

Deterministic filtering for optimal attitude estimation on $SO(3)$ using max-plus methods

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Abstract—In this article we introduce the use of recently developed min/max-plus techniques in order to solve the optimal attitude estimation problem in filtering for nonlinear systems on the special orthogonal ($SO(3)$) group. This work helps synthesize deterministic filters for nonlinear systems – i.e. optimal filters which estimate the system state using a related optimal control problem. The technique indicated herein is validated using a set of optimal attitude estimation example problems on $SO(3)$.

I. INTRODUCTION

Optimal filtering is one of the major themes of research interest in the areas of systems and control. There are two distinct approaches to filter design which have been applied in the design of filters for systems subjected to process/measurement noise. The most well known approach to filter design is the Kalman filter developed for linear systems in the 1960's [1]. The Kalman filter uses the statistics of the noise/measurement processes in order to compute the coefficients of the equations used to update the state estimate as new measurements are observed. An alternative approach is the minimum energy filtering technique developed by Mortensen [2]. This interpretation views the optimal filtering problem as a optimal control problem where the objective is to minimize the energy of the noise process required to explain the observations. The resulting filter obtained via both these approaches for the linear system case with white noise processes (and a quadratic energy function with L^2 noise processes) turns out to be the Kalman filter (KF) [3]–[5].

In the case of nonlinear systems (or non-normal noise) the congruence in these two approaches no longer holds. There are various extensions of the KF to nonlinear systems (e.g. unscented Kalman filter (UKF) [6], [7],

particle filter [8], extended Kalman filters (EKF) [9]). However the extensions of the deterministic filter to the nonlinear case and non Euclidean spaces have only recently garnered greater attention [10], [11]. The specific problem of practical interest motivating this work on filter design for nonlinear systems is the attitude estimation problem [12] on the group of rotation matrices (the special orthogonal group $SO(3)$).

Recent work by Coote et al. [13] and Zamani et al. [14] studied the deterministic optimal filtering problem for the case of systems evolving on $SO(2)$ and $SO(3)$ respectively. The filter obtained in the latter case is a near optimal solution to the filtering problem (i.e. it is shown to lie within a specific performance gap from the optimal filter). The motivation in this work on using a near optimal approximation has to do with the numerical intractability of solving the optimal control problem associated with the filter design. For nonlinear systems standard approaches using the dynamic programming method suffer from larger computational requirements (termed the curse of dimensionality) while computing numerical solutions to these classes of problems.

In the recent past, a new class of approaches to manage this computational intractability have been developed for both filtering [15] and control [16]–[18] for a variety of application domains. In this article we introduce and solve the Hamilton-Jacobi-Bellman (HJB) equation for the optimal control problem (associated with the filtering problem) using recently developed theoretical and numerical techniques for nonlinear filtering. These approaches (termed min/max-plus methods) utilize the fact that the dynamic programming operator for the HJB equation is a linear operator over the set of semi-convex functions. Hence, in a certain class

of problems, it is possible to solve the HJB equation without having to generate a mesh/grid. By applying these techniques to the current problem we indicate an approach to achieve dimensionality reduction in filter synthesis for nonlinear systems [19], [20] – specifically the attitude estimation problem; Thus we extend the max/min-plus filtering methods to the case of a compact Lie group ($SO(3)$).

This article is organized as follows: we first introduce the system description and the problem statement for the attitude estimation problem in Sec. II. This leads to a consideration of the variation in the parameterization of the value function as it is propagated in time using the dynamic programming operator. It is shown in Sec. III that this structure is preserved. Hence this permits the use of a specific expansion of the value function in terms of a basis whose form is preserved – allowing for the generation of reduced dimensionality methods for filtering. The theory developed is then applied to a class of example problems in Sec. IV thereby validating the techniques introduced. In Sec. V we conclude this article with a description of several interesting avenues for future research into different aspects of the min-plus approach to filtering for the classes of problems described herein.

II. SYSTEM DESCRIPTION

Let the original system dynamics be

$$\begin{aligned} \dot{R}(t) &= R(t)(A(t) + z(t)), \\ Y(t) &= R(t)\epsilon(t), \quad \epsilon(\cdot) \in SO(3), \end{aligned} \quad (1)$$

where Y, A are known signals and z is the unknown state disturbance signal and ϵ is the unknown measurement noise. Let the *backward* time state transition function be defined as

$$R_k = \psi(R_{k+1}, z_{k+1}, A_{k+1}) := R_{k+1}\Psi(z_{k+1}, A_{k+1}).$$

Thus the operator $\Psi(\cdot)$ is the state transition operator, which in this case would be time ordered exponential map. Given measurements Y , drift A , and an initial state estimate \hat{R}_0 the cost function for the filtering problem

is given by [14]

$$V_0(R) := \inf_{z \in \mathcal{Z}} \left\{ \int_0^T \frac{1}{2} \text{tr}(z^T z) ds + \int_0^T \frac{1}{4} \phi_{L^{-1}}(R^{-1}Y) ds + \frac{1}{4} \phi_{K^{-1}}(R(0; z, A) - \hat{R}_0) \right\} \quad (2)$$

where $\phi_{K^{-1}}$ and $\phi_{L^{-1}}$ are versions of the following function with different parameter values for M

$$\phi_M(R) := \text{tr}[(R - I)^T M (R - I)], \quad R \in SO(3)$$

and $R(0; z, A)$ denotes the state at time 0 given the choice z of the disturbance signals which lie in the space \mathcal{Z} of L^2 signals. Thus the cost function is a measure of the weighed sum of the energy in the disturbance in the system dynamics, the unexplained part of the measurements and the error in the initial estimate (versus the predicted initial estimate as obtained from the choice of the disturbance signal). Thus the terminal cost at a point R_0 with an initial state estimate \hat{R}_0 is a measure of the deviation of the final state from the expected value in (2) and is defined as

$$\begin{aligned} V_0(R_0) &= \frac{1}{4} \phi_{K^{-1}}(R_0 \hat{R}_0^T) \\ &= \frac{1}{4} \text{tr}[(R_0 \hat{R}_0^T - I)^T K^{-1} (R_0 \hat{R}_0^T - I)]. \end{aligned} \quad (3)$$

Using the orthogonality property of R_0 and \hat{R}_0

$$R_0^T R_0 = I = R_0 R_0^T, \quad \hat{R}_0^T \hat{R}_0 = I = \hat{R}_0 \hat{R}_0^T,$$

the following properties of the trace operator

$$\text{tr}[AB] = \text{tr}[BA], \quad \text{tr}[P] = \text{tr}[P^T],$$

and the symmetric nature of K , it follows that (3) is of the form

$$= \frac{1}{2} \text{tr}[(K^{-1} - K^{-1} R_0 \hat{R}_0^T)],$$

Hence, the value function has an affine structure of the form

$$\begin{aligned} V_0(R_0) &= c_0 + P_0(R_0), \quad \text{where } c_0 = (1/2)\text{tr}(K^{-1}), \\ P_0(R) &:= -(1/2)\text{tr}(\hat{R}_0^T K^{-1} R). \end{aligned} \quad (4)$$

The latter equation (4) is obtained using the circular property of the trace operator. Note that terminal cost function (4) is affine in R_0 ¹.

¹More precisely, the cost function is affine if R_0 is considered to be the element of $SO(3)$ after embedding in a vector space. Note also that the penalty function $\phi_M(R_0 \hat{R}_0^T)$ is equal to the alternative penalty function $\phi_M(R_0, \hat{R}_0) := \text{tr}[(R_0 - \hat{R}_0)^T M (R_0 - \hat{R}_0)]$, where R_0 and \hat{R}_0 are treated as elements of the vector space.

In order to solve for the value function we apply the dynamic programming method [21] from optimal control theory. For the cost function (2) the dynamic programming principle takes the form

$$V_s(R) = \inf_{z \in \mathcal{Z}[s, s+\tau]} \left\{ \int_s^{s+\tau} \frac{1}{2} \text{tr}(z^T z) dt + \int_s^{s+\tau} \frac{1}{4} \phi_{L^{-1}}(R^{-1} Y) dt + V_{s+\tau}(R(s+\tau; z, A)) \right\},$$

where for $t_1, t_2 \in \mathbb{R}^+$ s.t. $t_1 < t_2$, $\mathcal{Z}[t_1, t_2]$ denotes the restriction of the control signal space to the time horizon $[t_1, t_2]$.

We first indicate the effect of one step of propagation of the value function (2), using the dynamic programming equation (5), on its affine structure. In order to solve for the value function numerically we make the following assumption. We use a discretized and bounded version of the disturbance set \mathcal{Z} . For simplicity of notation we reuse the notation to denote the bounded discrete set. Note that the boundedness assumption is not overly restrictive for reasons that will be explained in Sec. III-A.

Assuming a discretization time Δt the application of the dynamic programming principle to the value function yields

$$\begin{aligned} V_1(R) &= \inf_{z \in \mathcal{Z}} \left\{ \frac{1}{2} \text{tr}(z^T z) \Delta t + \frac{1}{4} \phi_{L^{-1}}(R^{-1} Y) \Delta t + V_0(R(0; z, A, Y)) \right\}, \\ &= \inf_{z \in \mathcal{Z}} \left\{ \frac{1}{2} \text{tr}(z^T z) \Delta t + \frac{1}{4} \phi_{L^{-1}}(R^{-1} Y) \Delta t + \frac{1}{4} \phi_{K^{-1}}(\psi(R, z, A) \hat{R}_0^T) \right\}, \\ &= \inf_{z \in \mathcal{Z}} \left\{ \frac{1}{2} \text{tr}(z^T z) \Delta t + \frac{1}{2} \text{tr} \left[L^{-1} - L^{-1} R^{-1} Y \right] \Delta t + \frac{1}{2} \text{tr} \left[K^{-1} - \hat{R}_0^T K^{-1} R \Psi(A_1, z) \right] \right\}. \end{aligned} \quad (5)$$

If z belongs to a discretization of the control set $\mathfrak{so}(3)$, then the above expression is affine in $R \in SO(3)$ for each element z in the algebra. It takes the form

$$V_1(R) = \inf_{z \in \mathcal{Z}} \left\{ c_z + P_z(R) \right\} = \bigoplus_{z \in \mathcal{Z}} c_z \otimes P_z(R),$$

where

$$\begin{aligned} c_z &:= \frac{1}{2} (z^T z) \Delta t + \frac{1}{2} \text{tr}(L^{-1}) \Delta t + \frac{1}{2} \text{tr}(K^{-1}), \\ P_z(R) &:= -\frac{1}{2} \text{tr}(L^{-1 T} Y^T R \Delta t + \hat{R}_0^T K^{-1} R \Psi(A_1, z)), \\ &= -\frac{1}{2} \text{tr} \left([L^{-1 T} Y^T + \Psi(A_1, z) \hat{R}_0^T K^{-1}] R \right). \end{aligned}$$

Thus the min-plus affine structure of the value function is preserved after one step (from the initial time step). This motivates the following more general proof of the invariance of this affine property. under propagation by the dynamic programming operator.

III. THE MIN-PLUS EXPANSION AND THE PROPAGATION OF THE COST FUNCTION

We first indicate the structure preservation and then describe how this approach to solving for the optimal cost function can be used to obtain the state estimate subsequent to propagation of the solution for the duration of the filter time horizon.

A. Propagation of the form of the cost function

Let the form of the max-plus basis expansion of the value function at time step k be

$$V_k(R) := \bigoplus_{\lambda_k \in \Lambda_k} [c_{\lambda_k} \otimes P_{\lambda_k}(R)], \quad (6)$$

where $P_{\lambda_k}(R) = -\frac{1}{2} \text{tr}(M_k R)$. Now, from the dynamic programming principle

$$\begin{aligned} V_{k+1}(R) &:= \bigoplus_{z \in \mathcal{Z}} \left(\frac{1}{2} \text{tr}(z^T z) \Delta t + \frac{1}{2} \text{tr} \left[L^{-1} - L^{-1} R^{-1} Y \right] \Delta t + V_k(R \Psi(A_{k+1}, z)) \right), \\ &= \bigoplus_{\lambda_{k+1} \in \Lambda_{k+1}} [c_{\lambda_{k+1}} + P_{\lambda_{k+1}}(R)]. \end{aligned}$$

where, by comparing coefficients, it is seen that

$$c_{\lambda_{k+1}} = c_{\lambda_k} + \frac{1}{2} \text{tr}(z^T z) \Delta t + \frac{1}{2} \text{tr}(L^{-1}) \Delta t,$$

$$P_{\lambda_{k+1}}(R) = -\frac{1}{2} \text{tr}(M_{k+1} R),$$

where

$$M_{k+1} = L^{-1 T} Y^T \Delta t + \Psi(A_{k+1}, z) M_k.$$

Note that for a bounded continuous value function on a compact set, the optimal value of the disturbance signal

z is bounded. This is the reason for the aforementioned non-restrictiveness of the boundedness assumption. The basis function parameterization set evolves via the recursive relationship

$$\Lambda_{k+1} = \Lambda_k \times \mathcal{Z}.$$

This expression provides the mechanism via which the basis function sequences $\lambda_{(\cdot)}$ and their corresponding components $c_{(\cdot)}$ and $P_{(\cdot)}$ are generated at each time step in response to the new measurements being gathered therein.

B. Generating the state estimate

Having obtained the value function in terms of a min-plus basis expansion (6) we have the following expression for the optimal state estimate. Given the observation time horizon \mathcal{N} (the window over which we use the observations to generate a state estimate), the optimal state estimate is defined as follows

$$V_{\mathcal{N}}(R) := \bigoplus_{\lambda \in \Lambda_{\mathcal{N}}} [c_{\lambda} \otimes P_{\lambda}(\cdot)](R),$$

where $\Lambda_{\mathcal{N}}$ is the collection of affine functions generated at the time step \mathcal{N} via the recursive propagation as described in the previous section. The minimum value of the optimal cost (value) function is achieved at the optimal state estimate. Hence the estimated state is obtained from the expression

$$\hat{R}^* = \operatorname{argmin}_{R \in SO(3)} \bigoplus_{\lambda \in \Lambda_{\mathcal{N}}} [c_{\lambda} \otimes P_{\lambda}(R)]. \quad (7)$$

The above problem (7) can be solved faster, by breaking it down into a set of independent parallel tasks, each of which solves a problem of the form

$$\hat{R}_{\lambda}^* := \operatorname{argmin}_{R \in SO(3)} [c_{\lambda} \otimes P_{\lambda}(R)], \quad \lambda \in \Lambda_{\mathcal{N}}.$$

As each of the basis functions in the max-plus basis expansion give rise to a particular affine form (max-plus basis function), savings in computational burden as well as storage requirements are gained if the number of such affine forms are reduced where possible. Thus we can formulate a measure of the importance of each basis function – which takes the form of an optimization problem. As described in [22], albeit with appropriate changes for the case of $SO(3)$, this computation of the contribution of each basis can be relaxed to take the form

of a semi-definite programming problem (i.e., a convex optimization problem) [23] – a problem with efficient numerical solvers. We can then use the information about this metric in order to prune or keep various functions.

Further details of these steps and the associated error analysis will be deferred to future publications under progress. A similar approach for the case of the special unitary group/ manifold may be found in [22]

IV. EXAMPLE

In this section we describe an application of the technique developed in the previous sections to a specific set of examples of a tracking problem on $SO(3)$. This application will serve to indicate the performance of the proposed approach in the case of the bilinear model (1) introduced in Sec. I.

We use the following notation for the directions used to generate the disturbance set \mathcal{Z} . They lie along the positive and negative directions of the skew-symmetric basis elements of $\mathfrak{so}(3)$.

$$\begin{aligned} H_1 &:= \begin{pmatrix} 0 & +1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ H_3 &:= \begin{pmatrix} 0 & 0 & +1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \\ H_5 &:= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & +1 \\ 0 & -1 & 0 \end{pmatrix}, \\ H_2 &:= -H_1, \quad H_4 := -H_3, \quad H_6 := -H_5. \end{aligned}$$

The system dynamics considered in this section conform to the general form (1) and we apply min-plus approach for specific cases of the model in order to better understand the performance of this approach. The first three cases below start with a zero initial estimation error (however starting from the first time step an error arises which is the reason for the nonzero starting value of the error in the figures).

- 1) **Zero drift with collinear measurement and process disturbance:** In this case the dynamics is $\dot{R} = Rz$ where z is generated as a normal distribution along the direction H_1 . In addition the

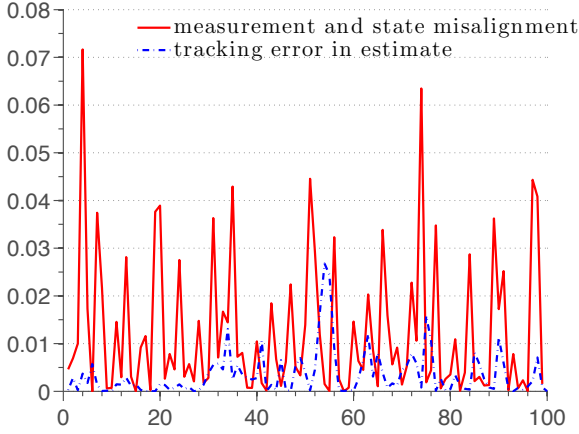


Fig. 1: The case of zero drift with the state disturbance and measurement noise along H_1, H_2 respectively.

measurement noise are along the same direction H_1 (ref Fig.1).

- 2) **Non-zero drift with collinear measurement and process disturbance:** In this case the dynamics is $\dot{R} = R(H_1 + z)$ and the state disturbance and measurement errors are normally distributed along H_1 (c.f. Fig. 2).

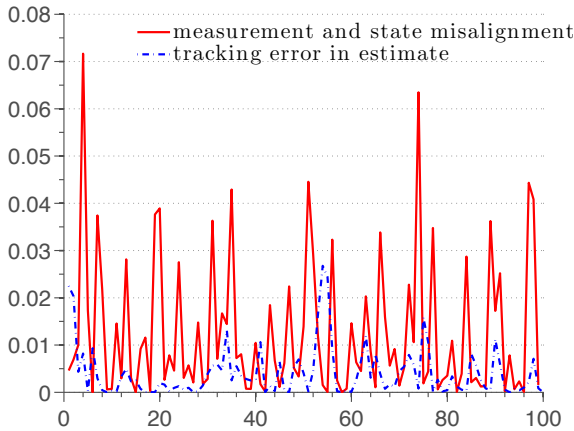


Fig. 2: The case of drift along H_1 with the state disturbance along H_1 and measurement noise along H_2 .

- 3) **Non-zero drift with initial estimation error and orthogonal measurement and process distur-**

bances: In this case, the drift direction is H_1 , there is an initial error in the state estimate; the measurement noise ϵ and disturbance z are along the H_1, H_2 and H_3, H_4 directions respectively (c.f. Fig. ??).

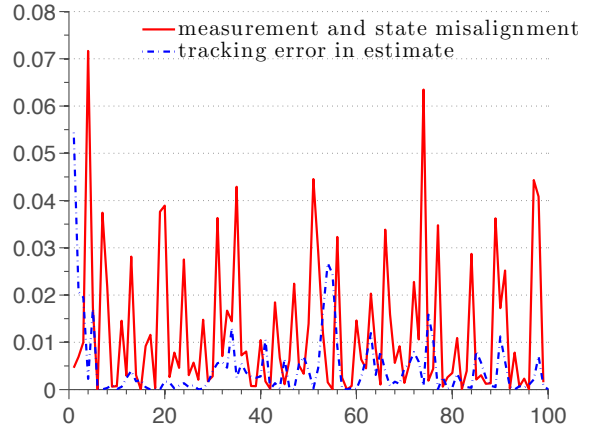


Fig. 3: The case of drift along H_1 with an initial error in the state estimate; the dynamic and measurement noise are along H_3, H_4 and H_1, H_2 respectively.

Note that in the figures, we analyze the performance of the filter by comparing the measurement noise and the error in the estimates. The metric which we use to indicate the level of noise in the measurements is

$$\text{measurement noise}(t) := \text{tr} \left[I - Y^T(t)R(t) \right],$$

at a time t ; the measure of the agreement between the estimate and the true state i.e., the error in the estimate is given by the tracking error in the estimate (TE)

$$\text{TE}(t) := \text{tr} \left[I - \hat{R}^T(t)R(t) \right].$$

In the cases above, we apply the min-plus reduced dimensionality techniques with fixed values of the weighting matrices (K, L) and with a fixed length of the time horizon over which the optimization is carried out (width of the sliding window).

Note that the parameters in the studies above were not selected in any optimal manner. They serve to be indicative of the levels of performance achievable with minimal effort and hence demonstrate the potential nominal performance achievable from these classes of

filters with little effort at tuning them. This also serves to guide some of the directions which such work could proceed along as described in the next section.

V. CONCLUSIONS AND FUTURE DIRECTIONS

In this article we describe the min-plus approaches for deterministic filtering on the $SO(3)$ group. The approach was then applied to example problems for a specific class of dynamics and noise processes. The results obtained reveal the need to further study these class of techniques. Some of the paths that this work can be directed along are: the study of the sensitivity of various filter weight matrices on the performance of the filter; the error analysis for a specific value of these parameters; an analysis of the length of the filtering window and weight matrices on the performance of the filter; analysis of the filter robustness to unknown noise/disturbance and unmodeled dynamics; the consideration of min-plus based deterministic filter design for alternative system models [12] for attitude tracking problems. Note that in the current article we assume that the number of terms in the basis expansion generated for the given filter time horizon is capable of being stored in memory. However in the case that this growth in the number of terms if unfeasible, there exist pruning methods [19] which can be considered in order to manage this growth (e.g. when the time horizon of the filter or the cardinality of the disturbance process set is larger than the available memory constraints). Thus this article provides a potential core for several diverse lines of investigation into the area of computationally efficient approaches to deterministic filtering for nonlinear systems evolving on manifolds.

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