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Time-varying generalisations of the gap and ν-gap metrics induce the same topology in continuous time

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***Abstract*— Recent robust stability analysis results for linear time-varying feedback interconnections are based on a time-varying generalisation of the ν-gap metric. The causality of closed-loop mappings is dealt with explicitly, rather than via well-posedness assumptions as is common in the literature. Here, an alternative time-varying gap metric is defined. It is shown that this gives rise to corresponding robust stability results. It is also established that the time-varying gap metric induces the same topology as the generalised ν-gap metric, this being the coarsest under which closed-loop stability and performance are both robust properties.**

***Index Terms*— linear time-varying systems, robustness, ν-gap metric, gap metric, strong graph representations**

I. Introduction

In perturbation theory, the 'aperture' or 'gap' between linear manifolds, is known to be a useful quantification of distance [1]. The gap metric was introduced to the systems and control literature in [2], and has since been developed as an important tool in the study of robust feedback stability for various classes of linear systems [3], [4], [5], [6]. In [7], [8], Vinnicombe took a profound departure from the traditional setting of studying robustness of systems on the semi-infinite time axis with the gap metric and introduced the so-called ν-gap metric on systems effectively defined on bi-infinite time, coupled with a frequency-domain winding number condition which ensures closed-loop causality [9]. He showed that for finite-dimensional linear time-invariant (LTI) systems, the ν-gap metric leads to the least conservative robust stability conditions, in addition to admitting a classical frequency-domain interpretation [8].

The ν-gap metric was recently generalised in [10] to causal linear time-varying (LTV) systems which admit *normalised strong graph representations*; the winding number condition is replaced by a more general Fredholm index condition. Some well-known feedback robustness results from [8] were shown to hold with respect to a generalised definition of closed-loop stability, defined over finite-energy signals with *arbitrary* forward semi-infinite time support. An important feature of this framework is the preservation of the notion of system *causality* via feedback, without which closed-loop stability is arguably ill-defined [11], [9]. It also avoids the so-called Georgiou-Smith Paradox [12], [9].

The gap metric for LTV systems was introduced much earlier in [4] within a continuous-time setting, involving signals defined on the fixed semi-infinite time axis and without explicitly considering causality. It is also developed in [13], [5] within a discrete-time setting. The apparent benefit of working in a discrete-time setting is that sharper results can be derived and exploited. For instance, since the inverse of a lower triangular matrix is also lower triangular, invertibility of causal systems is equivalent to causal invertibility. This property entails a well-defined notion of closed-loop stability for causal discrete-time systems. On the other hand, such a property is absent in the continuous-time setting.

This papers provides a generalised definition of the gap metric that is consistent with the closed-loop stability framework of [10] for *continuous-time* LTV systems with closed graphs over forward semi-infinite signals with arbitrary start time. In particular, it is shown that a sufficient robust stability condition in [10] also holds for this alternative time-varying gap metric. The main difference to that in [4] is that causality of the closed-loop operator is guaranteed here. For time-varying systems which admit strong graph representations, it is established that the gap and ν-gap metrics induce the same topology, as in the LTI case [8]. Specifically, this is the weakest topology in which perturbations within a sufficiently small neighbourhood do not result in feedback instability and the corresponding closed-loop mappings vary continuously in the induced-norm topology.

The paper evolves along the following lines. First, some preliminaries on operator theory are covered in the next section to facilitate the definition of the generalised ν-gap metric. Strong representations of system graphs and closed-loop stability are examined in Section III. In Section IV, an alternative LTV gap metric is defined and shown to give rise to a corresponding robust stability result. Finally, the topological equivalence of the generalised ν-gap and gap metrics is established in Section V.

II. Basic operator theory

The notation is defined and a few useful results from operator theory gathered in this section. To be specific, the relavant theory of Fredholm, Toeplitz, and Hankel operators on Hilbert spaces is summarised. The notion of causality, which is important in the development of this paper, is also discussed.

Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. Consider a linear operator $\mathbf{X} : \mathrm{dom}\,(\mathbf{X}) \subset \mathcal{H}_1 \to \mathcal{H}_2$. The image and kernel of $\mathbf{X}$ are denoted respectively $\mathrm{img}\,(\mathbf{X}) := \{w \in \mathcal{H}_2 : w = \mathbf{X}v;\, v \in \mathrm{dom}\,(\mathbf{X})\}$ and $\ker(\mathbf{X}) := \{v \in \mathrm{dom}\,(\mathbf{X}) : \mathbf{X}v = 0\}$.

Consider a bounded operator $\mathbf{X} : \mathcal{H}_1 \to \mathcal{H}_2$; i.e. $\bar{\gamma}(\mathbf{X}) := \sup_{v \neq 0} \|\mathbf{X}v\|_{\mathcal{H}_2} / \|v\|_{\mathcal{H}_1} < \infty$. Let $\underline{\gamma}(\mathbf{X}) := \inf_{v \neq 0} \|\mathbf{X}v\|_{\mathcal{H}_2} / \|v\|_{\mathcal{H}_1} \geq 0$. The Hilbert adjoint, denoted

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$\mathbf{X}^* : \mathcal{H}_2 \to \mathcal{H}_1$, is uniquely defined by the equation

$$\langle \mathbf{X}v,\, w\rangle_{\mathcal{H}_2} = \langle v,\, \mathbf{X}^* w\rangle_{\mathcal{H}_1} \;\forall v \in \mathcal{H}_1,\, w \in \mathcal{H}_2$$

and $\bar{\gamma}(\mathbf{X}) = \bar{\gamma}(\mathbf{X}^*)$. It holds that $\mathrm{img}\,(\mathbf{X})^\perp = \ker\,(\mathbf{X}^*)$ and $\ker\,(\mathbf{X}^*)^\perp = \mathrm{cl}\,\mathrm{img}\,(\mathbf{X})$, where cl denotes closure and given $\mathcal{W} \subset \mathcal{H}_2$, the orthogonal complement $\mathcal{W}^\perp := \{u \in \mathcal{H}_2 \;:\; \langle u,\, w\rangle_{\mathcal{H}_2} = 0 \,\forall w \in \mathcal{W}\}$, which is closed. The restriction of $\mathbf{X}$ to $\mathcal{V} \subset \mathcal{H}_1$ is denoted $\mathbf{X}|_{\mathcal{V}}$; this is bounded. If for every bounded sequence $\{v_k\} \subset \mathcal{H}_1$ there exists a subsequence of $\{\mathbf{X}v_k\}$ that is convergent in $\mathcal{H}_2$, then $\mathbf{X}$ is said to be compact. Henceforth, the Banach space of all bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ is denoted $\mathscr{L}(\mathcal{H}_1, \mathcal{H}_2)$.

Definition 2.1: An operator $\mathbf{X} \in \mathscr{L}(\mathcal{H}_1, \mathcal{H}_2)$ is of Fredholm type if both $\dim\ker\,(\mathbf{X})$ and $\dim\ker\,(\mathbf{X}^*)$ are finite, where $\dim$ denotes the dimension of a subspace. In this case, the Fredholm index of $\mathbf{X}$ is defined to be

$$\mathrm{ind}(\mathbf{X}) := \dim\ker\,(\mathbf{X}) - \dim\ker\,(\mathbf{X}^*)\,.$$

Note that $\mathrm{img}\,(\mathbf{X})$ is necessarily closed [1]. Moreover, a bijective $\mathbf{X}$ is necessarily Fredholm with

$$\mathrm{ind}(\mathbf{X}) = \dim\ker\,(\mathbf{X}) = \dim\ker\,(\mathbf{X}^*) = 0.$$

The following properties of Fredholm operators are well-known; e.g. see [14, Chapter XI].

Lemma 2.2: Let $\mathbf{X} \in \mathscr{L}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathbf{Z} \in \mathscr{L}(\mathcal{H}_2, \mathcal{H}_3)$ be Fredholm operators. The following hold:

(i) $\mathbf{X}^*$ is Fredholm and $\mathrm{ind}(\mathbf{X}^*) = -\mathrm{ind}(\mathbf{X})$;

(ii) $\mathbf{ZX}$ is Fredholm and $\mathrm{ind}(\mathbf{ZX}) = \mathrm{ind}(\mathbf{Z}) + \mathrm{ind}(\mathbf{X})$;

(iii) if $\mathbf{Y} \in \mathscr{L}(\mathcal{H}_1, \mathcal{H}_2)$ is such that $\underline{\gamma}(\mathbf{X}) > \bar{\gamma}(\mathbf{Y})$, then $\mathbf{X} + \mathbf{Y}$ is Fredholm and $\mathrm{ind}(\mathbf{X} + \mathbf{Y}) = \mathrm{ind}(\mathbf{X})$;

In the subsequent stability analysis, continuous-time finite-energy signals are of primary concern. Define the Hilbert space

$$\boldsymbol{L}^2_{\mathbb{R}} := \left\{ \phi : \mathbb{R} \to \mathbb{R}^m \;\middle|\; \|\phi\|_2 := \langle \phi, \phi\rangle_2^{\frac{1}{2}} < \infty \right\},$$

where $\langle u,\, v\rangle_2 := \int_{-\infty}^{\infty} u(t)^T v(t)\, dt$. Also define the following two subspaces:

$$\boldsymbol{L}^2_{\mathbb{I}} := \left\{ \phi \in \boldsymbol{L}^2_{\mathbb{R}} \mid \phi(t) = 0 \,\forall\, t \in \mathbb{R} \setminus \mathbb{I} \right\};$$
$$\boldsymbol{L}^2_{+} := \textstyle\bigcup_{\tau \in \mathbb{R}} \boldsymbol{L}^2_{[\tau,\infty)}.$$

For a linear operator $\mathbf{X} : \mathrm{dom}\,(\mathbf{X}) \subset \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$, define its *graph* as

$$\mathscr{G}(\mathbf{X}) := \{ \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] : u \in \mathrm{dom}\,(\mathbf{X}) \text{ and } y = \mathbf{X}u \};$$
$$\mathscr{G}_\tau(\mathbf{X}) := \mathscr{G}(\mathbf{X}) \cap \boldsymbol{L}^2_{[\tau,\infty)},$$

and its inverse graph as

$$\mathscr{G}'(\mathbf{X}) := \{ \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] : y \in \mathrm{dom}\,(\mathbf{X}) \text{ and } u = \mathbf{X}y \};$$
$$\mathscr{G}'_\tau(\mathbf{X}) := \mathscr{G}'(\mathbf{X}) \cap \boldsymbol{L}^2_{[\tau,\infty)}.$$

In the sequel, open-loop systems are considered to be possibly unbounded operators that map from a domain in $\boldsymbol{L}^2_{+}$ to $\boldsymbol{L}^2_{+}$, which is dense in $\boldsymbol{L}^2_{\mathbb{R}}$. Note that

$$\boldsymbol{L}^2_{\mathbb{R}} = \boldsymbol{L}^2_{(-\infty,\tau)} \oplus \boldsymbol{L}^2_{[\tau,\infty)} \;\forall \tau \in \mathbb{R},$$

where $\oplus$ denotes orthogonal sum. The orthogonal projections from $\boldsymbol{L}^2_{\mathbb{R}}$ onto $\boldsymbol{L}^2_{(-\infty,\tau)}$, respectively $\boldsymbol{L}^2_{[\tau,\infty)}$, are denoted $\mathbf{\Pi}_\tau$, respectively $\mathbf{I} - \mathbf{\Pi}_\tau$, where $\mathbf{I}$ denotes the identity operator. Given a bounded $\mathbf{X} : \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$ and a $\tau \in \mathbb{R}$, the corresponding Toeplitz (a.k.a. Wiener-Hopf) operators

$$\mathbf{T}_\tau(\mathbf{X}) := (\mathbf{I} - \mathbf{\Pi}_\tau)\mathbf{X}|_{\boldsymbol{L}^2_{[\tau,\infty)}} \text{ and } \mathbf{B}_\tau(\mathbf{X}) := \mathbf{\Pi}_\tau \mathbf{X}|_{\boldsymbol{L}^2_{(-\infty,\tau)}}$$

and Hankel operators

$$\mathbf{H}_\tau(\mathbf{X}) := (\mathbf{I} - \mathbf{\Pi}_\tau)\mathbf{X}|_{\boldsymbol{L}^2_{(-\infty,\tau)}} \text{ and } \mathbf{J}_\tau(\mathbf{X}) := \mathbf{\Pi}_\tau \mathbf{X}|_{\boldsymbol{L}^2_{[\tau,\infty)}}$$

are also relevant in the subsequent analysis. In particular, these operators play an important role in ν-gap metric based stability analysis of time-varying feedback systems in conjunction with the theory of Fredholm operators [10]. Note that $\mathbf{T}_\tau(\mathbf{X})^* = \mathbf{T}_\tau(\mathbf{X}^*)$. The projections $\mathbf{\Pi}_\tau$ and $\mathbf{I} - \mathbf{\Pi}_\tau$ give rise to the following characterisation of causality.

Definition 2.3: An operator $\mathbf{X} : \mathrm{dom}\,(\mathbf{X}) \subset \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$ is *causal* if given any $\tau \in \mathbb{R}$ and $\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right] \in \mathbf{\Pi}_\tau \mathscr{G}(\mathbf{X}) \subset \boldsymbol{L}^2_{(-\infty,\tau)}$, $w = 0$ implies $v = 0$. When $\mathrm{dom}\,(\mathbf{X}) = \boldsymbol{L}^2_{\mathbb{R}}$, this is equivalent to $\mathbf{\Pi}_\tau \mathbf{X} \mathbf{\Pi}_\tau = \mathbf{\Pi}_\tau \mathbf{X} \,\forall \tau \in \mathbb{R}$, whereas if $\mathbf{\Pi}_\tau \mathbf{X} \mathbf{\Pi}_\tau = \mathbf{X} \mathbf{\Pi}_\tau \,\forall \tau \in \mathbb{R}$ instead, then $\mathbf{X}$ is *anti-causal*. Note that an $\mathbf{X} \in \mathscr{L}(\boldsymbol{L}^2_{\mathbb{R}}, \boldsymbol{L}^2_{\mathbb{R}})$ is causal if, and only if, $\mathbf{X}^*$ is anti-causal.

Lemma 2.4 ([10, Lem. 2.6]): Let $\mathbf{X}, \mathbf{Y} \in \mathscr{L}(\boldsymbol{L}^2_{\mathbb{R}}, \boldsymbol{L}^2_{\mathbb{R}})$.

(i) If $\mathbf{X}$ is causal then $\mathbf{T}_\tau(\mathbf{X}) = \mathbf{X}|_{\boldsymbol{L}^2_{[\tau,\infty)}}$ and $\mathbf{T}_\tau(\mathbf{X})$ is causal for all $\tau \in \mathbb{R}$;

(ii) $\mathbf{T}_\tau(\mathbf{YX}) = \mathbf{T}_\tau(\mathbf{Y})\mathbf{T}_\tau(\mathbf{X}) + \mathbf{H}_\tau(\mathbf{Y})\mathbf{J}_\tau(\mathbf{X}) \forall \tau \in \mathbb{R}$. If $\mathbf{X}$ is causal or $\mathbf{Y}$ is anti-causal, then $\mathbf{T}_\tau(\mathbf{YX}) = \mathbf{T}_\tau(\mathbf{Y})\mathbf{T}_\tau(\mathbf{X}), \forall \tau \in \mathbb{R}$;

(iii) If $\mathbf{X}$ is causal, then $\bar{\gamma}(\mathbf{X}) = \sup_{\tau \in \mathbb{R}} \bar{\gamma}(\mathbf{T}_\tau(\mathbf{X}))$.

III. Feedback stability

A. Feedback interconnection

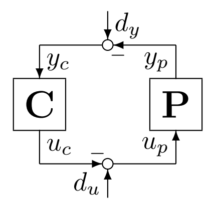

Fig. 1. Standard feedback configuration

The main object of study here is the feedback interconnection illustrated in Fig. 1, denoted $[\mathbf{P}, \mathbf{C}]$, where

$$d_y = y_c + y_p,\; d_u = u_p + u_c,\; y_p = \mathbf{P}u_p,\; u_c = \mathbf{C}y_c, \quad (1)$$

and $\mathbf{P} : \mathrm{dom}\,(\mathbf{P}) \subset \boldsymbol{L}^2_{+} \to \boldsymbol{L}^2_{+}$ and $\mathbf{C} : \mathrm{dom}\,(\mathbf{C}) \subset \boldsymbol{L}^2_{+} \to \boldsymbol{L}^2_{+}$ are two *causal* linear operators. Note that by causality, $\mathrm{img}\left(\mathbf{P}|_{\mathrm{dom}(\mathbf{P}) \cap \boldsymbol{L}^2_{[\tau,\infty)}}\right) \subset \boldsymbol{L}^2_{[\tau,\infty)}$ and $\mathrm{img}\left(\mathbf{C}|_{\mathrm{dom}(\mathbf{C}) \cap \boldsymbol{L}^2_{[\tau,\infty)}}\right) \subset \boldsymbol{L}^2_{[\tau,\infty)}$.

Definition 3.1: The feedback interconnection $[\mathbf{P}, \mathbf{C}]$ is said to be internally *stable* if for all $\tau \in \mathbb{R}$ the operator

$$\mathbf{F}_\tau(\mathbf{P}, \mathbf{C}) := \left[\begin{smallmatrix} \mathbf{I} & \mathbf{P} \\ \mathbf{C} & \mathbf{I} \end{smallmatrix}\right]|_{(\mathrm{dom}(\mathbf{C}) \times \mathrm{dom}(\mathbf{P})) \cap \boldsymbol{L}^2_{[\tau,\infty)}}$$

has an inverse on $\boldsymbol{L}^2_{[\tau,\infty)}$, with $\sup_{\tau\in\mathbb{R}} \bar{\gamma}(\mathbf{F}_\tau(\mathbf{P},\mathbf{C})^{-1}) < \infty$.

Lemma 3.2 ([10, Thm. 3.4]): If $[\mathbf{P},\mathbf{C}]$ is stable in the sense of Definition 3.1, then $\mathbf{F}_\tau^{-1}$ is necessarily causal for every $\tau \in \mathbb{R}$.

Remark 3.3: Lemma 3.2 illustrates that causality of the closed-loop mapping is encapsulated in the definition of feedback stability. In other words, causality is a property that is preserved through feedback interconnections that are stable in the generalised sense of Definition 3.1. This is consistent with the viewpoint in [11], [9], where it is argued that a proper definition of closed-loop stability must incorporate causality, or a so-called 'arrow of time'.

Remark 3.4: In Definition 3.1, bounded invertibility of the open-loop operator $\mathbf{F}_\tau(\mathbf{P},\mathbf{C})$ is required on a *singly infinite* space $\boldsymbol{L}^2_{[\tau,\infty)}$, for all possible 'initial times' $\tau \in \mathbb{R}$. A necessary condition is that the truncated graphs $\mathscr{G}_\tau(\mathbf{P})$ and $\mathscr{G}'_\tau(\mathbf{C})$ are closed subspaces of $\boldsymbol{L}^2_{[\tau,\infty)}$ [15, Thm 6.2][4, Prop. 1]. Invertibility over the *doubly infinite* $\boldsymbol{L}^2_{\mathbb{R}}$ is not considered as this gives rise to the so-called Georgiou-Smith paradox [12], [16]. This viewpoint is consistent with the 'behavioural' theory of [17], where system behaviour is effectively defined on doubly-infinite time-axis while closed-loop stability requires boundedness mappings of signals of semi-infinite support.

For causal linear operators $\mathbf{P}$ and $\mathbf{C}$ such that the feedback interconnection $[\mathbf{P},\mathbf{C}]$ is stable, for all $\tau \in \mathbb{R}$ let

$$\begin{aligned}\mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P})\|\mathscr{G}'_\tau(\mathbf{C})} &:= \begin{bmatrix} d_y \\ d_u \end{bmatrix} \in \boldsymbol{L}^2_{[\tau,\infty)} \mapsto \begin{bmatrix} y_p \\ u_p \end{bmatrix} \in \mathscr{G}_\tau(\mathbf{P}) \\ &= \begin{bmatrix} -\mathbf{I} & 0 \\ 0 & \mathbf{I} \end{bmatrix} \mathbf{F}_\tau(\mathbf{P},\mathbf{C})^{-1} + \begin{bmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{bmatrix}; \\ \mathbf{\Pi}_{\mathscr{G}'_\tau(\mathbf{C})\|\mathscr{G}_\tau(\mathbf{P})} &:= \begin{bmatrix} d_y \\ d_u \end{bmatrix} \in \boldsymbol{L}^2_{[\tau,\infty)} \mapsto \begin{bmatrix} y_c \\ u_c \end{bmatrix} \in \mathscr{G}'_\tau(\mathbf{C}) \\ &= \begin{bmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{bmatrix} \mathbf{F}_\tau(\mathbf{P},\mathbf{C})^{-1} + \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{I} \end{bmatrix}. \end{aligned} \tag{2}$$

The notation reflects that these are parallel projection operators onto and along the restricted graphs $\mathscr{G}_\tau(\mathbf{P})$ and $\mathscr{G}'_\tau(\mathbf{C})$ [4], [18], [6]. Define

$$b_{\mathbf{P},\mathbf{C}} := \begin{cases} \left(\sup_{\tau\in\mathbb{R}} \bar{\gamma}\left(\mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P})\|\mathscr{G}'_\tau(\mathbf{C})} \right) \right)^{-1} & \text{if } [\mathbf{P},\mathbf{C}] \text{ is stable;} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $b_{\mathbf{P},\mathbf{C}} \leq 1$. As seen later, this is a measure of robust stability and nominal performance.

B. Strong graph representations

The definition of the ν-gap metric and the corresponding robust stability results in [10] are underpinned by the following assumptions. These concern the existence of 'strong graph representations' for causal linear systems. Several classes of systems satisfy the assumptions, including finite-dimensional LTV state-space systems, distributed-parameter time-invariant systems, and generic periodic systems with finite-dimensional 'realisations'; see [10], [19] for more details.

Assumption 3.5: Given a *causal* operator $\mathbf{P} : \operatorname{dom}(\mathbf{P}) \subset \boldsymbol{L}^2_+ \to \boldsymbol{L}^2_+$, there exist *causal* operators $\mathbf{N}, \mathbf{M}, \tilde{\mathbf{N}}, \tilde{\mathbf{M}}, \mathbf{X}, \mathbf{Y}, \tilde{\mathbf{X}}, \tilde{\mathbf{Y}} \in \mathscr{L}(\boldsymbol{L}^2_{\mathbb{R}}, \boldsymbol{L}^2_{\mathbb{R}})$ satisfying the following properties:

1) the double Bezout identity
$$\begin{bmatrix} \mathbf{Y} & \mathbf{X} \\ \tilde{\mathbf{M}} & -\tilde{\mathbf{N}} \end{bmatrix} \begin{bmatrix} \mathbf{N} & \tilde{\mathbf{X}} \\ \mathbf{M} & -\tilde{\mathbf{Y}} \end{bmatrix} = \begin{bmatrix} \mathbf{N} & \tilde{\mathbf{X}} \\ \mathbf{M} & -\tilde{\mathbf{Y}} \end{bmatrix} \begin{bmatrix} \mathbf{Y} & \mathbf{X} \\ \tilde{\mathbf{M}} & -\tilde{\mathbf{N}} \end{bmatrix} = \mathbf{I};$$
2) $\mathscr{G}_\tau(\mathbf{P}) := \mathscr{G}(\mathbf{P}) \cap \boldsymbol{L}^2_{[\tau,\infty)} = \operatorname{img}(\mathbf{T}_\tau(\mathbf{G})) = \ker\left(\mathbf{T}_\tau(\tilde{\mathbf{G}})\right)$ for all $\tau \in \mathbb{R}$, where
$$\mathbf{G} := \begin{bmatrix} \mathbf{N} \\ \mathbf{M} \end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{G}} := \begin{bmatrix} -\tilde{\mathbf{M}} & \tilde{\mathbf{N}} \end{bmatrix}$$
are respectively called right and left *strong graph representations* for $\mathbf{P}$, respectively.

The term 'strong' in part 2 of Assumption 3.5 is borrowed from [15] to emphasise that right (resp. left) graph representations have left (resp. right) bounded causal inverses. In this paper, graph representations are always taken to be strong.

Remark 3.6: By the properties of strong graph representations in Assumption 3.5 and Lemma 2.4(ii), it follows that $\mathbf{T}_\tau(\mathbf{G})$ has a causal left inverse $\mathbf{T}_\tau(\mathbf{Z})$, and $\mathbf{T}_\tau(\tilde{\mathbf{G}})$ has a causal right inverse $\mathbf{T}_\tau(\tilde{\mathbf{Z}})$, where $\mathbf{Z} := \begin{bmatrix} \mathbf{Y} & \mathbf{X} \end{bmatrix}$ and $\tilde{\mathbf{Z}} := \begin{bmatrix} -\tilde{\mathbf{X}} \\ \tilde{\mathbf{Y}} \end{bmatrix}$. Note that the left-bounded-invertibility of $\mathbf{T}_\tau(\mathbf{G})$ implies $\mathscr{G}_\tau(\mathbf{P}) = \operatorname{img}(\mathbf{T}_\tau(\mathbf{G}))$ is a closed subspace [1, Thm. IV.5.2], as is necessary for feedback stabilisability [15, Thm. 6.2][4, Prop. 1]; see Remark 3.4. Moreover, note that $\tilde{\mathbf{G}}\mathbf{G} = 0$ and $\mathbf{G}\mathbf{Z} + \tilde{\mathbf{Z}}\tilde{\mathbf{G}} = \mathbf{I}$, whereby it holds that $\operatorname{img}(\mathbf{G}) = \ker\left(\tilde{\mathbf{G}}\right)$.

Assumption 3.7: $\mathbf{G}^*\mathbf{G} = \mathbf{I}$ and $\tilde{\mathbf{G}}\tilde{\mathbf{G}}^* = \mathbf{I}$, i.e. strong right and left graph representations can be taken to be normalised.

The following assumption is important in the derivation of the robust feedback stability result in terms of the ν-gap metric in [10].

Assumption 3.8: $\mathbf{H}_\tau(\mathbf{G})$ and $\mathbf{H}_\tau(\tilde{\mathbf{G}})$ are compact for all $\tau \in \mathbb{R}$.

C. Characterising feedback stability

By convention, throughout this paper $\mathbf{G} := \begin{bmatrix} \mathbf{N} \\ \mathbf{M} \end{bmatrix}$ denotes a normalised strong right graph representation for $\mathbf{P}$ and $\tilde{\mathbf{G}} := \begin{bmatrix} -\tilde{\mathbf{M}} & \tilde{\mathbf{N}} \end{bmatrix}$ a normalised strong left graph representation. On the other hand, the notation $\mathbf{K} := \begin{bmatrix} \mathbf{V} \\ \mathbf{U} \end{bmatrix}$ and $\tilde{\mathbf{K}} := \begin{bmatrix} -\tilde{\mathbf{U}} & \tilde{\mathbf{V}} \end{bmatrix}$, is adopted for the strong right and left (inverse) graph representations for $\mathbf{C}$; i.e. $\mathscr{G}'_\tau(\mathbf{C}) = \operatorname{img}(\mathbf{T}_\tau(\mathbf{K})) = \ker\left(\mathbf{T}_\tau(\tilde{\mathbf{K}})\right)$ for every $\tau \in \mathbb{R}$.

The stability of the feedback interconnection $[\mathbf{P},\mathbf{C}]$ can be conveniently characterised in terms of strong right and left graph representations for $\mathbf{P}$ and $\mathbf{C}$. The following result can be found in [10, Thm 3.6] and [19, Thm 3.2.19].

Lemma 3.9: Given causal operators $\mathbf{P}$ and $\mathbf{C}$, suppose that Assumption 3.5 holds, then the following are equivalent:

1) $[\mathbf{P},\mathbf{C}]$ is stable;
2) $\underline{\gamma}(\tilde{\mathbf{K}}\mathbf{G}) > 0$ and $\mathbf{T}_\tau(\tilde{\mathbf{K}}\mathbf{G})$ is boundedly invertible for all $\tau \in \mathbb{R}$;
3) $\underline{\gamma}(\tilde{\mathbf{G}}\mathbf{K}) > 0$ and $\mathbf{T}_\tau(\tilde{\mathbf{G}}\mathbf{K})$ is boundedly invertible for all $\tau \in \mathbb{R}$;

4) $\tilde{\mathbf{K}}\mathbf{G}$ has a bounded causal inverse;
5) $\tilde{\mathbf{G}}\mathbf{K}$ has a bounded causal inverse.

Moreover, when $[\mathbf{P},\mathbf{C}]$ is stable, for all $\tau \in \mathbb{R}$, $\mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P})\|\mathscr{G}'_\tau(\mathbf{C})} = \mathbf{T}_\tau(\mathbf{G})\mathbf{T}_\tau(\tilde{\mathbf{K}}\mathbf{G})^{-1}\mathbf{T}_\tau(\tilde{\mathbf{K}})$ and $\mathbf{\Pi}_{\mathscr{G}'_\tau(\mathbf{C})\|\mathscr{G}_\tau(\mathbf{P})} = \mathbf{T}_\tau(\mathbf{K})\mathbf{T}_\tau(\tilde{\mathbf{G}}\mathbf{K})^{-1}\mathbf{T}_\tau(\tilde{\mathbf{G}})$. Suppose further that Assumption 3.7 holds, then the robust performance margin

$$b_{\mathbf{P},\mathbf{C}} = \underline{\gamma}(\tilde{\mathbf{K}}\mathbf{G}) = \underline{\gamma}(\tilde{\mathbf{G}}\mathbf{K}) > 0.$$

IV. A Time-Varying Generalisation of the Gap Metric

Let

$$\mathcal{C} := \left\{ \begin{array}{l} \mathbf{X} : \operatorname{dom}(\mathbf{X}) \subset \\ \boldsymbol{L}^2_+ \to \boldsymbol{L}^2_+ \end{array} \middle| \begin{array}{l} \mathbf{X} \text{ is causal and } \mathscr{G}_\tau(\mathbf{X}) \\ \text{is closed in } \boldsymbol{L}^2_{[\tau,\infty)}\ \forall \tau \in \mathbb{R} \end{array} \right\}.$$

The standard gap distance between two systems $\mathbf{P}_1 \in \mathcal{C}$ and $\mathbf{P}_2 \in \mathcal{C}$ is defined to be

$$\delta_0(\mathbf{P}_1,\mathbf{P}_2) := \bar{\gamma}\left(\mathbf{\Pi}_{\mathscr{G}_0(\mathbf{P}_1)} - \mathbf{\Pi}_{\mathscr{G}_0(\mathbf{P}_2)}\right),$$

where $\mathbf{\Pi}_{\mathscr{G}_0(\mathbf{P}_i)} : \boldsymbol{L}^2_{[0,\infty)} \to \mathscr{G}_0(\mathbf{P}_i)$ denotes the orthogonal projection [1], [3], [4], [5], [6]. In line with the time-varying nature of the dynamical systems concerned in this paper, define

$$\delta_\tau(\mathbf{P}_1,\mathbf{P}_2) := \bar{\gamma}\left(\mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P}_1)} - \mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P}_2)}\right) \le 1.$$

Note [1]

$$\delta_\tau(\mathbf{P}_1,\mathbf{P}_2) = \max\left\{\vec{\delta}_\tau(\mathbf{P}_1,\mathbf{P}_2), \vec{\delta}_\tau(\mathbf{P}_2,\mathbf{P}_1)\right\},$$

where the directed gap

$$\begin{aligned} \vec{\delta}_\tau(\mathbf{P}_1,\mathbf{P}_2) &:= \bar{\gamma}\left(\mathbf{\Pi}_{(\mathscr{G}_\tau(\mathbf{P}_2))^\perp}\mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P}_1)}\right) \\ &= \bar{\gamma}\left(\mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P}_1)}\mathbf{\Pi}_{(\mathscr{G}_\tau(\mathbf{P}_2))^\perp}\right) \\ &= \sup_{x_1 \in \mathscr{G}_\tau(\mathbf{P}_1)} \inf_{x_2 \in \mathscr{G}_\tau(\mathbf{P}_2)} \frac{\|x_1 - x_2\|_2}{\|x_1\|_2}. \end{aligned} \tag{3}$$

If $\delta_\tau(\mathbf{P}_1,\mathbf{P}_2) < 1$, then $\delta_\tau(\mathbf{P}_1,\mathbf{P}_2) = \vec{\delta}_\tau(\mathbf{P}_1,\mathbf{P}_2) = \vec{\delta}_\tau(\mathbf{P}_2,\mathbf{P}_1)$ [1].

Definition 4.1: Let the time-varying generalisation of the gap metric on $\mathcal{C}$ be defined as

$$\delta(\mathbf{P}_1,\mathbf{P}_2) := \sup_{\tau\in\mathbb{R}} \delta_\tau(\mathbf{P}_1,\mathbf{P}_2) \le 1.$$

Theorem 4.2: Given a stable feedback interconnection $[\mathbf{P}_1,\mathbf{C}]$, suppose $b_{\mathbf{P}_1,\mathbf{C}} > \delta(\mathbf{P}_1,\mathbf{P}_2)$, then $[\mathbf{P}_2,\mathbf{C}]$ is stable.

Proof: According to Definition 3.1, it needs to be established that

(i) $\mathbf{F}_\tau(\mathbf{P}_2,\mathbf{C}) : (\operatorname{dom}(\mathbf{C})\times\operatorname{dom}(\mathbf{P}))\cap\boldsymbol{L}^2_{[\tau,\infty)} \to \boldsymbol{L}^2_{[\tau,\infty)}$ is invertible for all $\tau \in \mathbb{R}$; and
(ii) $\sup_{\tau\in\mathbb{R}} \bar{\gamma}(\mathbf{F}_\tau(\mathbf{P}_2,\mathbf{C})^{-1})$ is finite.

For any $\tau \in \mathbb{R}$, first note that [4, Eq. (25)]

$$\bar{\gamma}\left(\mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P}_1)\|\mathscr{G}'_\tau(\mathbf{C})}\right)^{-1} = \inf_{x_1\in\mathscr{G}_\tau(\mathbf{P}_1)} \inf_{x_2\in\mathscr{G}'_\tau(\mathbf{C})} \frac{\|x_1 - x_2\|_2}{\|x_1\|_2}.$$

Furthermore, invertibility of $\mathbf{F}_\tau(\mathbf{P},\mathbf{C})$ is equivalent to [4, Prop. 2]

$$\mathscr{G}_\tau(\mathbf{P}_1) + \mathscr{G}'_\tau(\mathbf{C}) = \boldsymbol{L}^2_{[\tau,\infty)} \quad \text{and} \quad \mathscr{G}_\tau(\mathbf{P}_1) \cap \mathscr{G}'_\tau(\mathbf{C}) = \{0\},$$

which is in turn equivalent to $\operatorname{codim}(\mathscr{G}_\tau(\mathbf{P}_1) + \mathscr{G}'_\tau(\mathbf{C})) = 0$ and $\dim(\mathscr{G}_\tau(\mathbf{P}_1) \cap \mathscr{G}'_\tau(\mathbf{C})) = 0$, where codim denotes the co-dimension of a subspace. By the hypothesis and the definition of $b_{\mathbf{P}_1,\mathbf{C}}$, application of [1, Thm. 4.18 and Thm. 4.24] yields that $\operatorname{codim}(\mathscr{G}_\tau(\mathbf{P}_2) + \mathscr{G}'_\tau(\mathbf{C})) = 0$ and $\dim(\mathscr{G}_\tau(\mathbf{P}_2) \cap \mathscr{G}'_\tau(\mathbf{C})) = 0$, which is equivalent to $\mathbf{F}_\tau(\mathbf{P}_2,\mathbf{C})$ being invertible, again by [4, Prop. 2]. Hence (i) holds. An alternative proof to (i) is given in the necessity proof of [4, Thm. 3].

To see that (ii) is true, note that for any $\tau \in \mathbb{R}$, the inequality in [6, Prop. III.1] implies

$$\bar{\gamma}\left(\mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P}_2)\|\mathscr{G}'_\tau(\mathbf{C})}\right)^{-1} \ge \bar{\gamma}\left(\mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P}_1)\|\mathscr{G}'_\tau(\mathbf{C})}\right)^{-1} - \delta_\tau(\mathbf{P}_1,\mathbf{P}_2),$$

where $\mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P}_2)\|\mathscr{G}'_\tau(\mathbf{C})}$ is well-defined by (2) since $\mathbf{F}_\tau(\mathbf{P}_2,\mathbf{C})$ is invertible. Taking the infimum on both sides over all $\tau \in \mathbb{R}$ then results in

$$\inf_{\tau\in\mathbb{R}} \gamma\left(\mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P}_2)\|\mathscr{G}'_\tau(\mathbf{C})}\right)^{-1} \ge b_{\mathbf{P}_1,\mathbf{C}} - \delta(\mathbf{P}_1,\mathbf{P}_2) > 0,$$

from which it follows by (2) again that $\sup_{\tau\in\mathbb{R}} \bar{\gamma}(\mathbf{F}_\tau(\mathbf{P}_2,\mathbf{C})^{-1}) < \infty$, i.e. (ii) holds. ■

Remark 4.3: The time-varying gap metric in Definition 4.1 resembles that in [13], [5] for discrete-time systems defined over signals on the *semi-infinite* time axis, where a generalisation of Georgiou's formula [20] is derived via Arveson's distance formula and inner/outer factorisations in *nest algebras* not known to hold in continuous time.

V. The Graph Topology

This section shows that the gap metric defined in the previous section and the ν-gap metric from [10] induce the same topology on the class of linear systems which admit normalised strong graph representations. Following [10], a generalised ν-gap is now defined for possibly LTV and infinite-dimensional systems, in terms of normalised strong graph representations.

Definition 5.1: Denote by $\mathcal{S}$ the set of causal operators for which all of Assumptions 3.5, 3.7, and 3.8 are satisfied.

Note that by Remark 3.6, $\mathcal{S} \subset \mathcal{C}$ defined at the beginning of Section IV. Throughout, $\tilde{\mathbf{G}}_i \in \mathcal{S}$ and $\mathbf{G}_i \in \mathcal{S}$ denote normalised left and right strong graph representations for $\mathbf{P}_i$, respectively, and $\tilde{\mathbf{K}}$ and $\mathbf{K}$ those for $\mathbf{C} \in \mathcal{S}$.

Definition 5.2: The ν-gap metric $\delta_\nu : \mathcal{S}\times\mathcal{S} \to [0,1]$ is defined as

$$\delta_\nu(\mathbf{P}_1,\mathbf{P}_2) := \begin{cases} \bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1) & \text{if for all } \tau \in \mathbb{R}, \\ & \mathbf{T}_\tau(\mathbf{G}_2^*\mathbf{G}_1) \text{ is Fredholm} \\ & \text{and } \operatorname{ind}(\mathbf{T}_\tau(\mathbf{G}_2^*\mathbf{G}_1)) = 0 \\ 1 & \text{otherwise.} \end{cases}$$

That the ν-gap is a metric is shown in [21]. By [10, Cor. 4.3], Theorem 4.2 also holds with $\delta(\cdot,\cdot)$ replaced by $\delta_\nu(\cdot,\cdot)$ for systems in $\mathcal{S}$. In general, the gap and the ν-gap between two systems are not equal, even when they are LTI; see [8, Chapter 7].

The following result is important in establishing the topological equivalence of the gap and ν-gap metrics. Let

$$\mathbb{Q} := \left\{ \mathbf{Q} \in \mathscr{L}(\boldsymbol{L}^2_{\mathbb{R}}, \boldsymbol{L}^2_{\mathbb{R}}) \;\middle|\; \begin{array}{l} \mathbf{Q} \text{ is boundedly invertible} \\ \text{with } \mathbf{Q}, \mathbf{Q}^{-1} \text{ causal} \end{array} \right\}.$$

Lemma 5.3: Given any $\mathbf{P}_1, \mathbf{P}_2 \in \mathcal{S}$, it holds that

$$\delta(\mathbf{P}_1, \mathbf{P}_2) \leq \inf_{\mathbf{Q}\in\mathbb{Q}} \bar{\gamma}(\mathbf{G}_1 - \mathbf{G}_2\mathbf{Q}).$$

Remark 5.4: The inequality of Lemma 5.3 can actually be shown to be an equality for LTI systems [20] (i.e. the Georgiou's formula) and discrete-time LTV systems [13], [5]. The LTI result [20] exploits the commutant lifting theorem [16] for shift-invariant operators and the discrete-time result [5] makes use of matrix representations of systems and the Arveson's distance formula for discrete nest algebras. These are not known to be hold for continuous LTV systems, as are considered in this paper.

Proof: [of Lemma 5.3] It suffices to show the above inequality with $\delta(\mathbf{P}_1, \mathbf{P}_2)$ replaced by $\delta_\tau(\mathbf{P}_1, \mathbf{P}_2)$ for every $\tau \in \mathbb{R}$.

Using (3), Lemma 2.4(i), Assumptions 3.5 and 3.7,

$$\begin{aligned} \vec{\delta}_\tau(\mathbf{P}_1, \mathbf{P}_2) &= \sup_{v_1 \in \boldsymbol{L}^2_{[\tau,\infty)}} \inf_{v_2 \in \boldsymbol{L}^2_{[\tau,\infty)}} \frac{\|\mathbf{G}_1 v_1 - \mathbf{G}_2 v_2\|_2}{\|v_1\|_2} \\ &\leq \sup_{v_1 \in \boldsymbol{L}^2_{[\tau,\infty)}} \frac{\|\mathbf{G}_1 v_1 - \mathbf{G}_2 \mathbf{Q} v_1\|_2}{\|v_1\|_2} \\ &\leq \sup_{v_1 \in \boldsymbol{L}^2_{\mathbb{R}}} \frac{\|\mathbf{G}_1 v_1 - \mathbf{G}_2 \mathbf{Q} v_1\|_2}{\|v_1\|_2}, \end{aligned}$$

for all $\mathbf{Q} \in \mathbb{Q}$. Thus, $\vec{\delta}_\tau(\mathbf{P}_1, \mathbf{P}_2) \leq \inf_{\mathbf{Q}\in\mathbb{Q}} \bar{\gamma}(\mathbf{G}_1 - \mathbf{G}_2\mathbf{Q})$.

The proof is complete if $\delta_\tau(\mathbf{P}_1, \mathbf{P}_2) < 1$, since then the two directed gaps are equal. Thus suppose $\delta_\tau(\mathbf{P}_1, \mathbf{P}_2) = 1$, but $\vec{\delta}_\tau(\mathbf{P}_1, \mathbf{P}_2) = \bar{\gamma}\left(\mathbf{\Pi}_{(\mathscr{G}_\tau(\mathbf{P}_2))^\perp} \mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P}_1)}\right) < 1$. The latter implies that $\mathscr{G}_\tau(\mathbf{P}_1) \cap \mathscr{G}_\tau(\mathbf{P}_2)^\perp = \{0\}$, as otherwise $\vec{\delta}_\tau(\mathbf{P}_1, \mathbf{P}_2) = 1$. Since $\ker\left((\mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P}_1)}|_{\mathscr{G}_\tau(\mathbf{P}_2)})^*\right) = \ker\left((\mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P}_2)}|_{\mathscr{G}_\tau(\mathbf{P}_1)})\right) = \mathscr{G}_\tau(\mathbf{P}_1) \cap \mathscr{G}_\tau(\mathbf{P}_2)^\perp$, it follows that $\dim\ker\left((\mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P}_1)}|_{\mathscr{G}_\tau(\mathbf{P}_2)})^*\right) = 0$. Recall from [4, Prop. 3] that $\delta_\tau(\mathbf{P}_1, \mathbf{P}_2) < 1$ is equivalent to the invertibility of $\mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P}_1)}|_{\mathscr{G}_\tau(\mathbf{P}_2)}$. The latter is in turn equivalent to

$$\begin{aligned} &\mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P}_1)}|_{\mathscr{G}_\tau(\mathbf{P}_2)} \text{ is Fredholm with} \\ &\qquad \dim\ker\left(\mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P}_1)}|_{\mathscr{G}_\tau(\mathbf{P}_2)}\right) = 0 \quad \text{and} \\ &\qquad \dim\ker\left((\mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P}_1)}|_{\mathscr{G}_\tau(\mathbf{P}_2)})^*\right) = 0. \end{aligned} \tag{4}$$

Therefore, the hypothesis $\delta_\tau(\mathbf{P}_1, \mathbf{P}_2) = 1$ implies either $\mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P}_1)}|_{\mathscr{G}_\tau(\mathbf{P}_2)}$ is not Fredholm or $\dim\ker\left(\mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P}_1)}|_{\mathscr{G}_\tau(\mathbf{P}_2)}\right) \neq 0$. Now, when there exists a $\mathbf{Q} \in \mathbb{Q}$ such that $\bar{\gamma}(\mathbf{G}_1 - \mathbf{G}_2\mathbf{Q}) < 1$, it is shown below that

$$\begin{aligned} &\mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P}_1)}|_{\mathscr{G}_\tau(\mathbf{P}_2)} \text{ is Fredholm with} \\ &\qquad \operatorname{ind}(\mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P}_1)}|_{\mathscr{G}_\tau(\mathbf{P}_2)}) = 0 \;\forall \tau \in \mathbb{R}, \end{aligned} \tag{5}$$

leading to a contradiction. As such, the hypothesis that $\delta_\tau(\mathbf{P}_1, \mathbf{P}_2) = 1$ and $\vec{\delta}_\tau(\mathbf{P}_1, \mathbf{P}_2) < 1$ implies $\bar{\gamma}(\mathbf{G}_1 - \mathbf{G}_2\mathbf{Q}) \geq 1$ for any $\mathbf{Q} \in \mathbb{Q}$. In this case, it is apparent that

$$\delta(\mathbf{P}_1, \mathbf{P}_2) \leq \inf_{\mathbf{Q}\in\mathbb{Q}} \bar{\gamma}(\mathbf{G}_1 - \mathbf{G}_2\mathbf{Q}),$$

as required.

The remainder of the proof establishes (5). Note that $\bar{\gamma}(\mathbf{G}_1 - \mathbf{G}_2\mathbf{Q}) < 1$ implies

$$\bar{\gamma}(\mathbf{G}_1 - \mathbf{G}_2\mathbf{Q}) = \bar{\gamma}(\mathbf{G}_1^*(\mathbf{G}_1 - \mathbf{G}_2\mathbf{Q})) = \bar{\gamma}(\mathbf{I} - \mathbf{G}_1^*\mathbf{G}_2\mathbf{Q}) < 1.$$

Consequently, as $\mathbf{G}_1^*\mathbf{G}_2\mathbf{Q} = \mathbf{I} - (\mathbf{I} - \mathbf{G}_1^*\mathbf{G}_2\mathbf{Q})$, $\underline{\gamma}(\mathbf{G}_1^*\mathbf{G}_2\mathbf{Q}) > 0$ and $\mathbf{T}_\tau(\mathbf{G}_1^*\mathbf{G}_2\mathbf{Q})$ is Fredholm with $\operatorname{ind}(\mathbf{T}_\tau(\mathbf{G}_1^*\mathbf{G}_2\mathbf{Q})) = 0 \,\forall \tau \in \mathbb{R}$, where the latter holds by Lemma 2.2(iii). Because $\mathbf{Q}$ has a bounded causal inverse, $\underline{\gamma}(\mathbf{G}_1^*\mathbf{G}_2\mathbf{Q}) > 0$ implies $\underline{\gamma}(\mathbf{G}_1^*\mathbf{G}_2) > 0$. Moreover, $\mathbf{T}_\tau(\mathbf{Q})$ is boundedly invertible, hence for all $\tau \in \mathbb{R}$, $\operatorname{ind}(\mathbf{T}_\tau(\mathbf{Q})) = 0$ and Fredholmness of $\mathbf{T}_\tau(\mathbf{G}_1^*\mathbf{G}_2\mathbf{Q}) = \mathbf{T}_\tau(\mathbf{G}_1^*\mathbf{G}_2)\mathbf{T}_\tau(\mathbf{Q})$ implies that of $\mathbf{T}_\tau(\mathbf{G}_1^*\mathbf{G}_2)$. Applying Lemma 2.2(ii) again, it follows that

$$\begin{aligned} 0 = \operatorname{ind}(\mathbf{T}_\tau(\mathbf{G}_1^*\mathbf{G}_2\mathbf{Q})) &= \operatorname{ind}(\mathbf{T}_\tau(\mathbf{G}_1^*\mathbf{G}_2)) + \operatorname{ind}(\mathbf{T}_\tau(\mathbf{Q})) \\ &= \operatorname{ind}(\mathbf{T}_\tau(\mathbf{G}_1^*\mathbf{G}_2)) \,\forall \tau \in \mathbb{R}, \end{aligned}$$

as claimed. ∎

The following generalises the LTI result of [8, Thm. 7.5]. Define the maximal robustness margin

$$b_{\text{opt}}(\mathbf{P}) := \sup_{\mathbf{C}\in\mathcal{S}\,:\,[\mathbf{P},\mathbf{C}] \text{ is stable}} b_{\mathbf{P},\mathbf{C}}.$$

Theorem 5.5: Given any $\mathbf{P}_1, \mathbf{P}_2, \mathbf{C} \in \mathcal{S}$, the following inequality holds:

$$\delta_\nu(\mathbf{P}_1, \mathbf{P}_2) \leq \delta(\mathbf{P}_1, \mathbf{P}_2) \leq \frac{\delta_\nu(\mathbf{P}_1, \mathbf{P}_2)}{b_{\text{opt}}(\mathbf{P}_1)}.$$

Proof: First we show $\delta_\nu(\mathbf{P}_1, \mathbf{P}_2) \leq \delta(\mathbf{P}_1, \mathbf{P}_2)$ by mimicking aspects of [8, Section 7.2]. Note that if $\delta(\mathbf{P}_1, \mathbf{P}_2) = 1$, the inequality is trivially true, and hence we assume $\delta(\mathbf{P}_1, \mathbf{P}_2) < 1$. This implies $\delta_\tau(\mathbf{P}_1, \mathbf{P}_2) < 1$ for all $\tau \in \mathbb{R}$, whereby (4) holds. Observe that $\mathscr{G}_\tau(\mathbf{P}_i) = \operatorname{img}(\mathbf{T}_\tau(\mathbf{G}_i))$ by Assumption 3.5, $\mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P}_1)} = \mathbf{T}_\tau(\mathbf{G}_1)\mathbf{T}_\tau(\mathbf{G}_1^*)$ by Assumption 3.5 and Lemma 2.4(ii) because $\mathbf{G}_1$ is normalised (cf. Assumption 3.7), and $\underline{\gamma}(\mathbf{T}_\tau(\mathbf{G}_1)) > 0$ (cf. Remark 3.6). It follows as in [8, Section 7.2] that

$$\mathbf{T}_\tau(\mathbf{G}_1^*)\mathbf{T}_\tau(\mathbf{G}_2) \text{ is Fredholm} \iff \mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P}_1)}|_{\mathscr{G}_\tau(\mathbf{P}_2)} \text{ is Fredholm},$$

in which case

$$\operatorname{ind}(\mathbf{T}_\tau(\mathbf{G}_1^*\mathbf{G}_2)) = \operatorname{ind}(\mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P}_1)}|_{\mathscr{G}_\tau(\mathbf{P}_2)}),$$

where Lemma 2.4(ii) has been exploited. As such, (4) implies that $\mathbf{T}_\tau(\mathbf{G}_1^*\mathbf{G}_2)$ is Fredholm and $\operatorname{ind}(\mathbf{T}_\tau(\mathbf{G}_1^*\mathbf{G}_2)) = 0$ for all $\tau \in \mathbb{R}$, whereby $\delta_\nu(\mathbf{P}_1, \mathbf{P}_2) = \bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1)$.

Now note that $\mathbf{\Pi}_{\operatorname{img}(\mathbf{G}_i)} = \mathbf{G}_i\mathbf{G}_i^*$ and $\mathbf{\Pi}_{\operatorname{img}(\mathbf{G}_i)^\perp} = \tilde{\mathbf{G}}_i^*\mathbf{G}_i$, since $\tilde{\mathbf{G}}_i^*\tilde{\mathbf{G}}_i + \mathbf{G}_i\mathbf{G}_i^* = \mathbf{I}$ by the properties of normalised strong graph representations [10, Eq. 3.2]. Using again Assumption 3.7,

$$\begin{aligned} \delta_\nu(\mathbf{P}_1, \mathbf{P}_2) = \bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1) &= \bar{\gamma}(\tilde{\mathbf{G}}_2^*\tilde{\mathbf{G}}_2\mathbf{G}_1\mathbf{G}_1^*) \\ &= \bar{\gamma}\left(\mathbf{\Pi}_{\operatorname{img}(\mathbf{G}_2)^\perp}\mathbf{\Pi}_{\operatorname{img}(\mathbf{G}_1)}\right). \end{aligned}$$

Since the ν-gap is a metric [21], it follows by the same lines

$$\delta_\nu(\mathbf{P}_1, \mathbf{P}_2) = \delta_\nu(\mathbf{P}_2, \mathbf{P}_1) = \bar{\gamma}\left(\mathbf{\Pi}_{\operatorname{img}(\mathbf{G}_1)^\perp}\mathbf{\Pi}_{\operatorname{img}(\mathbf{G}_2)}\right).$$

Comparing this with the directed gap expressions in (3) it follows that

$$\begin{aligned}\delta_\nu(\mathbf{P}_1,\mathbf{P}_2) &= \bar{\gamma}\left(\mathbf{\Pi}_{\mathrm{img}(\mathbf{G}_1)} - \mathbf{\Pi}_{\mathrm{img}(\mathbf{G}_2)}\right)\\ &= \bar{\gamma}(\mathbf{G}_1\mathbf{G}_1^* - \mathbf{G}_2\mathbf{G}_2^*);\end{aligned} \tag{6}$$

see also [8, Section 7.2]. Given any $v \in \boldsymbol{L}^2_{\mathbb{R}}$, note that

$$\lim_{\tau\to-\infty}(\mathbf{I}-\mathbf{\Pi}_\tau)v = v$$

in $\|\cdot\|_2$. As such, since $\mathbf{G}_i : \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$ is bounded and hence continuous, it follows that for any $v \in \boldsymbol{L}^2_{\mathbb{R}}$,

$$\lim_{\tau\to-\infty}\mathbf{G}_i(\mathbf{I}-\mathbf{\Pi}_\tau)\mathbf{G}_i^*(\mathbf{I}-\mathbf{\Pi}_\tau)v = \mathbf{G}_i\mathbf{G}_i^* v. \tag{7}$$

Moreover, recall that

$$\begin{aligned}&\delta_\tau(\mathbf{P}_1,\mathbf{P}_2) = \gamma\left(\mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P}_1)} - \mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P}_2)}\right)\\ &= \bar{\gamma}\left(\mathbf{G}_1(\mathbf{I}-\mathbf{\Pi}_\tau)\mathbf{G}_1^*|_{\boldsymbol{L}^2_{[\tau,\infty)}} - \mathbf{G}_2(\mathbf{I}-\mathbf{\Pi}_\tau)\mathbf{G}_2^*|_{\boldsymbol{L}^2_{[\tau,\infty)}}\right).\end{aligned}$$

Thus, for any $\tau \in \mathbb{R}$ and $u \in \boldsymbol{L}^2_{[\tau,\infty)}$ with $\|u\|_2 \le 1$,

$$\|\mathbf{G}_1(\mathbf{I}-\mathbf{\Pi}_\tau)\mathbf{G}_1^* u - \mathbf{G}_2(\mathbf{I}-\mathbf{\Pi}_\tau)\mathbf{G}_2^* u\|_2 \le \delta(\mathbf{P}_1,\mathbf{P}_2).$$

Combining this with (7) and (6) yields

$$\delta_\nu(\mathbf{P}_1,\mathbf{P}_2) = \bar{\gamma}\left(\mathbf{\Pi}_{\mathrm{img}(\mathbf{G}_1)} - \mathbf{\Pi}_{\mathrm{img}(\mathbf{G}_2)}\right) \le \delta(\mathbf{P}_1,\mathbf{P}_2),$$

as desired.

Now note that $\delta(\mathbf{P}_1,\mathbf{P}_2) \le \frac{\delta_\nu(\mathbf{P}_1,\mathbf{P}_2)}{b_{\mathrm{opt}}(\mathbf{P}_1)}$ is trivially satisfied if $\delta_\nu(\mathbf{P}_1,\mathbf{P}_2) \ge b_{\mathrm{opt}}(\mathbf{P}_1)$. Thus, consider $\delta_\nu(\mathbf{P}_1,\mathbf{P}_2) < b_{\mathrm{opt}}(\mathbf{P}_1)$, whereby there exists $\mathbf{C} \in \mathcal{S}$ such that $[\mathbf{P}_1,\mathbf{C}]$ is stable and $b_{\mathbf{P}_1,\mathbf{C}} > \delta_\nu(\mathbf{P}_1,\mathbf{P}_2)$, whereby $[\mathbf{P}_2,\mathbf{C}]$ is stable. Let $\mathbf{\Delta}_\tau := \mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P}_2)\|\mathscr{G}'_\tau(\mathbf{C})} - \mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P}_1)\|\mathscr{G}'_\tau(\mathbf{C})}$. It can be shown that $\mathbf{\Delta}_\tau = \mathbf{\Pi}_{\mathscr{G}'_\tau(\mathbf{C})\|\mathscr{G}_\tau(\mathbf{P}_1)}\mathbf{\Pi}_{\mathscr{G}_\tau(\mathbf{P}_2)\|\mathscr{G}'_\tau(\mathbf{C})}$ [6, Thm III.2]. From Lemma 3.9 , it follows that

$$\begin{aligned}\mathbf{\Delta}_\tau &= \mathbf{T}_\tau(\mathbf{G}_2)\mathbf{T}_\tau(\tilde{\mathbf{K}}\mathbf{G}_2)^{-1}\mathbf{T}_\tau(\tilde{\mathbf{K}}) - \\ &\qquad \mathbf{T}_\tau(\mathbf{G}_1)\mathbf{T}_\tau(\tilde{\mathbf{K}}\mathbf{G}_1)^{-1}\mathbf{T}_\tau(\tilde{\mathbf{K}}) \qquad (8)\\ &= \mathbf{T}_\tau(\mathbf{K})\mathbf{T}_\tau(\tilde{\mathbf{G}}_1\mathbf{K})^{-1}\mathbf{T}_\tau(\tilde{\mathbf{G}}_1\mathbf{G}_2)\mathbf{T}_\tau(\tilde{\mathbf{K}}\mathbf{G}_2)^{-1}\mathbf{T}_\tau(\tilde{\mathbf{K}}).\end{aligned} \tag{9}$$

Define

$$\mathbf{Q} := (\tilde{\mathbf{K}}\mathbf{G}_1)^{-1}\tilde{\mathbf{K}}\mathbf{G}_2 \in \mathbb{Q};$$

see Lemma 3.9. It follows from (8) and Lemma 2.4(ii) that

$$\mathbf{\Delta}_\tau = (\mathbf{T}_\tau(\mathbf{G}_2) - \mathbf{T}_\tau(\mathbf{G}_1\mathbf{Q}))\mathbf{T}_\tau(\tilde{\mathbf{K}}\mathbf{G}_2)^{-1}\mathbf{T}_\tau(\tilde{\mathbf{K}}).$$

Comparing this with (9) and applying Lemma 2.4(iii) yields

$$\bar{\gamma}(\mathbf{G}_2 - \mathbf{G}_1\mathbf{Q}) \le \bar{\gamma}((\tilde{\mathbf{G}}\mathbf{K})^{-1})\bar{\gamma}(\tilde{\mathbf{G}}_1\mathbf{G}_2) = \frac{\delta_\nu(\mathbf{P}_1,\mathbf{P}_2)}{b_{\mathbf{P}_1,\mathbf{C}}}.$$

Applying Lemma 5.3 gives

$$\delta(\mathbf{P}_1,\mathbf{P}_2) \le \bar{\gamma}(\mathbf{G}_2 - \mathbf{G}_1\mathbf{Q}) \le \frac{\delta_\nu(\mathbf{P}_1,\mathbf{P}_2)}{b_{\mathbf{P}_1,\mathbf{C}}}.$$

Since this inequality holds for any $\mathbf{C} \in \mathcal{S}$ satisfying $b_{\mathbf{P}_1,\mathbf{C}} > \delta_\nu(\mathbf{P}_1,\mathbf{P}_2)$, one can choose a sequence $\{\mathbf{C}_n\}_{n=1}^\infty \in \mathcal{S}$ such that $b_{\mathbf{P}_1,\mathbf{C}_n} \to b_{\mathrm{opt}}(\mathbf{P}_1)$ as $n \to \infty$ to arrive at

$$\delta(\mathbf{P}_1,\mathbf{P}_2) \le \frac{\delta_\nu(\mathbf{P}_1,\mathbf{P}_2)}{b_{\mathrm{opt}}(\mathbf{P}_1)};$$

see [22] for a use of the same argument above. ■

Remark 5.6: The inequality in Theorem 5.5 shows that the gap and the ν-gap metrics are equivalent on $\mathcal{S}$, since $b_{\mathrm{opt}}(\mathbf{X}) > 0$ for any $\mathbf{X} \in \mathcal{S}$ by Assumption 3.5. They generate the same topology, which is often termed the *graph topology* in the literature [4], [8], [6]. It can be shown as in the LTI case [23], [8] that this is the coarsest topology with respect to which both feedback stability and performance are robust properties [19, Section 3.3].

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