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Achieving a large domain of attraction with short-horizon linear MPC via polyhedral Lyapunov functions

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***Abstract*— Polyhedral control Lyapunov functions (PCLFs) are exploited in this paper to propose a linear model predictive control (MPC) formulation that guarantees a "large" domain of attraction (DoA) even for short horizon. In particular, the terminal region of the proposed finite-horizon MPC formulation is chosen as a level set of an appropriate PCLF. For small dimensional systems, this terminal region can be explicitly computed as an arbitrarily close approximation to the entire (infinite-horizon) stabilizable set. Global stability of the origin is guaranteed by using an "inflated" PCLF as terminal cost. The proposed MPC scheme can be formulated as a (small dimensional) quadratic programming problem by introducing one additional scalar variable. Numerical examples show the main benefits and achievements of the proposed formulation in terms of trade-off between volume of the DoA, computational time and closed-loop performance.**

I. Introduction

Model predictive control (MPC) algorithms solve a finite-horizon optimal control problem (FHOCP) that includes constraints on inputs and states over the predicted trajectory. The first input of the optimal control sequence is injected into the system, and at the successor decision time the FHOCP is solved starting from the new current state. In order to ensure nominal stability of the origin of the resulting closed-loop system several approaches can be used, e.g. inclusion of a suitable terminal constraint and/or a suitable terminal penalty [1, Chapter 2], or enforcing the contraction of a suitable control Lyapunov function (CLF) [2], [3]. When a terminal constraint is enforced, there is a well defined set of initial states for which the FHOCP is feasible, which is the set of states that can be driven to the terminal region in N steps, where N is the finite horizon. Such a set represents the domain of attraction (DoA) of the controller. The terminal region is often computed assuming (implicitly) that a linear state feedback control law is employed within such region [4], [5]. For linear systems, explicit computation of the maximal terminal region, under linear control, is possible [6] although the actual computation is practical only for system with a small-moderate number of states. A particular case of terminal region is represented by a terminal equality constraint [7], [8]. The inclusion of a terminal constraint, however, also has some disadvantages, typically associated to the fact that the DoA can be small if a short horizon is used. In fact, it follows trivially that the DoA can be enlarged by increasing the prediction horizon. Clearly, longer horizons imply higher computational times, and therefore a trade-off between size of the DoA and computational limits is usually necessary.

Terminal penalties are usually employed to take into account (exactly or an upper bound to) the infinite-horizon cost-to-go [5], [8]. In this way, the optimal value function of FHOCP can be shown to be a Lyapunov function for the closed-loop system, thus implying stability of the origin [9]. Moreover, for linear systems with a quadratic cost function, if the terminal penalty is chosen as the solution of the Riccati equation, it is possible to show that the FHOCP yields a solution identical to that of the corresponding infinite-horizon controller [10], [11].

The objective of this paper is to propose a linear MPC formulation with the following features: (i) the DoA is a "large" set irrespectively of the horizon (e.g., even for horizon $N = 1$); (ii) for small dimensional systems, the DoA can be explicitly computed as an arbitrarily close approximation to the infinite-horizon stabilizable set; (iii) the resulting FHOCP can be posed as a Quadratic Programming (QP) problem. To achieve the above goals, we exploit the properties of polyhedral control Lyapunov functions (PCLFs) in the formulation of the FHOCP.

The use of PCLFs for the constrained stabilization of a linear system traces back to [4], [12]. The main advantage of considering PCLF-based stabilization schemes for linear systems is that the maximal (possibly asymmetric) stabilizable set can be approximated with arbitrary precision. Constructive algorithms for PCLFs are based on (iterative) linear programming (LP) [13].

To the best of the authors' knowledge, only a few contributions are available in the literature regarding the use of PCLFs in linear (receding-horizon) MPC formulations. In [14], [15] infinity norms, namely symmetric polyhedral functions, are employed both in the stage cost and in the terminal cost. This paper is motivated by [16], where general Minkowski functionals are used as terminal cost. We indeed focus on achieving a large, possibly asymmetric, DoA even for short-horizon MPC formulations.

The paper is organized as follows. The problem statement is presented in Section II, together with the basic technical preliminaries. A PCLF-based MPC formulation is proposed in Section III, while nominal stability analysis is discussed in Section IV. Numerical implementation of the proposed MPC is presented in Section V. Simulation results are shown in Section VI. The achieved results are summarized in Section VII.

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Notation: $\mathbb{N}$ is the set of natural numbers; given $a, b \in \mathbb{N}$ with $a \le b$, $\mathbb{N}_{a:b} := \{a, \dots, b\} \subset \mathbb{N}$. $\mathbb{R}$, $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ denote the sets of real, strictly positive real, and non-negative real numbers, respectively. $\mathbb{B}$ denotes the unitary ball in $\mathbb{R}^n$. Given $x, y \in \mathbb{R}^n$, $x \le y$ denotes the component-wise inequality. $\max(x)$ denotes the maximum element of $x \in \mathbb{R}^n$. I denotes the identity matrix and $\mathbf{1}$ is a vector with all 1. $A \succ 0$ and $A \succcurlyeq 0$ mean positive definite and semi-definite matrix, respectively; moreover, $\overline{\lambda}_A$ denotes its largest eigenvalue. The interior of a set S is denoted by $\mathrm{int}(S)$.

II. Technical background

A. The infinite-horizon constrained optimal control problem

We consider discrete-time linear time-invariant systems:

$$x^+ = Ax + Bu, \tag{1}$$

in which $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state and input at a given time, and $x^+ \in \mathbb{R}^n$ is the successor state. States and inputs are subject to constraints

$$x(k) \in \mathbb{X} \subset \mathbb{R}^n, \qquad u(k) \in \mathbb{U} \subset \mathbb{R}^m \quad \forall k \in \mathbb{N}, \tag{2}$$

where $\mathbb{X}$ and $\mathbb{U}$ are compact and convex polyhedral sets containing the origin. In particular, we assume that $\mathbb{X}$ contains the origin in its interior, while this is not necessarily required for $\mathbb{U}$. We assume that the pair (A, B) is stabilizable.

Let $\mathbf{u}$ denote a (finite or infinite) control sequence $\{u(k) \mid k \in \mathbb{N}\}$, and let $\phi(k; x, \mathbf{u})$ denote the solution to (1) if the state at time 0 is x and the control sequence is $\mathbf{u}$. The *maximal* stabilizable set $\mathcal{X}_\infty$ is defined as

$$\mathcal{X}_\infty := \{x \in \mathbb{R}^n \mid \exists \mathbf{u} : u(k) \in \mathbb{U}, \\ \phi(k; x, \mathbf{u}) \in \mathbb{X} \ \ \forall k \in \mathbb{N}, \text{ and } \lim_{k \to \infty} \phi(k; x, \mathbf{u}) = 0\}.$$

For any $x \in \mathcal{X}_\infty$, we can also define the set of infinite-horizon admissible control sequences as

$$\mathcal{U}_\infty(x) := \{\mathbf{u} \mid u(k) \in \mathbb{U}, \ \phi(k; x, \mathbf{u}) \in \mathbb{X} \ \ \forall k \in \mathbb{N}, \\ \text{and } \lim_{k \to \infty} \phi(k; x, \mathbf{u}) = 0\}.$$

The control objective is the state-feedback stabilization of $0 \in \mathbb{X}$ for (1), starting from any $x \in \mathcal{X}_\infty$, trying to minimize the quadratic performance cost

$$V_\infty(x, \mathbf{u}) := \sum_{k=0}^{\infty} \ell(\phi(k; x, \mathbf{u}), u(k)),$$

in which $\ell(x, u) := x^\top Q x + u^\top R u$ where $Q \succcurlyeq 0$, $R \succ 0$. Thus, we consider an infinite-horizon optimal control problem (IHOCP):

$$\mathbb{P}_\infty(x) : \qquad \min_{\mathbf{u}} V_\infty(x, \mathbf{u}) \qquad \text{s. t.} \quad \mathbf{u} \in \mathcal{U}_\infty(x). \tag{3}$$

Let $\mathbf{u}^0(x)$ be the optimal solution of the problem $\mathbb{P}_\infty(x)$ and $\kappa(x) := u^0(0; x)$ its first component. Moreover, let $V^0_\infty(x) := V_\infty(x, \mathbf{u}^0(x))$ denote the optimal value of problem $\mathbb{P}_\infty(x)$.

B. Basic finite-horizon constrained formulation

The basic *sub-optimal* solution of the infinite-horizon problem (3) is the following finite-horizon constrained formulation. Let $\mathbf{u}$ be a finite-horizon control sequence of length N. Define the set of admissible initial states as

$$\mathcal{X}_N := \{x \in \mathbb{R}^n \mid \exists \mathbf{u} : u(k) \in \mathbb{U}, \\ \phi(k; x, \mathbf{u}) \in \mathbb{X} \ \ \forall k \in \mathbb{N}_{0:N-1}, \text{ and } \phi(N; x, \mathbf{u}) \in \mathbb{X}_f\}, \tag{4}$$

in which $\mathbb{X}_f \subset \mathcal{X}_\infty$ is a *terminal set*, containing the origin, later defined. For any $x \in \mathcal{X}_N$, we can define the set of finite-horizon admissible control sequences and the cost as

$$\mathcal{U}_N(x) := \{\mathbf{u} \mid u(k) \in \mathbb{U}, \ \phi(k; x, \mathbf{u}) \in \mathbb{X} \ \ \forall k \in \mathbb{N}_{0:N-1}, \\ \text{and } \phi(N; x, \mathbf{u}) \in \mathbb{X}_f\}, \tag{5}$$

$$V_N(x, \mathbf{u}) := V_f(\phi(N; x, \mathbf{u})) + \sum_{k=0}^{N-1} \ell(\phi(k; x, \mathbf{u}), u(k)),$$

in which V_f is a positive definite function, referred to as *terminal cost*. Consequently, the *basic* finite-horizon optimal control problem (FHOCP) considered is

$$\mathbb{P}_N(x) : \qquad \min_{\mathbf{u}} V_N(x, \mathbf{u}) \qquad \text{s. t.} \quad \mathbf{u} \in \mathcal{U}_N(x). \tag{6}$$

Remark 1: In order to ensure exponential stability of the origin of (1), V_f must satisfy the invariance condition that for all $x \in \mathbb{X}_f$, there exists $u \in \mathbb{U}$ such that $Ax + Bu \in \mathbb{X}_f$ and $V_f(Ax + Bu) - V_f(x) \le -\ell(x, u)$.

For instance, a common choice [10], [11] for such a function is $V_f(x) := x^\top P x$, where $P \succ 0$ is the (unique) positive definite solution to the discrete-time Algebraic Riccati Equation $A^\top P A - P + Q - A^\top P B \left(B^\top P B + R\right)^{-1} B^\top P A = 0$. Several options are available to construct an associated terminal set $\mathbb{X}_f$. Given the Riccati-optimal QCLF $V_f(x) = x^\top P x$ and the associated feedback gain $K := -\left(B^\top P B + R\right)^{-1} B^\top P A$, one possible choice for $\mathbb{X}_f$ is the maximal constraint-admissible invariant set [6] for the autonomous system $x^+ = (A + BK)x$, which is described by a (possibly large) number of linear inequalities. For larger systems is typically simpler to define $\mathbb{X}_f$ as a sub-level set of V_f, thus considering $\mathbb{X}_f$ as an ellipsoidal subset of the maximal constraint-admissible invariant set.

Remark 2: When the terminal constraint $\phi(N; x, \mathbf{u}) \in \mathbb{X}_f$ is omitted from the definition of $\mathcal{U}_N(x)$ *but* the solution to $\mathbb{P}_N(x)$, $\mathbf{u}^0(x)$, is such that $\phi(N; x, \mathbf{u}^0(x)) \in \mathbb{X}_f$ holds, then it is possible to show that $\mathbb{P}_N(x)$ in (6) and the $\mathbb{P}_\infty(x)$ in (3) yield the same solution [17], [10], [11].

Having active control constraints at $0 \in \mathbb{U}$ is not uncommon in MPC. It arises, e.g., when there is a steady-state optimization layer which pushes the input equilibrium targets (i.e. the origin in deviation variables) towards the boundary of the admissible region, in order to maximize the overall profit [18]. If $\mathbb{U}$ contains the origin on its boundary, the choice of $\mathbb{X}_f$ as the maximal constraint-admissible invariant set for the closed-loop system $x^+ = (A + BK)x$ generally provides a "small" polyhedral invariant set due to the fact that, for any $\rho > 0$, the unconstrained optimal control law $u = Kx$ is not feasible for all $x \in \rho\mathbb{B}$. A valid QCLF for the case in which $\mathbb{U}$ contains the origin on the boundary is discussed in [19], [20], which yields the *optimal* solution to (3) in the limit of *large* control horizon [20]. On the other hand, such possibility does not impose any limitation in our problem formulation. This further motivates the investigation of (asymmetric) PCLF-based MPC schemes, especially for *short or moderate* control horizons, as discussed later on.

III. Proposed MPC method

A. Polyhedral set and associated PCLF

Let $\mathbb{X}_p \subseteq \mathcal{X}_\infty$ be a compact polyhedral set defined as

$$\mathbb{X}_p := \{x \in \mathbb{R}^n \mid Fx \leq \mathbf{1}\}, \tag{7}$$

in which $F \in \mathbb{R}^{r \times m}$, is such that $\max(Fx) > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. We make the following assumption.

Assumption 3: $\mathbb{X}_f \subseteq \mathbb{X}_p$, and for any $x \in \mathbb{X}_p$ there exists $u \in \mathbb{U}$ and $\lambda \in [0, 1)$ such that:

$$F(Ax + Bu) \leq \lambda \max(Fx)\mathbf{1}. \tag{8}$$

Notice that condition (8) states that $\mathbb{X}_p$ is controlled invariant because $x \in \mathbb{X}_p$ implies $\max(Fx) \leq 1$, and thus $x^+ = Ax + Bu \in \mathbb{X}_p$ because $Fx^+ \leq \lambda\mathbf{1} \leq \mathbf{1}$. Therefore, it is natural to inquire about the use of $\mathbb{X}_p$ as a terminal set in a finite-horizon MPC formulation.

Remark 4: The construction of a valid set $\mathbb{X}_p$ with a prescribed $\lambda \in [0, 1)$ can be performed via sequential LP both for discrete-time systems [21] and "equivalently" for the class of continuous-time systems [22].

If we choose $\lambda = 1 - \epsilon$, with $\epsilon > 0$ small, the obtained set $\mathbb{X}_p$ is a close approximation to the entire stabilizable set $\mathcal{X}_\infty$. The numerical computation of $\mathbb{X}_p$ according to algorithm described in [21] is typically possible for small dimensional (e.g., $n \leq 4$) systems. A further trade-off between volume and complexity of the controlled invariant set $\mathbb{X}_p$ can be computed [23, Section 5].

Given $\mathbb{X}_p$ satisfying Assumption 3, we define an associated positive definite function $V_p : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ as

$$V_p(x) := (\max(Fx))^2. \tag{9}$$

We notice that V_p is a "second-order" PCLF [24] [25] for system (1) with controlled DoA $\mathbb{X}_p$.

Useful known results about PCLFs applied to specific definitions of V_p and $\mathbb{X}_p$ are reported in Appendix A.

B. MPC formulation

In this section we show how to exploit both V_p and $\mathbb{X}_p$ in order to define a finite-horizon MPC algorithm with a large domain of attraction even for very short horizons (e.g., $N = 1$). In particular, we show that the polyhedral function V_p, suitably weighted, can be used as a valid *terminal cost* in conjunction with $\mathbb{X}_p$ as a *terminal set*. We define the set of admissible initial states as

$$\begin{aligned}\mathcal{X}_N^p := \{x \in \mathbb{R}^n \mid \exists \mathbf{u} : u(k) \in \mathbb{U},\ \phi(k; x, \mathbf{u}) \in \mathbb{X} \\ \forall k \in \mathbb{N}_{0:N-1},\ \text{and}\ \phi(N; x, \mathbf{u}) \in \mathbb{X}_p\},\end{aligned} \tag{10}$$

and, for any $x \in \mathcal{X}_N^p$, the set of admissible sequences $\mathbf{u}$ is

$$\begin{aligned}\mathcal{U}_N^p(x) := \{\mathbf{u} \mid u(k) \in \mathbb{U},\ \phi(k; x, \mathbf{u}) \in \mathbb{X}\ \ \forall k \in \mathbb{N}_{0:N-1}, \\ \text{and}\ \phi(N; x, \mathbf{u}) \in \mathbb{X}_p\}.\end{aligned} \tag{11}$$

Since $\mathbb{X}_f \subseteq \mathbb{X}_p$, for all $N \in \mathbb{N}$ we have $\mathcal{X}_N \subseteq \mathcal{X}_N^p$; moreover for any $x \in \mathcal{X}_N$ there holds $\mathcal{U}_N(x) \subseteq \mathcal{U}_N^p(x)$.

We can now present our PCLF-based formulation. To this aim, we introduce a weighting factor $\beta > 0$, and we define the finite-horizon cost as

$$V_{N,\beta}(x, \mathbf{u}) := \beta V_p(\phi(N; x, \mathbf{u})) + \sum_{k=0}^{N-1} \ell(\phi(k; x, \mathbf{u}), u(k)),$$

so that the FHOCP, consequently, is

$$\mathbb{P}_N^\beta(x) : \qquad \min_{\mathbf{u}} V_{N,\beta}(x, \mathbf{u}) \quad \text{s. t.} \quad \mathbf{u} \in \mathcal{U}_N^p(x). \tag{12}$$

Let $V_{N,\beta}^0(x)$ denote the optimal value of $\mathbb{P}_N^\beta(x)$. As discussed later in Section IV, if β is chosen suitably large, stability of the origin of the closed-loop system can be proved by showing that $V_{N,\beta}^0$ acts as a Lyapunov function for the closed-loop system.

Remark 5: A PCLF-based variant to the problem (12) is the following one-step contraction scheme, with $\lambda \in (0, 1)$:

$$\begin{aligned}\mathbb{P}_N^\lambda(x) : \qquad \min_{\mathbf{u}} V_N(x, \mathbf{u}) \quad \text{s. t.} \quad \mathbf{u} \in \mathcal{U}_N^p(x), \\ \max\left(F\phi(1; x, \mathbf{u})\right) \leq \lambda \max\left(Fx\right),\end{aligned}$$

in which we remark that $V_N(\cdot)$ is used as cost function.

IV. Stability analysis

We first present some supporting results.

Lemma 6: For any $x \in \mathbb{X}_p$ and $\beta \in \mathbb{R}_{\geq 0}$ satisfying

$$\beta \geq \beta^* := \frac{\overline{\lambda}_Q + c^2\, \overline{\lambda}_R}{\alpha_3}, \tag{13}$$

there exists an input $u \in \mathbb{U}$ satisfying $(Ax + Bu) \in \mathbb{X}_p$ and

$$\beta V_p(Ax + Bu) - \beta V_p(x) \leq -\ell(x, u). \tag{14}$$

Proof: The fact that, for any $x \in \mathbb{X}_p$ there exists $u \in \mathbb{U}$ such that $x^+ = Ax + Bu \in \mathbb{X}_p$ comes from Assumption 3. In view of Lemma 14 in Appendix A, there always exists $u \in \mathbb{U}$, satisfying $\|u\| \leq c\|x\|$, such that: $\beta V_p(x) - \beta V_p(Ax + Bu) \geq \beta\alpha_3 \|x\|^2 \geq (\overline{\lambda}_Q + c^2\, \overline{\lambda}_R) \|x\|^2 \geq \ell(x, u)$ holds for any $\beta \geq \beta^* := \frac{\overline{\lambda}_Q + c^2\, \overline{\lambda}_R}{\alpha_3}$. ∎

Lemma 7: For any β satisfying condition (13) of Lemma 6 and any $N \in \mathbb{N}$, the following inequalities hold for any $x \in \mathbb{X}_p$: $V_\infty^0(x) \leq \cdots \leq V_{N+1,\beta}^0(x) \leq V_{N,\beta}^0(x)$.

Proof: We first prove that the inequality $V_{N+1,\beta}^0(x) \leq V_{N,\beta}^0(x)$ holds for any $N \in \mathbb{N}$. Let $\mathbf{u}_{N,\beta}^0(x) := (u_\beta^0(0; x), u_\beta^0(1; x), \ldots, u_\beta^0(N-1; x))$ be the optimal solution to problem $\mathbb{P}_N^\beta(x)$. Let $x_{N,\beta}^0 := \phi(N; x, \mathbf{u}_{N,\beta}^0) \in$

$\mathbb{X}_p$. From Lemma 6, choose any $u^* \in \mathbb{U}$ such that $(Ax^0_{N,\beta} + Bu^*) \in \mathbb{X}_p$ and $\beta V_p(Ax^0_{N,\beta} + Bu^*) + \ell(x^0_{N,\beta}, u^*) \leq \beta V_p(x^0_{N,\beta})$. Define the following candidate sequence for problem $\mathbb{P}^\beta_{N+1}(x)$: $\mathbf{u}_{N+1,\beta} := (u^0_\beta(0;x), u^0_\beta(1;x), \ldots, u^0_\beta(N-1;x), u^*)$ and notice that $\mathbf{u}_{N+1,\beta} \in \mathcal{U}^p_{N+1}(x)$. Since $\mathbf{u}_{N+1,\beta}$ is not necessarily optimal for problem $\mathbb{P}^\beta_{N+1}(x)$, we have that:

$$V^0_{N+1,\beta}(x) \leq V_{N+1,\beta}(x, \mathbf{u}_{N+1,\beta}) = V^0_{N,\beta}(x) - \beta V_p(x^0_{N,\beta}) + \ell(x^0_{N,\beta}, u^*) + \beta V_p(Ax^0_{N,\beta} + Bu^*) \leq V^0_{N,\beta}(x).$$

To prove $V^0_\infty(x) \leq V^0_{N,\beta}(x)$, we note that the sequence $\{V^0_{N,\beta}(x)\}$ is monotonically non-increasing with N and bounded below by 0. Thus, it converges to some point $V^0_{\infty,\beta}(x)$. Hence $V^0_{\infty,\beta}(x) := \lim_{N\to\infty} V^0_{N,\beta}(x)$ is equal to

$$\lim_{N\to\infty} \sum_{k=0}^{N-1} \ell(\phi(k;x,\mathbf{u}^0_{N,\beta}(x)), u^0_{N,\beta}(k;x)) + \lim_{N\to\infty} \beta V_p(\phi(N;x,\mathbf{u}^0_{N,\beta}(x)) \geq \lim_{N\to\infty} \sum_{k=0}^{N-1} \ell(\phi(k;x,\mathbf{u}^0_{N,\beta}(x)), u^0_{N,\beta}(k;x)) \geq \sum_{k=0}^{\infty} \ell(\phi(k;x,\mathbf{u}^0(x)), u^0(k;x)) = V^0_\infty(x),$$

from which $V^0_\infty(x) \leq V^0_{N,\beta}(x)$ follows $\forall N \in \mathbb{N}$. ■

Lemma 8: There exist $\gamma_1, \gamma_2 \in \mathbb{R}_{>0}$ such that:

$$\gamma_1 \|x\|^2 \leq V^0_{N,\beta}(x) \leq \gamma_2 \|x\|^2. \tag{15}$$

Proof: From Lemma 7 we have $V^0_\infty(x) \leq V^0_{N,\beta}(x) \leq V^0_{0,\beta}(x) := \beta V_p(x)$. The optimal cost $x^\top P x$ for the unconstrained system (1) obviously satisfies $x^\top P x \leq V^0_\infty(x)$. Therefore, from Lemma 14, it follows that: $x^\top P x \leq V^0_{N,\beta}(x) \leq \beta\alpha_2 \|x\|^2$, from which (15) trivially follows for some positive constants γ_1, γ_2. ■

In our stability analysis, we use the following notion of exponential stability (ES).

Definition 9 (Exponential Stability): Let $\psi(k;x)$ be the solution at time k of the difference equation $x^+ = f(x)$, with initial state $x(0) = x$, and let $f(0) = 0$. The origin of $x^+ = f(x)$ is exponentially stable (ES) on the set $\mathcal{X} \subseteq \mathbb{R}^n$ if there exist $b \in \mathbb{R}_{>0}$ and $\lambda \in (0,1)$ such that for any initial state $x \in \mathcal{X}$ there holds:

$$\psi(k;x) \in \mathcal{X}, \qquad \|\psi(k;x)\| \leq b\lambda^k \|x\| \quad \forall k \in \mathbb{N}.$$

From now on, let $\mathbf{u}^0_\beta(x)$ be the solution of problem $\mathbb{P}^\beta_N(x)$ in (12), and let $\kappa_\beta(x) := u^0_\beta(0;x)$ be its first component.

Theorem 10: For any β satisfying condition (13), the origin of $x^+ = Ax + B\kappa_\beta(x)$ is ES on $\mathcal{X}^p_N$.

Proof: This result can be proved by applying standard MPC stability results [1, Theorem 2.24, p. 123] and recalling the results of Lemma 6 and Lemma 8. ■

V. Numerical implementation

Problem $\mathbb{P}^\beta_N(x)$, as written in (12), is not posed as standard QP problem. However, we have the following results.

Lemma 11: For any $\xi \in \mathbb{R}_{\geq 0}$ there holds:

$$\{x \in \mathbb{R}^n \mid V_p(x) \leq \xi^2\} = \{x \in \mathbb{R}^n \mid Fx \leq \xi\mathbf{1}\}.$$

Proof: Let F_i be the i^{th} row of the matrix F. Then:

$$V_p(x) := (\max(Fx))^2 = \left(\max_{i\in\mathbb{N}_{1:r}} F_i x\right)^2 \leq \xi^2 \Leftrightarrow \max_{i\in\mathbb{N}_{1:r}} \{F_i x\} \leq \xi \Leftrightarrow F_i x \leq \xi \;\; \forall i \in \mathbb{N}_{1:r} \Leftrightarrow Fx \leq \xi\mathbf{1}.$$

■

Proposition 12: The nonlinear optimization problem $\mathbb{P}^\beta_N(x)$ in (12) is equivalent to the following QP problem:

$$\bar{\mathbb{P}}^\beta_N(x): \quad \min_{\mathbf{u},\xi} \bar{V}_{N,\beta}(x,\mathbf{u},\xi) \quad \text{s. t.} \quad \mathbf{u} \in \mathcal{U}^p_N(x), \quad \xi \in [0,1], \quad F\phi(N;x,\mathbf{u}) \leq \xi\mathbf{1}, \tag{16}$$

having cost function

$$\bar{V}_{N,\beta}(x,\mathbf{u},\xi) = \beta\xi^2 + \sum_{k=0}^{N-1} \ell(\phi(k;x,\mathbf{u}), u(k)). \tag{17}$$

Proof: Problem $\mathbb{P}^\beta_N(x)$ in (12) can be rewritten as

$$\min_{\mathbf{u},\xi} \bar{V}_{N,\beta}(x,\mathbf{u},\xi) \quad \text{s. t.} \quad \mathbf{u} \in \mathcal{U}^p_N(x), \; V_p(\phi(N;x,\mathbf{u})) \leq \xi^2$$

with cost function $\bar{V}_{N,\beta}$ as in (17). The QP formulation (16) is finally recovered in view of Lemma 11. ■

Remark 13: In Proposition 12, "equivalent" is intended in the following sense: the optimal values of $\mathbb{P}^\beta_N(x)$ in (12) and $\bar{\mathbb{P}}^\beta_N(x)$ in (16) are the same, and if $\mathbf{u}^0_\beta(x)$ solves $\mathbb{P}^\beta_N(x)$ and $(\bar{\mathbf{u}}^0_\beta(x), \xi^0(x))$ solves $\bar{\mathbb{P}}^\beta_N(x)$, then $\bar{\mathbf{u}}^0_\beta(x) = \mathbf{u}^0_\beta(x)$.

VI. Numerical examples

The examples show numerical comparisons and results regarding volume of the DoA and computational time.

We numerically compute $\mathbb{X}_p$ as a close approximation of the "maximal" stabilizable set $\mathcal{X}_\infty$ according to [21], namely with guaranteed decay rate $\lambda = 1 - 10^{-2}$, i.e. with $\epsilon := 10^{-2}$. Moreover, the set $\mathbb{X}_f$ is explicitly constructed as in [6, Algorithm 3.2]. Also the sets $\mathcal{X}_N$ in (4) and $\mathcal{X}^p_N$ in (10) are computed numerically (for $N = 1, 2, \cdots, 40$) to compare the volume of the DoA of the basic and of the proposed MPC formulations as the horizon N increases. In both examples, closed-loop simulations are performed for 100 time steps; the cost matrices are chosen as $Q = I_n$ and $R = 0.1 I_m$.

A. Example 1

Consider the open-loop unstable system

$$x^+ = \begin{bmatrix} 1.0250 & 0.0125 \\ 0.0250 & 1.0500 \end{bmatrix} x + \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix} u.$$

The constraint sets are $\mathbb{X} = \{x \in \mathbb{R}^2 \mid \|x\|_\infty \leq 1\}$, and $\mathbb{U} = \{u \in \mathbb{R}^2 \mid \|u\|_\infty \leq 1\}$. The weight β is chosen according to Lemma 6, with $\overline{\lambda}_Q = 1$, $c = 1$, $\overline{\lambda}_R = 0.1$, $\alpha_3 = (1-\lambda^2)\alpha_1 = (1-\lambda^2)\cdot 0.45$, namely $\beta = \beta^* \simeq 123$.

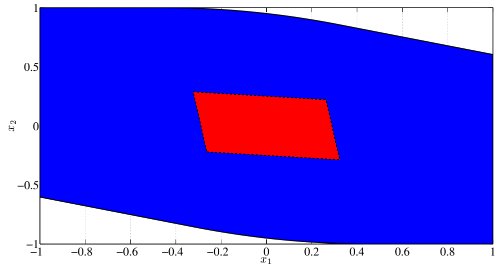

Fig. 1. Example 1. Comparison of terminal sets: $\mathbb{X}_f$ (red, boundary in dotted line), the maximal domain of attraction under the linear control law $u(x) = Kx$ [6], is used by the basic MPC formulation; $\mathbb{X}_p$ (blue, boundary in solid line) is used by the proposed MPC formulation.

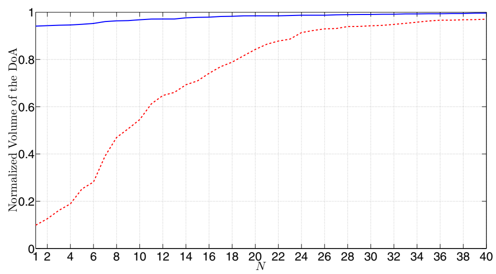

Fig. 2. Example 1. Comparison of the DoA normalized volume, as a function of the horizon N, for basic and proposed MPC formulations: $\mathcal{X}_N$ (red, dotted line) and $\mathcal{X}_N^p$ (blue, solid line).

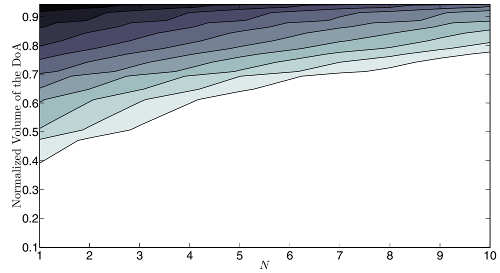

Fig. 3. Example 1. Ratio of solution times as a function of the horizon used by the proposed MPC and of the DoA normalized volume of the basic MPC. The darker the color the faster the proposed MPC is.

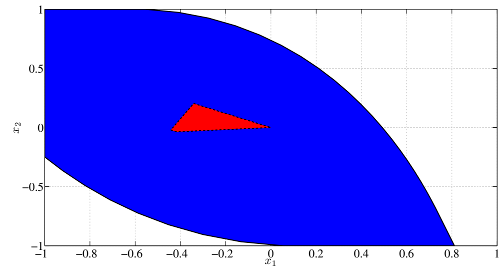

Fig. 4. Example 2. Comparison of terminal sets: $\mathbb{X}_f$ (red, boundary in dotted line), the maximal domain of attraction under the linear control law $u(x) = Kx$ [6], is used by the basic MPC formulation; $\mathbb{X}_p$ (blue, boundary in solid line) is used by the proposed MPC formulation.

The set $\mathbb{X}_p$ is shown in Figure 1, along with $\mathbb{X}_f$. It is clear that $\mathbb{X}_f \subset \mathbb{X}_p$, and that $\mathbb{X}_p$ is much larger. Since $\mathbb{X}_f$ and $\mathbb{X}_p$ are the terminal sets of, respectively, the basic and the proposed MPC formulations, for any horizon N the DoA of the latter is much larger than the one of the former, as shown in Figure 2. We can also appreciate that in the proposed formulation the DoA is almost the maximal one for any N, whereas in the basic formulation a long horizon is required if one wants to achieve a large DoA. For instance, a volume of 95% of the maximal DoA is achieved by the basic formulation with $N \geq 30$.

We next compare the required solution time for both MPC algorithms. Figure 3 displays the ratio of the solution times of the two MPCs as a function of the horizon $N \in [1, 10]$ used by the proposed formulation (on the x axis) and of the DoA normalized volume of the basic formulation (on the y axis). Results are averaged over 100 randomly-taken initial states. In order not to penalize the basic MPC formulation, the optimization problems are solved via an interior-point method, whose required solution time "scales linearly" with the used horizon [26]. The lowest curve indicates the points in which the two formulations perform equally fast. Below this curve, in the white area, the basic MPC is faster than the proposed one. Above this curve, our formulation is faster: the darker the color the more convenient our formulation is. In particular, the black area indicates point in which our formulation requires half solution time that the basic MPC. It is clear that the proposed MPC is convenient when one desires to control a large portion of the maximal DoA, e.g. at least $70 \div 80\%$ of it.

B. Example 2

We consider the open-loop unstable system

$$x^+ = \begin{bmatrix} 1.1 & 0 \\ 0.2 & 1.1 \end{bmatrix} x + \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0 \end{bmatrix} u.$$

The constraint sets are $\mathbb{X} = \{x \in \mathbb{R}^2 \mid \|x\|_\infty \leq 1\}$, and $\mathbb{U} = \{(u_1, u_2) \in \mathbb{R}^2 \mid u_1 \in [0,1],\ u_2 \in [-1,1]\}$. The set of admissible controls $\mathbb{U}$ has the origin on its boundary, because of the constraint $u_1 \geq 0$. As a consequence, the set $\mathbb{X}_f$ is particularly "small" (and asymmetric), see Figure 4. Notice that the asymmetry of $\mathbb{U}$ induces also the set $\mathbb{X}_p$ to be asymmetric, as also shown in Figure 4. However, $\mathbb{X}_p$ is much larger than $\mathbb{X}_f$.

The weight β is chosen according to Lemma 6, with $\overline{\lambda}_Q = 1$, $c = 1$, $\overline{\lambda}_R = 0.1$, $\alpha_3 = (1 - \lambda^2)\alpha_1 = (1 - \lambda^2) \cdot 0.7 = 0.0139$, namely $\beta = \beta^* \simeq 79$.

The comparisons between the basic and the proposed MPC formulations in this second example share the same qualitative behavior of the comparisons in Example 1, as shown in Figures 5, 6. We notice that the proposed MPC is again more convenient from a computational point of view whenever a large portion of the maximal DoA has to be controlled.

VII. Conclusions

Polyhedral control Lyapunov functions (PCLFs) can shape a "large" domain of attraction (DoA) for constrained linear systems. In particular, for small dimensional systems, an arbitrarily close approximation to the entire (infinite-horizon) stabilizable set can effectively computed via Linear

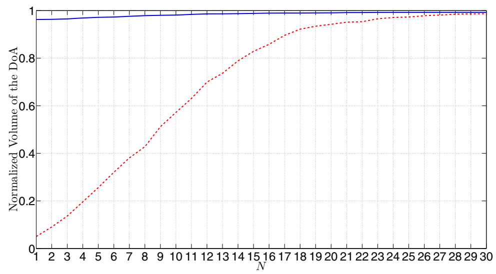

Fig. 5. Example 2. Comparison of the DoA normalized volume, as a function of the horizon N, for basic and proposed MPC formulations: $\mathcal{X}_N$ (red, dotted line) and $\mathcal{X}_N^p$ (blue, solid line).

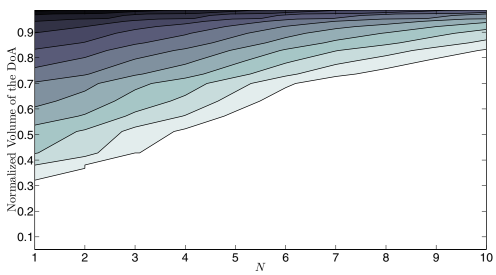

Fig. 6. Example 2. Ratio of solution times as a function of the horizon used by the proposed MPC and of the DoA normalized volume of the basic MPC. The darker the color the faster the proposed MPC is.

Programming. We exploit this property to propose a novel MPC formulation which guarantees (at least) such large stabilizable set, independently of the chosen finite horizon. This result is achieved as the terminal constraint region is the sub-level set of an appropriate PCLF. Closed-loop exponential stability of the origin is ensured by a suitably "inflated" PCLF-based terminal penalty. We also show that the optimal control problem can be solved as a Quadratic Programming problem as in standard MPC formulations.

Achieving a large DoA with short prediction horizon is an important goal because it allows the computational burden, due to long horizons, to be separated from feasibility and stability issues. In this way, the horizon only affects the closed-loop performance and solution time, *not* the DoA. This can be attractive when a long horizon cannot be used,for instance in fast, small-dimensional, systems.

Appendix

A. Known results on PCLFs and associated sets

Lemma 14: Let $V_p(x) := (\max(Fx))^2$ be the PCLF "shaping" the set $\mathbb{X}_p$ in (7). There exist $\alpha_1, \alpha_2 > 0$ such that: $\alpha_1 \|x\|^2 \leq V_p(x) \leq \alpha_2 \|x\|^2$. There also exist $\alpha_3, c > 0$ and $\lambda \in [0, 1)$ such that the following statement holds true. For any $x \in \mathbb{X}_p$ there exists $u \in \mathbb{U}$, such that $||u|| \leq c||x||$, $Ax + Bu \in \mathbb{X}_p$, $V_p(Ax + Bu) - V_p(x) \leq -\alpha_3 \|x\|^2$, and $V_p(Ax + Bu) \leq \lambda^2 V_p(x)$.

Remark 15: The condition $V_p(Ax + Bu) \leq \lambda^2 V_p(x)$ is equivalent to $F(Ax + Bu) \leq \lambda \max(Fx)\mathbf{1}$.

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