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Every Continuous Piecewise Affine Function Can Be Obtained by Solving a Parametric Linear Program

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***Abstract*— It is well-known that solutions to parametric linear or quadratic programs are continuous piecewise affine functions of the parameter. In this paper we prove the converse, i.e. that every continuous piecewise affine function can be identified with the solution to a parametric linear program. In particular, we provide a constructive proof that every piecewise affine function can be expressed as the linear mapping of the solution to a parametric linear program with at most twice as many variables as the dimension of the image of the piecewise affine function. Our method is illustrated via two small numerical examples.**

I. Introduction

Parametric linear programs (PLPs) are a common class of optimization problems that have been studied for some time [1], [2], with significant applications in optimization-based control [3]. A PLP can be written as

$$f(p) \quad \in \quad \arg\min_{z} \quad c^\top z \tag{1a}$$

$$\text{s.t.} \quad Fz \leq h + Gp, \tag{1b}$$

where $z \in \mathbb{R}^n$ are referred to as the *decision variables* and $p \in \mathbb{R}^m$ as the *parameters*, with all other problem data of compatible dimensions. It is well-known that there always exists a continuous PWA solution function $f : \Omega \to \mathbb{R}^n$ on a polyhedral partition for the PLP (1), even in those cases where the set of minimizers for (1) is not a singleton for every p [4]. The value function of (1) can also be shown to be convex and PWA over the same partition [1].

Any PWA solution function f can be written as

$$f(p) = f_i(p) := L_i p + l_i, \quad p \in \Omega_i,$$

where we refer to $f_i : \Omega_i \to \mathbb{R}^n$ as the *local function* on the convex polyhedral region Ω_i. The regions Ω_i form a partition of the convex polyhedral set $\Omega := \mathrm{dom}(f)$, i.e. $\mathrm{int}(\Omega_i) \cap \mathrm{int}(\Omega_j) = \emptyset$ for all $i \neq j$ and $\bigcup_{i=1}^{n_r} \Omega_i = \Omega$, where n_r is the total number of regions and $\Omega \subseteq \mathbb{R}^m$ is the set of all parameters for which the PLP (1) admits a solution. A method for generating such a PWA solution function is described in [5].

The inverse parametric linear programming problem addresses the following converse question: given a continuous PWA function $f : \mathbb{R}^m \to \mathbb{R}^n$, can one construct constraints and an objective function for a PLP in the form (1) that admits f as a parametric solution? In general the answer is negative, and we provide a counterexample in Section III-A. However, our main result is that it is *always* possible to construct a function $\hat{f} : \mathbb{R}^m \to \mathbb{R}^{\hat{n}}$ that *is* the solution to a PLP in the form (1) with $\hat{n} \leq 2n$, and for which $f(p) = T\hat{f}(p)$ for some matrix T for all $p \in \Omega$.

Previous research related to inverse optimization has either not considered the particular case of PWA functions, or has not furnished a constructive proof of the generality we claim. Solutions to nonparametric, polynomial inverse optimization problems have been described in [6]. The special case of finding a PLP for a given continuous PWA function was addressed in [7], in which the inverse PLP problem was posed as a bilevel programming problem. Finally, [8] showed how to construct a (possibly linear) objective function for a convex optimization problem such that a given function approximates its parametric solution at a finite number of points. All of these previous results require extensive knowledge of both the desired optimizer as well as the generating optimization problem. In particular, the constraints of the optimization problem are generally assumed to be known, which our method does not require.

Inverse optimization ideas have been used in a variety of applications in engineering and applied mathematics. In economics, inverse optimization has been used to identify hidden utility functions of market players from their behavior in the marketplace [9]. In [10], an inverse optimization approach is used to analyze human motion patterns with regards to the underlying optimization problem solved. The optimization problem identified is used to imitate human locomotion autonomously on a humanoid robot. Similar ideas are used in [11] to learn a task by observing an expert where the observed process is modeled as a Markov decision process.

This type of problem has also received considerable attention in control engineering over the last 50 years, e.g. in [12]–[16], often with respect to finding appropriate weight matrices such that a given control gain is LQR optimal. Furthermore, it was established in [17] that every continuous nonlinear control system can be identified with the solution to some parametric convex optimization problem. Although a generating parametric convex program (PCP) always exists for a given control policy, constructing such a problem is not straightforward in general.

In [18] the present authors made three distinct contributions to the particular problem of inverse parametric quadratic programming (PQP) that are also partly applicable to the inverse

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PLP problem. First, it was shown that a PQP with a given PWA solution can be constructed without explicit knowledge of the constraint set. Second, it was shown that the objective function for an inverse optimal PQP can be constructed by solving a set of linear matrix inequalities (LMIs). Taken together, these results allow one to compute a solution to an inverse PQP problem provided that a solution exists. Lastly, it was shown in [18] that inverse parametric programming provides a novel method for remodeling dynamical systems with PWA dynamics [19]. Numerical simulations indicate the potential for significantly shorter computation times when such an inverse PQP (instead of the original PWA) model is used for control based on real-time optimization.

The applicability of the method outlined in [18] was limited, insofar as not every continuous PWA function is the solution to a PQP problem. In this paper, we will address this shortcoming by presenting a way of constructing a PLP from any given continuous PWA function f, such that f can be obtained as the image of the explicit solution to the PLP. These results specialize the claim from [17] (every continuous nonlinear function is the solution to a parametric convex program) to the PWA case and give a straightforward procedure to construct such a generating PLP.

Notation: We use superscripts in square brackets to indicate elements of vectors and vector valued functions, e.g. $f^{[i]}$ is the i^{th} element of f. $\mathbf{1}$ denotes a vector of ones of appropriate dimension. $\mathbb{N}$ is the set of natural, $\mathbb{R}$ the set of real, $\mathbb{R}_+$ the set of positive real, and $\mathbb{R}_-$ the set of negative real numbers. The interior of a set $\Omega \subseteq \mathbb{R}^n$ is denoted $\mathrm{int}(\Omega)$ and the cardinality of a set $\mathcal{I} \subseteq \mathbb{N}$ by $|\mathcal{I}|$. All vector-valued inequalities are to be understood component-wise.

II. Extended Inverse Linear Programming

Since not every continuous PWA function is the solution to a PLP, we consider the following extended problem: given an arbitrary continuous PWA function f, can we find a PLP such that f can be obtained as the image of its solution under a linear operator? Formally, we define the extended inverse parametric linear programming problem as follows:

Definition 1 (Extended inverse PLP problem): Given a continuous PWA function $f\colon \Omega \to \mathbb{R}^n$, find some $\hat{n} \in \mathbb{N}$, a linear cost function $J\colon \mathbb{R}^{\hat{n}} \times \Omega \to \mathbb{R}$ as in (1a), a polyhedral constraint set $\Gamma \subseteq \mathbb{R}^{\hat{n}} \times \mathbb{R}^m$ as in (1b), and a matrix $T \in R^{n\times\hat{n}}$ such that

$$f(p) = T\hat{f}(p) \tag{2}$$

where

$$\begin{aligned}\hat{f}(p) \quad \in \quad \arg\min_{z} \quad & J(z,p) \\ \text{s.t.} \quad & (z,p) \in \Gamma.\end{aligned} \tag{3}$$

The desired function f is the image of the explicit solution function $\hat{f}$ to (3) under the linear operator T. We call $\hat{f}$ the *extended solution*.

We now state the main result of the paper:

Theorem 1 (Extended inverse PLP solution): Every continuous PWA function $f\colon \mathbb{R}^m \to \mathbb{R}^n$ can be obtained as the image under a linear operator T of the unique explicit solution $\hat{f}\colon \mathbb{R}^m \to \mathbb{R}^{\hat{n}}$ to a PLP in the form (1) of dimension $\hat{n}$ not higher than $2n$.

We provide a proof of this result in Section II-C, but present first a number of necessary technical lemmas.

A. Inverse PLP for Scalar-Valued Convex Functions

We consider initially a continuous scalar-valued PWA function $f\colon \Omega \to \mathbb{R}$. Our first basic result relates to the case when f additionally is convex:

Lemma 2 (Inverse PLP for convex PWA functions): Suppose that $f\colon \Omega \to \mathbb{R}$ is continuous, PWA and convex. Then there exists problem data (G,g) such that

$$\begin{aligned}f(p) \quad \in \quad \arg\min_{z} \quad & cz \\ \text{s.t.} \quad & Gp + g \leq z\mathbf{1}\end{aligned}$$

for any $c > 0$, where the solution set for the PLP is a singleton for every p.

Proof: Since f is a PWA convex function, it can be written as the pointwise supremum of a finite collection of N linear functions

$$f(p) = \sup_{i=1\ldots N} \left(G_i^\top p + g_i\right)$$

for some $N \in \mathbb{N}$. The result follows immediately by defining $G := [G_1,\ldots,G_N]^\top$ and $g := (g_1;\ldots,g_N)$, since the scalar c is assumed positive and the PLP has been recast to epigraph form. ∎

An analogous result holds in the case of a *concave* PWA function:

Corollary 3 (Inverse PLP for concave PWA functions): Suppose that $f\colon \Omega \to \mathbb{R}$ is continuous, PWA and concave. Then there exists problem data (H,h) such that

$$\begin{aligned}f(p) \quad \in \quad \arg\min_{z} \quad & cz \\ \text{s.t.} \quad & Hp + h \geq z\mathbf{1}\end{aligned}$$

for any $c < 0$, where the solution set for the PLP is a singleton for every p.

The proof is the same is in the preceding Lemma, but with f expressed as a pointwise infimum of linear functions so that the PLP is in hypograph form.

The two preceding results solve the inverse PLP problem for convex and concave PWA functions, respectively, without adding decision variables to the optimization problem, i.e. for the linear operator in (2) we get $T = 1$. The choice of PLP is not unique, e.g. one can vary the signed parameter c or add redundant inequalities to the PLP without varying its parametric solution. However, observe that the solution to the PLP we construct in Lemma 2 and Corollary 3 *is* unique, since it is a minimization (maximization) problem in epigraph (hypograph) form.

B. Inverse PLP for Scalar-Valued Non-Convex Functions

We next consider the case of a general, i.e. neither convex nor concave (but still continuous), PWA function. To handle this case we make use of ideas from so-called DC programming [20].

Lemma 4 (Convex decomposition of a PWA function): Every continuous PWA function $f\colon \Omega \to \mathbb{R}$ defined over a convex polyhedral partition of $\Omega \subseteq \mathbb{R}^m$ with full-dimensional elements Ω_k can be written as the difference of two convex PWA functions γ and η, i.e. $f(p) = \gamma(p) - \eta(p)$.

Proof: A proof can be found in [21]. We present here a considerably simplified proof that exploits a geometric result from [22]. Let

$$\mathcal{I} := \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid \dim(\Omega_i \cap \Omega_j) = n-1, i < j\} \quad (4)$$

be the set if tuples (i,j) indicating that the regions Ω_i and Ω_j are neighbors, i.e. they intersect along one hyperplane. For all $(i,j) \in \mathcal{I}$ define $\sigma_{i,j}\colon \Omega \to \mathbb{R}$ as

$$\sigma_{i,j}(p) := \max\{f_i(p), f_j(p)\},$$

over the whole domain $\Omega := \bigcup_{k=1}^{n_r} \Omega_k$. Observe that $\sigma_{i,j}(p)$ is the pointwise maximum of two affine function and therefore always convex. Define $\gamma_{i,j}\colon \Omega \to \mathbb{R}$ as

$$\gamma_{i,j}(p) := \begin{cases} \sigma_{i,j}(p) & \text{if } \sigma_{i,j}(p) = f(p) \text{ for all } p \in \Omega_i \cup \Omega_j \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $\gamma_{i,j}$ is a continuation of f on $\Omega_i \cup \Omega_j$ over the whole domain Ω if f is convex across the common boundary of Ω_i and Ω_j. If f is concave across this boundary then $\gamma_{i,j}$ is set to zero. Define

$$\gamma(p) := \sum_{(i,j)\in\mathcal{I}} \gamma_{i,j}(p).$$

As a sum of continuous convex PWA functions, γ is itself continuous, convex, and PWA. Define $\eta(p) := \gamma(p) - f(p)$, which is itself a continuous PWA function, and we must show that η is also convex. For this to be the case it is sufficient by Proposition 2.1 in [22] that η is convex along all line segments entirely contained within two neighboring regions, i.e. on any restriction to neighboring partition elements.

Observe that η may not naturally be defined on the same partition as f, but that the partition of η can always be further subdivided if necessary such that it is at least as fine as that of f. In other words, η can be defined on a partition $\bar{\Omega} = \bigcup_{i=1}^{n_{\bar{r}}} \bar{\Omega}_i$ such that

$$\forall k \quad \exists!\, i \in \{1,\dots,n_r\}\colon \bar{\Omega}_k \subseteq \Omega_i,$$

i.e. no region in the partition of η crosses the boundary of two regions defining the partition of f.

Analogously to (4), define

$$\bar{\mathcal{I}} := \{(k,l) \in \mathbb{N} \times \mathbb{N} \mid \dim(\bar{\Omega}_k \cap \bar{\Omega}_l) = n-1, k < l\},$$

as the set of tuples (k,l) indicating neighboring regions for the partition defining η. Now take any $(k,l) \in \bar{\mathcal{I}}$ and consider η over this pair of regions. Two cases must be considered. Either both $\bar{\Omega}_k$ and $\bar{\Omega}_l$ are contained in the same original region Ω_q, or they are contained in two different regions Ω_q and Ω_r:

Case 1: $(\bar{\Omega}_k \cup \bar{\Omega}_l) \subseteq \Omega_q$:

Observe that

$$\eta(p) = \sum_{(i,j)\in\mathcal{I}} \gamma_{i,j}(p) - f_q(p) \quad \forall p \in (\bar{\Omega}_k \cup \bar{\Omega}_l),$$

which is a sum of convex functions. Hence, η is convex over any two regions $\bar{\Omega}_k$ and $\bar{\Omega}_l$ contained in the same original region Ω_q.

Case 2: $\bar{\Omega}_k \subseteq \Omega_q, \bar{\Omega}_l \subseteq \Omega_r, (q,r) \in \mathcal{I}$:

Two subcases must be considered. If the restriction of f to $\Omega_q \cup \Omega_r$ is convex, then $\gamma_{q,r}(p) - f(p) = 0$ for all $p \in (\bar{\Omega}_k \cup \bar{\Omega}_l)$ and

$$\begin{aligned}\eta(p) &= \sum_{(i,j)\in\mathcal{I}} \gamma_{i,j}(p) - f(p) \\ &= \sum_{(i,j)\in\mathcal{I}\setminus(q,r)} \gamma_{i,j}(p), \quad \forall p \in (\bar{\Omega}_k \cup \bar{\Omega}_l),\end{aligned}$$

which is a sum of convex functions, hence convex. If f is concave on this restriction instead, then $-f$ is convex for $p \in (\bar{\Omega}_k \cup \bar{\Omega}_l)$ and

$$\eta(p) = \gamma(p) + (-f(p))$$

is again a sum of convex functions, hence still convex.

Since this reasoning holds for any $(k,l) \in \bar{\mathcal{I}}$, η is convex over any pair of neighboring regions and therefore convex over its whole domain. ■

Observe that the preceding proof is constructive, i.e. it constitutes an algorithm for decomposing any PWA function into a difference of convex PWA functions.

We are now in a position to demonstrate that every scalar-valued continuous PWA function, not necessarily convex, can be written as the image of the unique solution to a PLP under a linear operator.

Lemma 5 (Inverse PLP for continuous PWA functions): Suppose that $f\colon \Omega \to \mathbb{R}$ is continuous and PWA. Then there exists problem data (G,H,g,h) and a PWA function $\hat{f}\colon \Omega \to \mathbb{R}^2$ such that

$$f(p) = T\hat{f}(p) \quad \forall p \in \Omega$$

where

$$\begin{aligned}\hat{f}(p) \quad \in \quad \arg\min_{z} \quad & c_1 z^{[1]} + c_2 z^{[2]} \\ \text{s.t.} \quad & Gp + g \le z^{[1]}\mathbf{1} \\ & Hp + h \ge z^{[2]}\mathbf{1}\end{aligned} \quad (5)$$

for any $c_1 > 0$ and $c_2 < 0$, where $T = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and the solution set for the PLP (5) is a singleton for every p.

Proof: By Lemma 4 we can always obtain a decomposition $f(p) = \gamma(p) - \eta(p)$ with (γ, η) both continuous, PWA, and convex. Define the function $\hat{f}$ as

$$\hat{f}(p) = \begin{bmatrix} \gamma(p) \\ -\eta(p) \end{bmatrix}.$$

The result follows immediately by applying Lemma 2 and Corollary 3 (since $-\eta$ is concave) to form the separable PLP (5) and applying the summation operator T to the result. ■

We now have everything required to prove the main result of the paper:

C. Proof of Theorem 1

Recall that in general the function f can be vector-valued, and consider every component function $f^{[i]}\colon \Omega \to \mathbb{R}$ independently. Define the index sets

$$\begin{aligned} \mathcal{F}_+ &:= \{i \in \mathbb{N} \mid f^{[i]} \text{ convex}\}, \\ \mathcal{F}_- &:= \{i \in \mathbb{N} \mid f^{[i]} \text{ concave}\}, \\ \mathcal{F}_0 &:= \{i \in \mathbb{N} \mid f^{[i]} \text{ neither convex nor concave}\}. \end{aligned}$$

For every $i \in \mathcal{F}_+$, choose $c_i > 0$ and let $T_i = 1$. Analogously, for every $i \in \mathcal{F}_-$, choose $c_i < 0$ and again let $T_i = 1$.

Finally, for every $i \in \mathcal{F}_0$, find continuous convex PWA functions $\gamma_i, \eta_i\colon \Omega \to \mathbb{R}$ such that $f^{[i]}(p) = \gamma_i(p) - \eta_i(p)$, which is always possible by Lemma 4. Furthermore, choose $c_i \in \mathbb{R}_+ \times \mathbb{R}_-$ and let $T_i = \begin{bmatrix} 1 & 1 \end{bmatrix}$.

Define

$$c := \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, \quad z := \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}, \quad T := \begin{bmatrix} T_1 & & \\ & \ddots & \\ & & T_n \end{bmatrix},$$

where $z_i \in \mathbb{R}$ for all $i \in (\mathcal{F}_+ \cup \mathcal{F}_-)$ and $z_i \in \mathbb{R}^2$ for all $i \in \mathcal{F}_0$. Note that $c, z \in \mathbb{R}^{\hat{n}}$ and the block-diagonal matrix $T \in \mathbb{R}^{n \times \hat{n}}$ with $\hat{n} := n + |\mathcal{F}_0|$.

From Lemma 2, Corollary 3, and Lemma 5, it follows that there exist a collection of data matrices (G_i, H_i, g_i, h_i) such that $f(p) = T\hat{f}(p)$, where

$$\begin{aligned} \hat{f}(p) \;\in\; \arg\min_{z} \quad & c^\top z \\ \text{s.t.} \quad & G_i p + g \le z_i \mathbf{1}, && \forall i \in \mathcal{F}_+ \\ & H_i p + h \ge z_i \mathbf{1}, && \forall i \in \mathcal{F}_- \\ & \left.\begin{aligned} G_i p + g \le z_i^{[1]} \mathbf{1} \\ H_i p + h \ge z_i^{[2]} \mathbf{1} \end{aligned}\right\} && \forall i \in \mathcal{F}_0. \end{aligned} \tag{6}$$

The PLP problem (6) has exactly the form of (1). Note that $f(p) = T\hat{f}(p)$ since the problem (6) is separable in z_i. Since $|\mathcal{F}_0| \le n$, the dimension of the decision variable z in (6) is at most $2n$. Since the optimal solutions to the separated problems in z_i are all unique, so is the overall solution $\hat{f}(p)$. ■

Note that the proof of Theorem 1 is constructive, i.e. the PLP problem (6) can easily be formed from the problem data characterizing the PWA function f if it is decomposed into the difference of convex PWA functions using the algorithm described in the proof of Lemma 4. We do not claim that this method necessarily yields the smallest PLP that can generate the PWA function f. Consider as an example a solution to an unknown PLP where every component of its PWA solution is neither convex nor concave. Obviously, the generating PLP (of dimension n) would solve the extended inverse PLP problem for this function with $H = I$, while using the method from the proof of Theorem 1 would lead to a PLP of dimension $2n$.

On the other hand, our approach is guaranteed to always work and is much easier to implement compared to the method from [17], and furthermore does not require a value function or knowledge of the constraints *a priori*. Instead, the cost function coefficients can be chosen in such a way to satisfy another objective that may be application specific.

D. Application to Control Systems

The results from above can be used to remodel PWA dynamical systems in terms of an optimization problem. Let

$$x^+ = A_i x + B_i u + c_i \quad \text{for} \quad (x, u) \in \Omega_i$$

describe a PWA system with state $x \in \mathbb{R}^n$ and input $u \in \mathbb{R}^m$. Applying Theorem 1 to the PWA state transition function results in an equivalent inverse parametric linear programming model (IPLP):

$$x^+ = Ty \;\text{ where }\; y \in \arg\min_z c^\top z \;\text{ s.t. }\; Fz \le h + G \begin{bmatrix} x \\ u \end{bmatrix}$$

Application of the KKT conditions results in a system for which the next state is determined by a combination of linear equalities, inequalities, and complementarity conditions in a form similar to [23]. A related idea was used in [18] to speed up optimal control input calculations for the PWA system. While the procedure in [18] cannot be applied to every PWA system, Theorem 1 can be used to rewrite every PWA model in terms of a PLP.

III. Numerical Example

A. One-Dimensional Example

Assume we are given the function $f\colon \mathbb{R} \to \mathbb{R}$ shown in Figure 1 and want to find a PLP in the form (1) such that we can obtain f uniquely from its explicit solution. It is easy to see that f itself does not uniquely solve a PLP. To see this, assume that $f(p)$ did solve a PLP with $m = n = 1$ and $c > 0$. As p increases from -1 to 0, the optimizer (which will be the

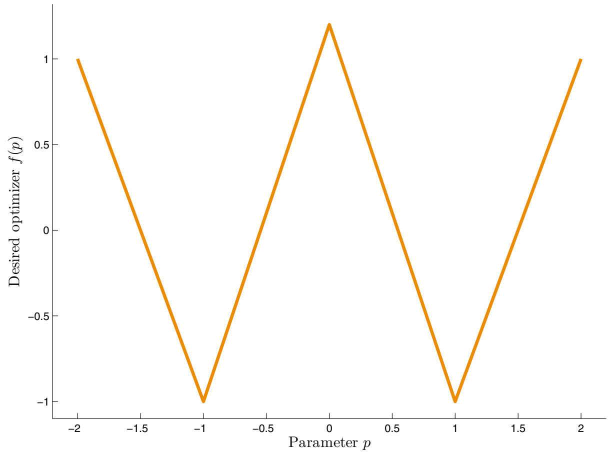

Fig. 1. A continuous 1D PWA function not solving a PLP.

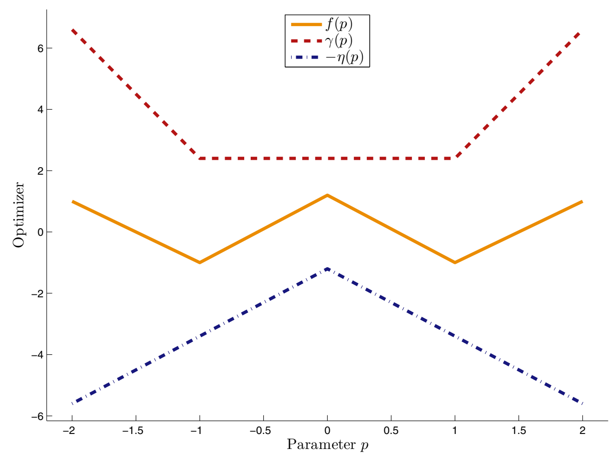

Fig. 2. Convex-concave decomposition of $f(p)$.

smallest feasible scalar) increases until we reach $p = 0$. For $p \in (0, 1)$, the optimizer decreases again, implying that the lower bound $\underline{z}(p)$ on the decision variable decreases as well, following the shape of $f(p)$. Obviously $f(p)$ and therefore $\underline{z}(p)$ is not convex in this region, which is a contradiction to the convexity assumption for the linear constraints (1b). A similar argument can be made when $c < 0$ is assumed. Finally, for $c = 0$ the explicit solution to the PLP is not unique. Therefore, the given PWA function f cannot be the unique explicit solution to a convex PLP.

Using our formulation of the extended inverse PLP problem from Definition 1 and the construction from the proof of Theorem 1 will yield a two-dimensional PLP with f as a linear mapping of its unique solution. The explicit expression of the function is given as

$$f(p) = \begin{cases} -2p \quad - 3 & \text{for} \quad p \in [-2, -1] \\ 2.2p + 1.2 & \text{for} \quad p \in [-1, 0] \\ -2.2p + 1.2 & \text{for} \quad p \in [0, 1] \\ 2p \quad - 3 & \text{for} \quad p \in [1, 2] \end{cases}.$$

Figure 2 shows the same function f and its decomposition into two convex PWA functions γ and η. They can be written explicitly as follows:

$$\gamma(p) = \begin{cases} -4.2p - 1.8 & \text{for} \quad p \in [-2, -1] \\ 2.4 & \text{for} \quad p \in [-1, 1] \\ 4.2p - 1.8 & \text{for} \quad p \in [1, 2] \end{cases}$$

$$\eta(p) = \begin{cases} -2.2p + 1.2 & \text{for} \quad p \in [-2, 0] \\ 2.2p + 1.2 & \text{for} \quad p \in [0, 2] \end{cases}$$

In accordance with Lemma 5 (which corresponds to Theorem 1 for $n = 1$), we construct the following optimization problem:

$$\begin{aligned} \min_{z_1, z_2} \quad & z_1 - z_2 \\ \text{s.t.} \quad & z_1 \geq -1.8 - 4.2p \\ & z_1 \geq 2.4 \\ & z_1 \geq -1.8 + 4.2p \\ & z_2 \leq -1.2 + 2.2p \\ & z_2 \leq -1.2 - 2.2p \\ & p \in [-2, 2] \end{aligned}$$

Because of its separable nature we can again use Lemma 2 and Corollary 3 to obtain its explicit solution as $z_1^*(p) = \gamma(p)$ and $z_2^*(p) = -\eta(p)$. We then get the desired function from this solution as $f(p) = z_1^*(p) + z_2^*(p) = \gamma(p) - \eta(p)$ as claimed in Theorem 1.

B. Two-Dimensional Example

As a more complex example, consider the vector-valued function f shown in Figure 3. The partition over which the PWA pieces are defined is shown underneath its graph. Clearly, this function is neither convex nor concave, but it can be decomposed into a convex function γ and a concave function $-\eta$ as shown in Figure 4. The explicit expressions for the functions are omitted here due to space limitations.

Using Theorem 1, we can construct the optimization problem

$$\begin{aligned} \min_{z_1, z_2} \quad & z_1 - z_2 \\ \text{s.t.} \quad & z_1 \geq 1 & & z_2 \leq -4 \\ & z_1 \geq p^{[1]} & & z_2 \leq -p^{[1]} - p^{[2]} \\ & z_1 \geq -p^{[1]} & & z_2 \leq p^{[1]} - p^{[2]} \\ & z_1 \geq p^{[2]} & & z_2 \leq -p^{[1]} + p^{[2]} \\ & z_1 \geq -p^{[2]} & & z_2 \leq p^{[1]} + p^{[2]} \\ & p \in [-5, 5] \times [-5, 5] \end{aligned}$$

in the two scalar decision variables z_1 and z_2. Again, due to its separability we can obtain the explicit solution as $z_1^*(p) =$

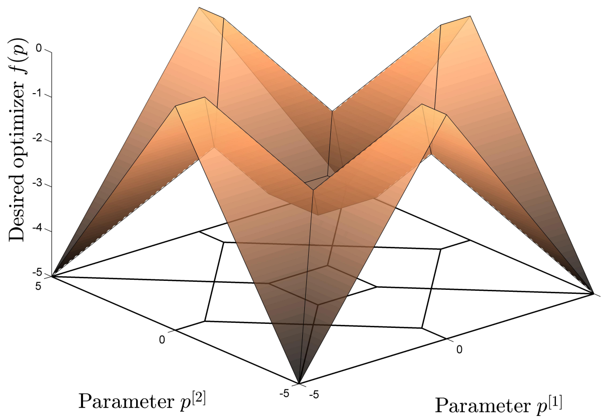

Fig. 3. Desired 2D function $f(p)$ with its partition.

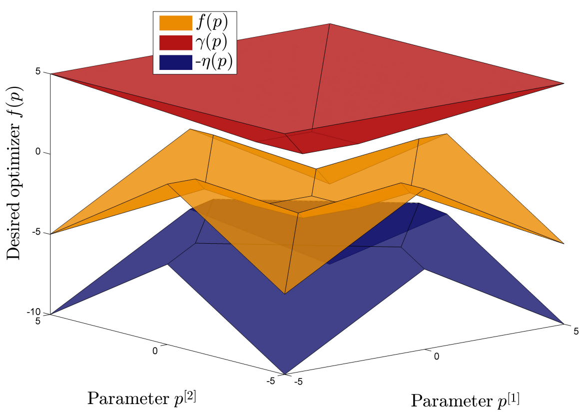

Fig. 4. Convex-concave decomposition of 2D function $f(p)$.

$\gamma(p)$ and $z_2^*(p) = -\eta(p)$ and the desired function shown in Figure 3 exactly as $f(p) = z_1^*(p) + z_2^*(p) = \gamma(p) - \eta(p)$.

IV. Conclusion and Outlook

In this paper we have shown how a PLP can be constructed from a given continuous PWA function f such that this function can be obtained as the image of the explicit solution to the PLP. This approach is applicable to any continuous PWA function and yields the desired function uniquely. The dimension of the constructed PLP is at most twice the dimension of the image of the desired function.

Not all continuous PWA functions are solutions to some PLP. This new result broadens the applicability of inverse optimization methods to these cases. Solving a PLP in a slightly higher dimensional space together with a simple linear operator yield exactly and uniquely the desired function.

Extending the results from this paper to more general, nonlinear DC functions is straightforward. Of more practical interest are related results that allow to obtain any continuous PWA function from the explicit solution to a PQP. Together with [18] this would unlock a new and promising toolbox for the control of PWA systems.

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