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Young Duality and Schmidt-pair for Linear Systems*

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***Abstract*— This paper studies the concepts of reachability, observability, controllability and constructibility focusing on the principle of duality with time reversal. This work takes into account linear time-varying systems in continuous time. A discussion is proposed about the Legendre transforms of energy functions and the significance of Schmidt-pairs.**

I. INTRODUCTION

The notion of duality is ubiquitous in control theory and is very helpful in the definition of some quantities such as reachability, observability, controllability and constructibility, for instance [6], [5], [7], [8]. Despite its common use, the duality principle often appears in an ad hoc way (mostly used by transposing matrices), without being presented as a general powerful tool. This sometimes leads to ambiguities about what should be called dual (or adjoint). For example, considering the following state-space representation (A, B, C, D) of a system Σ, the state-space of the so-called adjoint system $\Sigma^\star$ is $(A^\mathsf{T}, C^\mathsf{T}, B^\mathsf{T}, D^\mathsf{T})$ or $(-A^\mathsf{T}, -C^\mathsf{T}, B^\mathsf{T}, D^\mathsf{T})$ depending on the authors. A fundamental treatment of the notion of duality was given by Fuhrmann [4] in the context of polynomial models using the first expression of $\Sigma^\star$. Ten years later, van der Schaft [11] gave an intrinsic definition of the adjoint system and a characterization in terms of the system's external behaviour using the second expression of $\Sigma^\star$. In this paper, the latter form will be considered. As underlined by Antoulas [1, p.76]: "*One may think of the dual system $\Sigma^\star$ as the system Σ but with the role of the inputs and outputs interchanged, or with the flow of causality reversed and time running backward*".

After having restated known results, focusing on time reversal and the reachability, observability, controllability and constructibility operators and their adjoints (defined with respect to the standard inner product), the aim of this paper is to show that energy functions are also adjoint but in the sense of Young. Indeed, these functions are Legendre transforms of one another. Moreover, it will be established that it exists specific energy functions which are equal to their adjoint considering Schmidt-pair inputs.

The paper is organized as follows. Operators and their adjoints are reviewed in Section II and, in Section III, the adjoint of a continuous-time linear time-varying system is naturally derived. Then, in Section IV, the notion of Young duality of reachability and observability energy function of a system and of its adjoint is presented. At last, in Section , for time-invariant systems, energetic relation and an expression of Schmidt-pairs is proposed.

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Notations

Let Σ be a continuous-time linear time-varying (CT-LTV) system with the following possible state-space realization $\mathbb{S}$:

$$\begin{cases} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t) \end{cases}$$

with $\begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix} \in \mathbb{R}^{(n+p)\times(n+m)}$ and $u \in L_2^m(t_0, t_1)$, $x \in L_2^n(t_0, t_1)$ and $y \in L_2^p(t_0, t_1)$.

Three different objects have to be distinguished:

- $\mathbb{S} = \left(\begin{array}{c|c} A(t) & B(t) \\ \hline C(t) & D(t) \end{array}\right)$ the state-space itself.
- $\Sigma \colon \mathbb{S}$, the system Σ for which $\mathbb{S}$ is one possible state-space representation.
- In the time-invariant case, the associated transfer function is $C(sI - A)^{-1}B + D = G(s) \stackrel{s}{=} \mathbb{S}$.

Stability and minimality of the state-space are assumed throughout the paper. The state solution is given by $x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)\mathrm{d}\tau$ where $\Phi(t, \tau)$ is the state transition matrix which verifies: $\frac{\partial}{\partial t}\Phi(t, \tau) = A(t)\Phi(t, \tau)$, $\frac{\partial}{\partial \tau}\Phi(t, \tau) = -\Phi(t, \tau)A(\tau)$ and $\Phi^{-1}(t_1, t_2) = \Phi(t_2, t_1)$. The state transition matrix can be defined by $\Phi(t, \tau) = X(t)X^{-1}(\tau)$ where $X(t)$ is a fundamental matrix (i.e. a regular matrix solution of $\dot{X}(t) = A(t)X(t)$).

II. OPERATORS AND THEIR ADJOINTS

This Section recalls the reachability, observability, controllability and constructibility operators and their adjoints defined with respect to the standard inner product $\langle \cdot , \cdot \rangle$ (since L_2 spaces are Hilbert spaces). If $\Psi \colon H_1 \to H_2$ is an application between two Hilbert spaces, then the adjoint operator $\Psi^\star$ is the application $H_2 \to H_1$ such that $\langle x_1 , \Psi^\star(x_2) \rangle_{H_1} = \langle \Psi(x_1) , x_2 \rangle_{H_2}$, $\forall x_i \in H_i$ and $i \in \{1, 2\}$.

Definition 1: The reachability operator is defined by

$$\left| \begin{array}{rcl} {}^{t_1}_{t_0}\Psi_{\mathcal{R}} \colon L_2^m(t_0, t_1) & \to & \mathbb{R}^n \\ u & \mapsto & \displaystyle\int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)\mathrm{d}\tau \end{array} \right. \quad (1)$$

Remark 1: The state at time t_1 can be written: $x(t_1) = \Phi(t_1, t_0)x(t_0) + ({}^{t_1}_{t_0}\Psi_{\mathcal{R}} u(t))$.

Proposition 1: The adjoint reachability operator associated with Σ is the linear operator

$$\left| \begin{array}{rcl} {}_{t_0}^{t_1}\Psi_{\mathcal{R}}^{\star} : \mathbb{R}^n & \to & L_2^m(t_0, t_1) \\ a & \mapsto & B^{\mathsf{T}}(t)\Phi^{\mathsf{T}}(t_1, t)a \end{array} \right. \tag{2}$$

Proof: $\forall a \in \mathbb{R}^n$ and $\forall u \in L_2^m(t_0, t_1)$:

$$\begin{aligned} \left\langle a\,,\, {}_{t_0}^{t_1}\Psi_{\mathcal{R}} u \right\rangle_{\mathbb{R}^n} &= \int_{t_0}^{t_1} a^{\mathsf{T}}\Phi(t_1, \tau)B(\tau)u(\tau)\mathrm{d}\tau \\ &= \int_{t_0}^{t_1} u^{\mathsf{T}}(\tau)\left[B^{\mathsf{T}}(\tau)\Phi^{\mathsf{T}}(t_1, \tau)a\right]\mathrm{d}\tau \\ &= \left\langle u\,,\, {}_{t_0}^{t_1}\Psi_{\mathcal{R}}^{\star} a\right\rangle_{L_2^m(t_0,t_1)} \end{aligned}$$

■

Definition 2: The observability operator is

$$\left| \begin{array}{rcl} {}_{t_0}^{t_1}\Psi_{\mathcal{O}} : \mathbb{R}^n & \to & L_2^p(t_0, t_1) \\ a & \mapsto & C(t)\Phi(t, t_0)a \end{array} \right. \tag{3}$$

Remark 2: The output at time t_1 of the autonomous system can be written $y(t_1) = {}_{t_0}^{t_1}\Psi_{\mathcal{O}} x(t_0)$.

Proposition 2: The adjoint observability operator is

$$\left| \begin{array}{rcl} {}_{t_0}^{t_1}\Psi_{\mathcal{O}}^{\star} : L_2^p(t_0, t_1) & \to & \mathbb{R}^n \\ u & \mapsto & \displaystyle\int_{t_0}^{t_1} \Phi^{\mathsf{T}}(\tau, t_0)C^{\mathsf{T}}(\tau)u(\tau)\mathrm{d}\tau \end{array} \right. \tag{4}$$

Proof: The proof follows the same scheme as previously. ■

Since $x(t_1) = \Phi(t_1, t_0)x(t_0) + {}_{t_0}^{t_1}\Psi_{\mathcal{R}} u$ and $\Phi(t_0, t_1) = \Phi^{-1}(t_1, t_0)$ the following quantity is obtained: $x(t_0) = \Phi(t_0, t_1)x(t_1) - \Phi(t_0, t_1){}_{t_0}^{t_1}\Psi_{\mathcal{R}} u$ and noting that

$$\begin{aligned} -\Phi(t_0, t_1){}_{t_0}^{t_1}\Psi_{\mathcal{R}} u &= -\Phi(t_0, t_1)\int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)\mathrm{d}\tau \\ &= \int_{t_1}^{t_0} \Phi(t_0, \tau)B(\tau)u(\tau)\mathrm{d}\tau \\ &= {}_{t_1}^{t_0}\Psi_{\mathcal{R}} u \end{aligned}$$

the following definition can be proposed.

Definition 3: The controllability operator is

$$\left| \begin{array}{rcl} {}_{t_0}^{t_1}\Psi_{\mathcal{C}} : L_2^m(t_0, t_1) & \to & \mathbb{R}^n \\ u & \mapsto & {}_{t_0}^{t_1}\Psi_{\mathcal{C}}(u) = {}_{t_1}^{t_0}\Psi_{\mathcal{R}}(u) \\ & & = \int_{t_1}^{t_0} \Phi(t_0, \tau)B(\tau)u(\tau)\mathrm{d}\tau \end{array} \right. \tag{5}$$

Proposition 3: The adjoint controllability operator is

$$\left| \begin{array}{rcl} {}_{t_0}^{t_1}\Psi_{\mathcal{C}}^{\star} : \mathbb{R}^n & \to & L_2^m(t_0, t_1) \\ a & \mapsto & {}_{t_0}^{t_1}\Psi_{\mathcal{C}}^{\star}(a) = -{}_{t_1}^{t_0}\Psi_{\mathcal{R}}^{\star}(a) = -B^{\mathsf{T}}(t)\Phi^{\mathsf{T}}(t_0, t)a \end{array} \right. \tag{6}$$

Proof: The proof follows the same scheme as previously. ■

Definition 4: The constructibility operator is

$$\left| \begin{array}{rcl} {}_{t_0}^{t_1}\Psi_{\mathcal{N}} : \mathbb{R}^n & \to & L_2^p(t_0, t_1) \\ a & \mapsto & {}_{t_0}^{t_1}\Psi_{\mathcal{N}}(a) = -{}_{t_1}^{t_0}\Psi_{\mathcal{O}}(a) = -C(t)\Phi(t, t_1)a \end{array} \right. \tag{7}$$

Proposition 4: The adjoint constructibility operator is

$$\left| \begin{array}{rcl} {}_{t_0}^{t_1}\Psi_{\mathcal{N}}^{\star} : L_2^p(t_0, t_1) & \to & \mathbb{R}^n \\ u & \mapsto & {}_{t_0}^{t_1}\Psi_{\mathcal{N}}^{\star}(u) = {}_{t_1}^{t_0}\Psi_{\mathcal{O}}^{\star}(u) \\ & & = \displaystyle\int_{t_1}^{t_0} \Phi^{\mathsf{T}}(\tau, t_1)C^{\mathsf{T}}(\tau)u(\tau)\mathrm{d}\tau \end{array} \right. \tag{8}$$

Proof: The proof follows the same scheme as previously. ■

The definitions of the eight operators are summarized in TABLE I. It can be noted that the bounds of two integrals have been switched by adding minus signs in order to exhibit a strong symmetry between the different quantities. This symmetry will be used in the next Section to carry out substitutions.

A last operator which maps the past inputs of the system to the future outputs is introduced as follow.

Definition 5: The Hankel operator is defined by

$$\left| \begin{array}{rcl} \Gamma : L_2^m(-\infty, 0) & \to & L_2^p(0, \infty) \\ u & \mapsto & {}_{0}^{\infty}\Psi_{\mathcal{O}} \circ {}_{-\infty}^{0}\Psi_{\mathcal{R}} u \\ & & = \displaystyle\int_{-\infty}^{0} C(t)\Phi(t, \tau)B(\tau)u(\tau)\mathrm{d}\tau \end{array} \right. \tag{9}$$

Proposition 5: The adjoint Hankel operator is

$$\left| \begin{array}{rcl} \Gamma^{\star} : L_2^p(0, \infty) & \to & L_2^m(-\infty, 0) \\ y & \mapsto & {}_{-\infty}^{0}\Psi_{\mathcal{R}}^{\star} \circ {}_{0}^{\infty}\Psi_{\mathcal{O}}^{\star} y \\ & & = -\displaystyle\int_{\infty}^{0} B^{\mathsf{T}}(t)\Phi^{\mathsf{T}}(\tau, t)C^{\mathsf{T}}(\tau)y(\tau)\mathrm{d}\tau \end{array} \right. \tag{10}$$

Proof: The proof follows the same scheme as previously. ■

III. ADJOINT SYSTEM AND GRAMIANS

A. Adjoint system

In this Section, the definition of adjoint system is derived from a term by term identification step. The reader may refer to [11] for a more formal approach.

Proposition 6: If $X(t)$ is a regular solution of $\dot{X}(t) = A(t)X(t)$ then $Z(t) = X^{-\mathsf{T}}(t)$ is a regular solution of $\dot{Z}(t) = -A^{\mathsf{T}}(t)Z(t)$.

Proof: Using the fact that $X(t)X^{-1}(t) = I$, it can be written $\frac{\mathrm{d}}{\mathrm{d}t}\left(X(t)X^{-1}(t)\right) = \frac{\mathrm{d}X(t)}{\mathrm{d}t}X^{-1}(t) + X(t)\frac{\mathrm{d}}{\mathrm{d}t}X^{-1}(t)$, so, the time derivative of the inverse matrix is $\frac{\mathrm{d}}{\mathrm{d}t}X^{-1}(t) = -X^{-1}(t)\frac{\mathrm{d}X(t)}{\mathrm{d}t}X^{-1}(t) = -X^{-1}(t)A(t)$. ■

From a fundamental matrix X, considering the definition of the state transition matrix $\Phi(t, \tau) = X(t)X(\tau)$, one has $\Phi^{-\mathsf{T}}(t, \tau) = X^{-\mathsf{T}}(t)X^{\mathsf{T}}(\tau) = Z(t)Z(\tau)$.

Using the results of the previous Section, the quantity ${}_{t_0}^{t_1}\Psi_{\mathcal{O}}^{\star}$ (resp. ${}_{t_0}^{t_1}\Psi_{\mathcal{N}}^{\star}$) can be obtained from the quantity ${}_{t_1}^{t_0}\Psi_{\mathcal{R}}$ (resp. ${}_{t_1}^{t_0}\Psi_{\mathcal{C}}$) through the following substitution $\diamond$:

$$\begin{array}{rcll} \diamond : A & \longmapsto & -A^{\mathsf{T}} & (\Leftrightarrow \Phi \longmapsto \Phi^{-\mathsf{T}} \triangleq (\Phi^{-1})^{\mathsf{T}}) \\ B & \longmapsto & -C^{\mathsf{T}} & \\ C & \longmapsto & B^{\mathsf{T}} & \\ D & \longmapsto & D^{\mathsf{T}} & \\ t_0 & \longmapsto & t_1 & \\ t_1 & \longmapsto & t_0 & \end{array}$$

This set of substitutions naturally leads to the following definition.

TABLE I

Summary of operators and their adjoints

	Operator	Adjoint operator
Reachability	$u \overset{\Psi_{\mathcal{R}}}{\mapsto} \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)u(\tau)\mathrm{d}\tau$	$a \overset{\Psi_{\mathcal{R}}^\star}{\mapsto} B^\mathsf{T}(t)\Phi^\mathsf{T}(t_1,t)a$
Observability	$a \overset{\Psi_{\mathcal{O}}}{\mapsto} C(t)\Phi(t,t_0)a$	$u \overset{\Psi_{\mathcal{O}}^\star}{\mapsto} -\int_{t_1}^{t_0} \Phi^\mathsf{T}(\tau,t_0)C^\mathsf{T}(\tau)u(\tau)\mathrm{d}\tau$
Controllability	$u \overset{\Psi_{\mathcal{C}}}{\mapsto} \int_{t_1}^{t_0} \Phi(t_0,\tau)B(\tau)u(\tau)\mathrm{d}\tau$	$a \overset{\Psi_{\mathcal{C}}^\star}{\mapsto} -B^\mathsf{T}(t)\Phi^\mathsf{T}(t_0,t)a$
Constructibility	$a \overset{\Psi_{\mathcal{N}}}{\mapsto} -C(t)\Phi(t,t_1)a$	$u \overset{\Psi_{\mathcal{N}}^\star}{\mapsto} -\int_{t_0}^{t_1} \Phi^\mathsf{T}(\tau,t_1)C^\mathsf{T}(\tau)u(\tau)\mathrm{d}\tau$

Definition 6: The adjoint system of the system Σ is

$$\Sigma^\star : \mathbb{S}^\diamond = \left(\begin{array}{c|c} -A^\mathsf{T}(t) & -C^\mathsf{T}(t) \\ \hline B^\mathsf{T}(t) & D^\mathsf{T}(t) \end{array}\right)$$

Remark 3: Some authors give this denotation to the following system $\left(\begin{array}{c|c} A^\mathsf{T}(t) & C^\mathsf{T}(t) \\ \hline B^\mathsf{T}(t) & D^\mathsf{T}(t) \end{array}\right)$ and call the other one conjugate or paraconjugate denoted $\Sigma^\sim$ (see, for instance, [12, p. 67]).

Remark 4: Four substitutions are needed to obtain identity: $\diamond^4 = \mathrm{Id}$. $\left(\begin{array}{c|c} A(t) & B(t) \\ \hline C(t) & D(t) \end{array}\right) \overset{\diamond}{\mapsto} \left(\begin{array}{c|c} -A^\mathsf{T}(t) & -C^\mathsf{T}(t) \\ \hline B^\mathsf{T}(t) & D^\mathsf{T}(t) \end{array}\right) \overset{\diamond}{\mapsto} \left(\begin{array}{c|c} A(t) & -B(t) \\ \hline -C(t) & D(t) \end{array}\right) \overset{\diamond}{\mapsto} \left(\begin{array}{c|c} -A^\mathsf{T}(t) & C^\mathsf{T}(t) \\ \hline -B^\mathsf{T}(t) & D^\mathsf{T}(t) \end{array}\right)$. It can be underlined that $\mathbb{S}\colon \Sigma = \Sigma^{\star\star}\colon \mathbb{S}^{\diamond\diamond} = \diamond^2(\mathbb{S})$. Indeed, Σ and $\Sigma^{\star\star}$ have the same Gramians and the same Hankel operator. Moreover, in the time-invariant case, they have the same transfer function and the same convolution operator.

Therefore, it can be written

$$\boxed{\begin{array}{c} {}_{t_0}^{t_1}\Psi_{\mathcal{O}}^\star\{\Sigma\} = {}_{t_1}^{t_0}\Psi_{\mathcal{R}}\{\Sigma^\star\}\ ;\ {}_{t_0}^{t_1}\Psi_{\mathcal{R}}^\star\{\Sigma\} = {}_{t_1}^{t_0}\Psi_{\mathcal{O}}\{\Sigma^\star\} \\ \\ {}_{t_0}^{t_1}\Psi_{\mathcal{N}}^\star\{\Sigma\} = {}_{t_1}^{t_0}\Psi_{\mathcal{C}}\{\Sigma^\star\}\ ;\ {}_{t_0}^{t_1}\Psi_{\mathcal{C}}^\star\{\Sigma\} = {}_{t_1}^{t_0}\Psi_{\mathcal{N}}\{\Sigma^\star\} \end{array}} \tag{11}$$

Proposition 7 ([11]): Let a system with, respectively, x, u and y the state, the input and the output and η, μ and γ the respective adjoint state, input and output. They satisfy the following instantaneous condition:

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta^\mathsf{T}(t)x(t) = \gamma^\mathsf{T}(t)u(t) - \mu^\mathsf{T}(t)y(t) \tag{12}$$

Proof: The proof is straightforward, considering the system and its adjoint. ■

Remark 5 ([1, p.132]): [1, p.76] If the system is autonomous then $\frac{\mathrm{d}}{\mathrm{d}t}\eta^\mathsf{T}(t)x(t) = 0$ which means that $\eta^\mathsf{T}(t)x(t) = \eta^\mathsf{T}(t_0)x(t_0)\ \forall t \geq t_0$.

B. Gramians

Definition 7: The finite reachability Gramian is

$$\begin{aligned} W_{\mathcal{R}}(t_0,t) &\triangleq {}_{t_0}^{t}\Psi_{\mathcal{R}} \circ {}_{t_0}^{t}\Psi_{\mathcal{R}}^\star \\ &= \int_{t_0}^{t} \Phi(t,\tau)B(\tau)B^\mathsf{T}(\tau)\Phi^\mathsf{T}(t,\tau)\mathrm{d}\tau \end{aligned} \tag{13}$$

Remark 6: It can be written $\left\|{}_{t_0}^{t_1}\Psi_{\mathcal{R}}^\star x(t_0)\right\|^2_{L_2^m(t_0,t_1)} = \left\langle x(t_0)\,,\, {}_{t_0}^{t_1}\Psi_{\mathcal{R}}^{\star\star} \circ {}_{t_0}^{t_1}\Psi_{\mathcal{R}}^\star x(t_0)\right\rangle_{\mathbb{R}^n} = x^\mathsf{T}W_{\mathcal{R}}(t_0,t_1)x$ where ${}_{t_0}^{t_1}\Psi_{\mathcal{R}}^{\star\star} = {}_{t_0}^{t_1}\Psi_{\mathcal{R}}$.

Definition 8: The finite observability Gramian

$$\begin{aligned} W_{\mathcal{O}}(t,t_1) &\triangleq {}_{t}^{t_1}\Psi_{\mathcal{O}}^\star \circ {}_{t}^{t_1}\Psi_{\mathcal{O}} \\ &= \int_{t}^{t_1} \Phi^\mathsf{T}(\tau,t)C^\mathsf{T}(\tau)C(\tau)\Phi(\tau,t)\mathrm{d}\tau \end{aligned} \tag{14}$$

Remark 7: It can be written $\left\|{}_{t_0}^{t_1}\Psi_{\mathcal{O}} x(t_0)\right\|^2_{L_2^p(t_0,t_1)} = \left\langle x(t_0)\,,\, {}_{t_0}^{t_1}\Psi_{\mathcal{O}}^\star \circ {}_{t_0}^{t_1}\Psi_{\mathcal{O}} x(t_0)\right\rangle_{\mathbb{R}^n} = x^\mathsf{T}W_{\mathcal{O}}(t_0,t_1)x$

Definition 9: The finite controllability Gramian

$$\begin{aligned} W_{\mathcal{C}}(t,t_1) &\triangleq {}_{t}^{t_1}\Psi_{\mathcal{C}} \circ {}_{t}^{t_1}\Psi_{\mathcal{C}}^\star \\ &= \int_{t}^{t_1} \Phi(t,\tau)B(\tau)B^\mathsf{T}(\tau)\Phi^\mathsf{T}(t,\tau)\mathrm{d}\tau \end{aligned} \tag{15}$$

Definition 10: The finite constructibility Gramian

$$\begin{aligned} W_{\mathcal{N}}(t_0,t) &\triangleq {}_{t_0}^{t}\Psi_{\mathcal{N}}^\star \circ {}_{t_0}^{t}\Psi_{\mathcal{N}} \\ &= \int_{t_0}^{t} \Phi^\mathsf{T}(\tau,t)C^\mathsf{T}(\tau)C(\tau)\Phi(\tau,t)\mathrm{d}\tau \end{aligned} \tag{16}$$

The different Gramians are linked as follows

$$W_{\mathcal{O}}(t_0,t_1) = \Phi^\mathsf{T}(t_1,t_0)W_{\mathcal{N}}(t_0,t_1)\Phi(t_1,t_0)$$

and

$$W_{\mathcal{R}}(t_0,t_1) = \Phi(t_1,t_0)W_{\mathcal{C}}(t_0,t_1)\Phi^\mathsf{T}(t_1,t_0).$$

As the state transition matrix Φ is always regular in the continuous time case, the matrices $W_{\mathcal{R}}(t_0,t_1)$ and $W_{\mathcal{C}}(t_0,t_1)$ (resp. $W_{\mathcal{O}}(t_0,t_1)$ and $W_{\mathcal{N}}(t_0,t_1)$) are congruent.

Consider the following change of basis $x = T\tilde{x}$ (where the superscript "$\sim$" denotes the quantity in the new basis). TABLE II summarizes the change of basis of operators, their adjoints and the "contragredient transformation" [9] of the Gramians.

IV. Energy functions and Young duality

This Section presents the input and output energy functions and establishes a duality relation (in the sense of Young) between them.

A. Optimal input energy

Definition 11: The optimal input energy J_{opt} is the solution of the following optimisation problem:

$$J_{\mathrm{opt}} = \min_{\substack{u \in L_2^m(t_0,t_1) \\ x(t_0)=x_0 \\ x(t_1)=x_1}} J(u) = \frac{1}{2}\|u\|_{L_2^m(t_0,t_1)} = \frac{1}{2}\int_{t_0}^{t_1} u^\mathsf{T}(\tau)u(\tau)\mathrm{d}\tau$$

subject to $\dot{x}(t) = Ax(t) + Bu(t)$.

TABLE II

Change of basis: operator, adjoint and Gramian

	Operator	Adjoint operator	Gramian
Reachability	$\tilde{\Psi}_{\mathcal{R}} = T^{-1}\Psi_{\mathcal{R}}$	$\tilde{\Psi}^{\star}_{\mathcal{R}} = \Psi^{\star}_{\mathcal{R}} T^{-\mathsf{T}}$	$\tilde{W}_{\mathcal{R}} = T^{-1} W_{\mathcal{R}} T^{-\mathsf{T}}$
Observability	$\tilde{\Psi}_{\mathcal{O}} = \Psi_{\mathcal{O}} T$	$\tilde{\Psi}^{\star}_{\mathcal{O}} = T^{\mathsf{T}} \Psi^{\star}_{\mathcal{O}}$	$\tilde{W}_{\mathcal{O}} = T^{\mathsf{T}} W_{\mathcal{O}} T$
Controllability	$\tilde{\Psi}_{\mathcal{C}} = T^{-1}\Psi_{\mathcal{C}}$	$\tilde{\Psi}^{\star}_{\mathcal{C}} = \Psi^{\star}_{\mathcal{C}} T^{-\mathsf{T}}$	$\tilde{W}_{\mathcal{C}} = T^{-1} W_{\mathcal{C}} T^{-\mathsf{T}}$
Constructibility	$\tilde{\Psi}_{\mathcal{N}} = \Psi_{\mathcal{N}} T$	$\tilde{\Psi}^{\star}_{\mathcal{N}} = T^{\mathsf{T}} \Psi^{\star}_{\mathcal{N}}$	$\tilde{W}_{\mathcal{N}} = T^{\mathsf{T}} W_{\mathcal{N}} T$

Proposition 8: The optimal input energy can be expressed with the reachability gramian or the controlability gramian as follow

$$\begin{aligned} J_{\text{opt}} &= \tfrac{1}{2}(x_1 - \Phi(t_1,t_0)x_0)^{\mathsf{T}} W_{\mathcal{R}}^{-1}(t_0,t_1)(x_1 - \Phi(t_1,t_0)x_0) \\ &= \tfrac{1}{2}(x_0 - \Phi(t_0,t_1)x_1)^{\mathsf{T}} W_{\mathcal{C}}^{-1}(t_0,t_1)(x_0 - \Phi(t_0,t_1)x_1) \end{aligned}$$

Proof: The solution in open loop is to minimize $\|u\|_{L_2^m(t_0,t_1)}$ subject to ${}_{t_0}^{t_1}\Psi_{\mathcal{R}}[u(t)] = x(t_1) - \Phi(t_1,t_0)x(t_0)$. Using the pseudo-inverse leads to the result. ■

Two remarkable cases can be highlighted:

- Starting from an initial state x_0 and steering the system to the origin $x_1 = 0$ leads to:

$$\boxed{J_{\text{opt}} = \frac{1}{2} x_0^{\mathsf{T}} W_{\mathcal{C}}^{-1}(t_0,t_1) x_0}$$

- Starting from the origin $x_0 = 0$ and steering the system to the state x_1 leads to

$$\boxed{J_{\text{opt}} = \frac{1}{2} x_1^{\mathsf{T}} W_{\mathcal{R}}^{-1}(t_0,t_1) x_1}$$

B. Output energy

When the system is autonomous ($u(t) = 0 \ \forall t \in [t_0, t_1]$), the output energy is given by:

$$\begin{aligned} J(y) &= \tfrac{1}{2}\int_{t_0}^{t_1} y^{\mathsf{T}}(\tau)y(\tau)\mathrm{d}\tau \\ &= \tfrac{1}{2}x_0^{\mathsf{T}} W_{\mathcal{O}}(t_0,t_1)x_0 \end{aligned}$$

Moreover, $x_0 = \Phi(t_0,t_1)x_1$, thus

$$\begin{aligned} J(y) &= \tfrac{1}{2}x_1^{\mathsf{T}}\Phi^{\mathsf{T}}(t_0,t_1)W_{\mathcal{O}}(t_0,t_1)\Phi(t_0,t_1)x_1 \\ &= \tfrac{1}{2}x_1^{\mathsf{T}} W_{\mathcal{N}}(t_0,t_1)x_1 \end{aligned}$$

C. Energy functions and Young duality

From the optimal input energy and the output energy, one can define energy functions (depending of the state or the co-state) as follow.

Definition 12: The reachability energy function of system Σ in time $[t_0;t_1]$ is defined by

$$L_{\mathcal{R}}(x) \triangleq \min_{\substack{u \in L_2^m(t_0,t_1) \\ x(t_0)=0 \\ x(t_1)=x}} \frac{1}{2}\int_{t_0}^{t_1} u^{\mathsf{T}}(\tau)u(\tau)\mathrm{d}\tau = \frac{1}{2}x^{\mathsf{T}} W_{\mathcal{R}}^{-1}(t_0,t_1)x$$

Definition 13: The observability energy function of the system Σ in time $[t_0;t_1]$ is defined by

$$L_{\mathcal{O}}(x) \triangleq \frac{1}{2}\int_{t_0}^{t_1} y^{\mathsf{T}}(\tau)y(\tau)\mathrm{d}\tau = \frac{1}{2}x^{\mathsf{T}} W_{\mathcal{O}}(t_0,t_1)x$$

Noting that the two previous functions can be written as follows $L_{\mathcal{R}}(x) = \frac{1}{2}\left\langle x \,,\, \left({}_{t_0}^{t_1}\Psi_{\mathcal{R}} \circ {}_{t_0}^{t_1}\Psi^{\star}_{\mathcal{R}}\right)^{-1} x\right\rangle_{\mathbb{R}^n}$ and $L_{\mathcal{O}}(x) = \frac{1}{2}\left\langle x \,,\, {}_{t_0}^{t_1}\Psi^{\star}_{\mathcal{O}} \circ {}_{t_0}^{t_1}\Psi_{\mathcal{O}} x\right\rangle_{\mathbb{R}^n}$. By substituting the operator according to equations (11), the following definitions can be written.

Definition 14: The reachability energy function of the adjoint system $\Sigma^{\star}$ in time $[t_0;t_1]$ is defined by

$$L^{\dagger}_{\mathcal{R}}(\eta) \triangleq \min_{\substack{\mu \in L_2^p(t_0,t_1) \\ \eta(t_1)=0 \\ \eta(t_0)=\eta}} \frac{1}{2}\int_{t_0}^{t_1} \mu^{\mathsf{T}}(\tau)\mu(\tau)\mathrm{d}\tau = \frac{1}{2}\eta^{\mathsf{T}} W_{\mathcal{O}}^{-1}(t_0,t_1)\eta$$

Definition 15: The observability energy function of the adjoint system $\Sigma^{\star}$ in time $[t_0;t_1]$ is defined by

$$L^{\dagger}_{\mathcal{O}}(\eta) \triangleq \frac{1}{2}\int_{t_0}^{t_1} \gamma^{\mathsf{T}}(\tau)\gamma(\tau)\mathrm{d}\tau = \frac{1}{2}\eta^{\mathsf{T}} W_{\mathcal{R}}(t_0,t_1)\eta$$

These functions are linked together through the Legendre transform.

Definition 16: The Legendre transform $E^{\dagger}$ of a convex function E is

$$E^{\dagger}(\eta) = \max_x \eta^{\mathsf{T}}x - E(x)$$

Definition 17 ([3, p.64]): Two functions which are Legendre transforms of one another are called *dual in the sense of Young*.

Proposition 9: $L^{\dagger}_{\mathcal{O}}(\eta)$ (resp. $L^{\dagger}_{\mathcal{R}}(\eta)$) is the dual function (in the sense of Young) of $L_{\mathcal{R}}(x)$ (resp. $L_{\mathcal{O}}(x)$).

Proof: From the definition of the Legendre transform, consider $L^{\dagger}_{\mathcal{O}}(\eta) = \max_x \eta^{\mathsf{T}}x - L_{\mathcal{R}}(x)$. The maximum is obtained for $\eta = \nabla_x L_{\mathcal{R}}(x) = W_{\mathcal{R}}^{-1}(t_0,t_1)x$. Substituting x leads to $L^{\dagger}_{\mathcal{O}}(\eta) = \eta^{\mathsf{T}}x - \frac{1}{2}\eta^{\mathsf{T}} W_{\mathcal{R}}(t_0,t_1) W_{\mathcal{R}}^{-1}(t_0,t_1) W_{\mathcal{R}}(t_0,t_1)\eta = \frac{1}{2}\eta^{\mathsf{T}} W_{\mathcal{R}}(t_0,t_1)\eta$. ■

V. Schmidt-pair properties

In this Section, some results are derived for the specific case where the matrices A, B, C and D are time-invariant.

A. Adjoint System and Gramian

Let Σ be the following CT-LTI

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases}$$

Remark 8: Let $G_{\Sigma}(s) = C(sI - A)^{-1}B + D$ be the transfer function of system Σ; the transfer function of the adjoint system then is:

$$G_{\Sigma^{\star}}(s) \triangleq G_{\Sigma}^{\sim}(s) = G_{\Sigma}^{\mathsf{T}}(-s) = G_{\Sigma}^{*}(-\bar{s}).$$

In other words, the paraconjugate of the system's transfer function is the transfer function of the adjoint system.

The state transition matrix is given by $\Phi(t,\tau) = \exp(A(t-\tau))$ and the impulse response is $g(t) = C\exp(At)B$.

The (asymptotic) reachability Gramian $\mathcal{P}$, observability Gramian $\mathcal{Q}$, controllability Gramian $\mathcal{C}$ and constructibility Gramian $\mathcal{N}$ are respectively given by:

$$\mathcal{P} = W_{\mathcal{R}}(-\infty,t) = \int_0^{\infty} \exp(A\tau)BB^{\mathsf{T}}\exp(A^{\mathsf{T}}\tau)\mathrm{d}\tau,$$

$$\mathcal{Q} = W_{\mathcal{O}}(t,\infty) = \int_0^{\infty} \exp(A^{\mathsf{T}}\tau)C^{\mathsf{T}}C\exp(A\tau)\mathrm{d}\tau,$$

$$\mathcal{C} = W_{\mathcal{C}}(t,-\infty) = -\int_0^{\infty} \exp(A\tau)BB^{\mathsf{T}}\exp(A^{\mathsf{T}}\tau)\mathrm{d}\tau$$

and

$$\mathcal{N} = W_{\mathcal{N}}(\infty,t) = -\int_0^{\infty} \exp(A^{\mathsf{T}}\tau)C^{\mathsf{T}}C\exp(A\tau)\mathrm{d}\tau.$$

They are solutions of the following respective Lyapunov equations:

$$\begin{array}{ll} A\mathcal{P}+\mathcal{P}A^{\mathsf{T}}+BB^{\mathsf{T}}=0, & A^{\mathsf{T}}\mathcal{Q}+\mathcal{Q}A+C^{\mathsf{T}}C=0,\\ A\mathcal{C}+\mathcal{C}A^{\mathsf{T}}-BB^{\mathsf{T}}=0, & A^{\mathsf{T}}\mathcal{N}+\mathcal{N}A-C^{\mathsf{T}}C=0.\end{array}$$

B. Hankel singular values and Schmidt pairs

Definition 18: Considering Γ the Hankel operator of a minimal representation, a vector pair (u_i,μ_i) verifying

$$\mu_i \triangleq \frac{1}{\sigma_i}\Gamma u_i$$

is called σ_i-Schmidt pair. The Hankel singular values σ_i are strictly positive since the state-space representation is assumed to be minimal.

Remark 9: The σ_i-Schmidt pair (u_i,v_i), satisfying by definition $\Gamma u_i = \sigma_i\mu_i$, also satisfies $\Gamma^{\star}\mu_i = \sigma_i u_i$.

Proposition 10: Let (u_i,μ_i) be a σ_i-Schmidt-pair, then the state x_i and the co-state η_i verify:

$$\left\{\begin{array}{lcl} \sigma_i^2 x_i &=& W_{\mathcal{R}}(-\infty,0)W_{\mathcal{O}}(0,\infty)x_i,\\ \sigma_i^2\eta_i &=& W_{\mathcal{O}}(0,\infty)W_{\mathcal{R}}(-\infty,0)\eta_i.\end{array}\right.$$

Proof: The proof is straightforward by expressing the state and the co-state according to each other:

$$x_i = {}_{-\infty}^{\;\;0}\Psi_{\mathcal{R}}u_i = \frac{1}{\sigma_i}\,{}_{-\infty}^{\;\;0}\Psi_{\mathcal{R}}\circ{}_{-\infty}^{\;\;0}\Psi_{\mathcal{R}}^{\star}\eta_i = \frac{1}{\sigma_i}W_{\mathcal{R}}(-\infty,0)\eta_i$$

and

$$\eta_i = {}_{0}^{\infty}\Psi_{\mathcal{O}}^{\star}\mu_i = \frac{1}{\sigma_i}\,{}_{0}^{\infty}\Psi_{\mathcal{O}}^{\star}\circ{}_{0}^{\infty}\Psi_{\mathcal{O}}x_i = \frac{1}{\sigma_i}W_{\mathcal{O}}(0,\infty)x_i$$

■

C. Energy of a Schmidt-pair

Proposition 11: Let (u_i,μ_i) be a σ_i-Schmidt-pair and x and η the respective associated state and co-state.

- For negative time, the co-energy $E_i^{\dagger}(\eta(t)) = \frac{1}{2}\eta^{\mathsf{T}}(t)\frac{\mathcal{P}}{\sigma_i}\eta(t)$ is the Legendre transform of the energy $E_i(x(t)) = \frac{1}{2}x^{\mathsf{T}}(t)\left(\frac{\mathcal{P}}{\sigma_i}\right)^{-1}x(t)$ and their values are equal.
- For positive time, the co-energy $\mathcal{E}_i^{\dagger}(\eta(t)) = \frac{1}{2}\eta^{\mathsf{T}}(t)\left(\frac{\mathcal{Q}}{\sigma_i}\right)^{-1}\eta(t)$ is the Legendre transform of the energy $\mathcal{E}_i(x(t)) = \frac{1}{2}x^{\mathsf{T}}(t)\frac{\mathcal{Q}}{\sigma_i}x(t)$ and their values are equal.

Proof: The first part of the proof is the same as the proof of proposition 9: according to the definition of the Legendre transform, consider $E_i^{\dagger}(\eta(t)) = \max_{x(t)}\eta(t)^{\mathsf{T}}x(t) - E_i(x(t))$. The maximum is obtained for $\eta(t) = \nabla_{x(t)}E_i(x(t)) = \left(\frac{\mathcal{P}}{\sigma_i}\right)^{-1}x(t)$, thus, $E_i^{\dagger}(\eta(t)) = \frac{1}{2}\eta^{\mathsf{T}}(t)\frac{\mathcal{P}}{\sigma_i}\eta(t)$. The equality of E and $E^{\dagger}$ can be proved using proposition 10. Indeed, according to this proposition the state and co-state verify $x_i = \frac{\mathcal{P}}{\sigma_i}\eta_i$, which is the condition to obtain the maximum in the Legendre transformation. ■

D. Expression of Schmidt-pair

Proposition 12: Let (u_i,μ_i) be a σ_i-Schmidt-pair and x_i be an eigenvector associated to the eigenvalue σ_i^2 of the matrix $\mathcal{P}\mathcal{Q}$ then

$$\begin{array}{lcll} u_i(t) &=& \alpha_i B^{\mathsf{T}}e^{-A^{\mathsf{T}}t}\mathcal{Q}x_i & t\leq 0\\ \mu_i(t) &=& \alpha_i\sigma_i Ce^{At}x_i & t\geq 0\end{array}$$

with $\alpha_i \in \mathbb{R}^*$ a scaling factor.

Proposition 13: The eigenvectors x_i of the matrix $\mathcal{P}\mathcal{Q}$ are orthogonal according to the matrix $\mathcal{P}^{-1}$ and $\mathcal{Q}$:

$$\forall i\neq j \quad x_i^{\mathsf{T}}\mathcal{P}^{-1}x_j = 0 \quad\text{and}\quad x_i^{\mathsf{T}}\mathcal{Q}x_j = 0$$

Proof: Assume $\sigma_i \neq \sigma_j$, by defintion

$$\begin{array}{lcl} \mathcal{Q}x_i - \sigma_i^2\mathcal{P}^{-1}x_i &=& 0\\ \mathcal{Q}x_j - \sigma_j^2\mathcal{P}^{-1}x_j &=& 0\end{array}$$

Premultipling the first equation by x_j^{T} and second by x_i^{T} and performing the difference leads to $(\sigma_j^2-\sigma_i^2)x_j^{\mathsf{T}}\mathcal{P}^{-1}x_i = 0$. Thus $x_j^{\mathsf{T}}\mathcal{P}^{-1}x_i = 0$ and substituting it in the first equations leads to $x_j^{\mathsf{T}}\mathcal{Q}x_i = 0$.

If $\sigma_i = \sigma_j$, the eigenspace is orthogonal to the other ones. It is always possible to find an orthogonal basis of this subspace. Thus, after this small tuning, the condition $\sigma_i \neq \sigma_j$ can be replaced by $i \neq j$. ■

Moreover, it is possible to scale the eigenvectors in order to obtain the following relationships:

$$\left\{\begin{array}{lcl} x_i^{\mathsf{T}}\mathcal{P}^{-1}x_j &=& \delta_{ij}\\ x_i^{\mathsf{T}}\mathcal{Q}x_j &=& \sigma_i^2\delta_{ij}\end{array}\right. \tag{17}$$

Then, setting the scale factor α_i of the proposition 12 to $\alpha_i = \frac{1}{\sigma_i^2}$, the Schmidt-pair can be expressed as

$$\boxed{\begin{array}{rcll} u_i(t) & = & B^{\mathsf{T}} e^{-A^{\mathsf{T}} t} \mathcal{P}^{-1} x_i, & t \leq 0 \\ \mu_i(t) & = & C e^{At} \mathcal{Q}^{-1} \eta_i, & t \geq 0 \end{array}} \tag{18}$$

and the families $\{u_i\}_{1 \leq i \leq n}$ and $\{\mu_i\}_{1 \leq i \leq n}$ are orthonormal:

$$\begin{array}{rcl} \langle u_i \,,\, u_j \rangle_{L_2^m(-\infty,0)} & = & \delta_{ij} \\ \langle \mu_i \,,\, \mu_j \rangle_{L_2^p(0,\infty)} & = & \delta_{ij}. \end{array}$$

The state and co-state's expressions are summarized in the following table:

	$t \leq 0$	$t \geq 0$
$x(t)$	$\mathcal{P} e^{-A^{\mathsf{T}} t} \mathcal{P}^{-1} x_i$	$e^{At} x_i = \sigma_i e^{At} \mathcal{Q}^{-1} \eta_i$
$\eta(t)$	$e^{-A^{\mathsf{T}} t} \eta_i = \sigma_i e^{-A^{\mathsf{T}} t} \mathcal{P}^{-1} x_i$	$\mathcal{Q} e^{At} \mathcal{Q}^{-1} \eta_i$

and they are linked as follow

$t \leq 0$	$t \geq 0$
$x(t) = \frac{1}{\sigma_i} \mathcal{P} \eta(t)$	$\eta(t) = \frac{1}{\sigma_i} \mathcal{Q} x(t)$

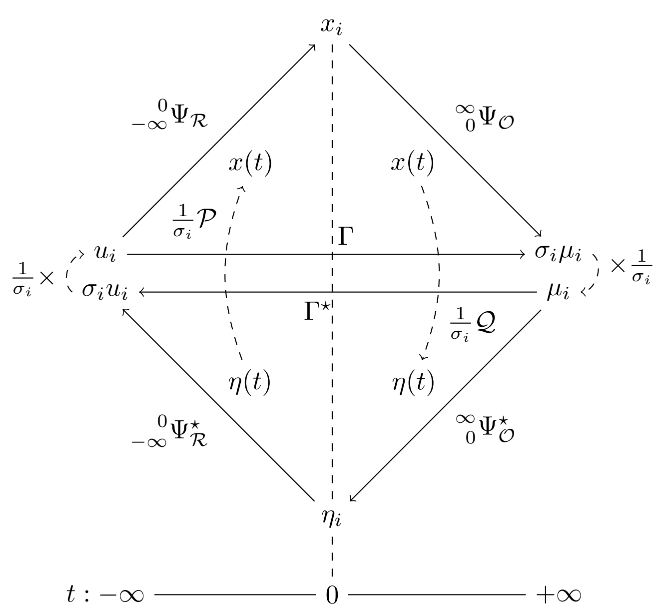

Fig. 1. *σ_i-Schmidt-pair's representation with the associated state and co-state*

Proposition 14: Let (u_i, μ_i) be a σ_i-Schmidt-pair and x_i and η_i the respective associated state and co-state. The following relations hold

$\forall t \leq 0$	$\forall t \geq 0$
$\frac{\mathrm{d}}{\mathrm{d}t}[x^{\mathsf{T}}(t) \mathcal{P}^{-1} x(t)] = u_i^{\mathsf{T}}(t) u_i(t)$	$\frac{\mathrm{d}}{\mathrm{d}t}[x^{\mathsf{T}}(t) \mathcal{Q} x(t)] = -\sigma_i^2 \mu_i^{\mathsf{T}}(t) \mu_i(t)$
$\frac{\mathrm{d}}{\mathrm{d}t}[\eta^{\mathsf{T}}(t) \mathcal{P} \eta(t)] = \sigma_i^2 u_i^{\mathsf{T}}(t) u_i(t)$	$\frac{\mathrm{d}}{\mathrm{d}t}[\eta^{\mathsf{T}}(t) \mathcal{Q}^{-1} \eta(t)] = -\mu_i^{\mathsf{T}}(t) \mu_i(t)$

Proof: The result can be obtain by a straightforward computation. The use of proposition 10 leads to

$$x(t) = \frac{1}{\sigma_i} \mathcal{P} \eta(t) \quad \text{and} \quad \eta(t) = \frac{1}{\sigma_i} \mathcal{Q} x(t)$$

thus

$$\eta^{\mathsf{T}}(t) x(t) = \sigma_i x^{\mathsf{T}}(t) \mathcal{P}^{-1} x(t) = \frac{1}{\sigma_i} x^{\mathsf{T}}(t) \mathcal{Q} x(t)$$

From equation (12),

$$\frac{\mathrm{d}}{\mathrm{d}t} \eta^{\mathsf{T}}(t) x(t) = \sigma_i [u_i^{\mathsf{T}}(t) u_i(t) - \mu_i^{\mathsf{T}}(t) \mu_i(t)]$$

and the proof is obtained. ■

FIGURE 1 summarizes the relations between the reachability, observability and Hankel operators with the infinite reachability and observability Gramians considering the σ_i-Schmidt-pair.

VI. CONCLUSION

This paper has reviewed the notion of duality for linear time-varying systems. Three different notions of duality are involved, namely the duality "$\star$" defined with respect to the standard inner product of Hilbert spaces for the operators, the paraconjugate "$\sim$" for the transfer matrices and the duality in the sens of Young "†" for the energy functions. To summarize, the reachability, observability, controllability and constructibility operators of a system and its adjoint are linked with respect to the standard inner product. on the other hand, the reachability, observability, controllability and constructibility energy functions are linked with respect to the Legendre transform. The following table summarizes the different quantities involved and their dualities.

System	Transfer function	State-space	Operator	Energy function
Σ	$G(s)$	$\mathbb{S}, \mathbb{S}^{\diamond\diamond}$	Ψ	L
$\Sigma^\star$	$G^\sim(s)$	$\mathbb{S}^\circ, \mathbb{S}^{\diamond\diamond\diamond\diamond}$	$\Psi^\star$	$L^\dagger$

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