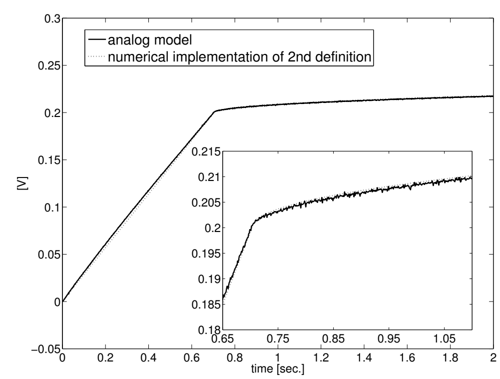

Fig. 5. Results of analog and numerical implementation of switching order derivative from $\alpha = -1$ to $\alpha = -0.48$

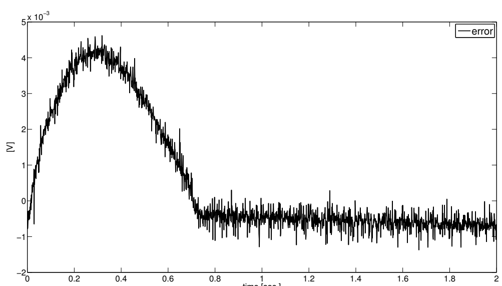

Fig. 6. Difference between analog and numerical implementation of switching order derivative from $\alpha = -1$ to $\alpha = -0.48$

2) Switching between order $\alpha = -0.48$ and $\alpha = -1$: The experimental results of integrator with switched orders from $\alpha = -0.48$ to $\alpha = -1$ compared with numerical results are presented in Fig. 7. The difference between the 2nd type of derivative definition and its analog implementation is presented in Fig. 8. As it could be seen, the analog realization is close to the 2nd type of definition (numerical realization). As it was expected in experimental setup description, the difference between analog and numerical implementations, after switching time, increase to obtain limited value.

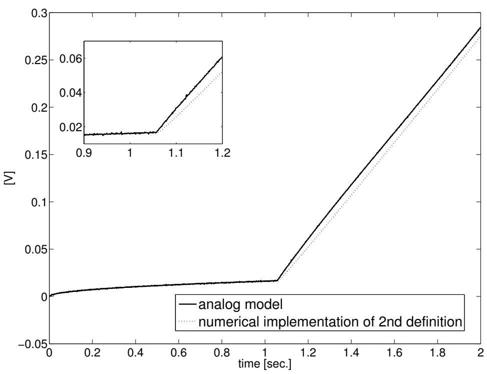

Fig. 7. Results of analog and numerical implementation of switching order derivative from $\alpha = -0.48$ to $\alpha = -1$

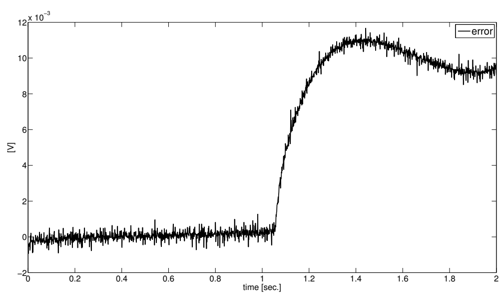

Fig. 8. Difference between analog and numerical implementation of switching order derivative from $\alpha = -0.48$ to $\alpha = -1$

VI. Conclusions

The paper presents a switching strategy for the second type of variable order derivative. The numerical scheme, based on matrix approach, for that switching scheme was introduced and investigated. It was shown that obtained numerical scheme is equivalent to the 2nd type of fractional variable order derivative. This switching scheme can also be used as an interpretation of the second type of the definition. It is also worth to notice that the rule given by the switching scheme can itself be a definition of a variable order derivative and the relation given by Def. 3 is just a consequence of this scheme. Presented interpretation allows to better understand behavior of the definition, and, in generally, switching process in variable order systems. Based on this, it can give rise to more appropriate choice of type of definitions, accordingly to particular application. Additionally, an analog modeling method of switched order integrator was introduced and examined. Presented method allows to switch between orders -0.5 and -1. Obtained results were compared with the numerical implementation and show high accuracy of introduced method.

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