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On the interplay between periodic switches and uncontrolled jumps in linear discrete-time systems

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***Abstract*— This paper deals with the analysis of stability and the characterization of input-output norms for dual switching linear systems in discrete-time. These systems are subject to two independent switching signals, taking values in finite sets, coming from different sources. The first switching signal is periodic while the second is uncontrolled and generate parameter jumps which are either completely arbitrary or satisfy a prescribed dwell-time constraint. The overall system is a linear time-varying system exhibiting a complex dynamic behavior due to the interplay between the periodic switches and the uncontrolled jumps. LMI conditions for stability and guaranteed H_2 norm input/output performances are provided. The results are obtained by merging the theory of linear periodic systems with recent developments on switched linear time-invariant systems.**

***Index Terms*— Switched systems, Periodic systems, Dwell-time, Stability, H_2 norm.**

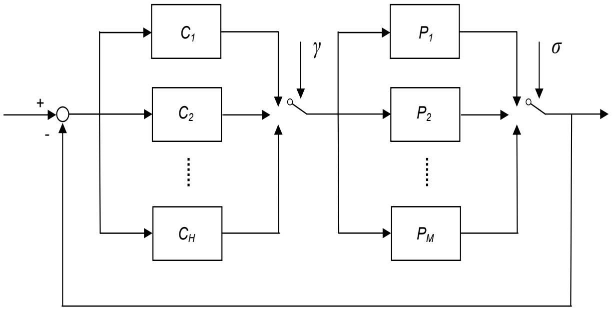

Fig. 1. A dual switching system consisting of a jumping process controlled by a switching linear time-invariant feedback controller.

I. INTRODUCTION

Switching linear systems are widely studied for their ability to describe the behavior of systems where the dynamics changes abruptly due to jumps in parameters taking values in a finite set. It is well known that the presence of jumps may significantly affect the performance and the stability of the switching system, see [12], [13] and the references therein. When the switching is exogenous, it is interesting to investigate on stability and performance under the constraint that the switching signal has a maximal commutation frequency. Results on these issues in discrete-time can be found, e.g., in [7], [11], [10].

Recently, the attention has been focused on *dual* switching systems, namely systems affected by two independent sources of switching. For instance, this is important in the analysis of closed-loop systems structured as in Figure 1, where the switching signal $\gamma(k)$ represents the action of a supervisor selecting at each time instant one controller within a bunch of available compensators, and the uncontrolled jumping signal $\sigma(k)$ selects the process under control that is active at time k. The interaction between the two switching signals makes the system dynamics quite complicate. According to the specific problem at hand, one can consider different assumptions on $\gamma(k)$ and $\sigma(k)$. For instance, there are situations where $\sigma(k)$ is naturally described as a stochastic process, e.g. taking values according to a Markov chain, and $\gamma(k)$ is a deterministic exogenous signal, possibly subject to constraints on the maximum frequency of switches (dwell-time constraints), see [4], [5]. In other contexts, the signal $\gamma(k)$ may be considered as an additional control variable to be designed in order to optimize the overall performance.

In this paper we restrict our attention to the particular case where $\gamma(k)$ is a *fixed* periodic switching signal and $\sigma(k)$ is either completely arbitrary or subject to a dwell-time constraint, i.e. the distance between two successive jumping time instants is greater than a prescribed constant integer $\Delta > 1$. With reference to Figure 1, this situation occurs when the compensators are selected according to a cyclic schedule and the process to be controlled is affected by uncertain spaced-out jumps in its parameters. We aim at studying the stability and the evaluation of the H_2 performance for such a class of systems. The analysis under arbitrary switching can be cast in the framework of periodic polytopic systems, studied in [10]. However, when the dwell-time is greater than 1, the picture becomes more involved and the parallelism with polytopic systems no longer holds. In the present paper, we provide sufficient conditions for stability in terms of periodic LMI's. The exact computation of the H_2 norm associated with the worst possible jumping signal is a formidable problem and no computationally viable solution exists. In the paper, we derive upper bonds of this norm that are easily computable through the use of periodic LMI's.

This paper relies on the results illustrated in the recent report [3] and all the proofs are therefore omitted.

The paper is organized as follows. After some preliminaries on the theory of discrete-time linear periodic systems (Section II), in Section III stability is analyzed, while Section IV deals with the computation of an upper bound for the H_2

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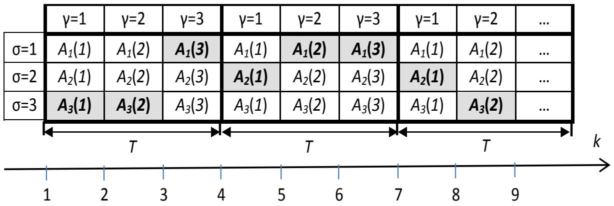
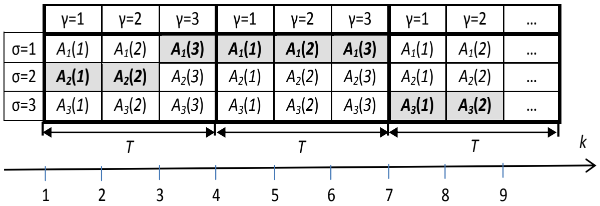


	γ=1	γ=2	γ=3	γ=1	γ=2	γ=3	γ=1	γ=2	…
σ=1	$A_1(1)$	$A_1(2)$	**$A_1(3)$**	$A_1(1)$	**$A_1(2)$**	**$A_1(3)$**	$A_1(1)$	$A_1(2)$	…
σ=2	$A_2(1)$	$A_2(2)$	$A_2(3)$	**$A_2(1)$**	$A_2(2)$	$A_2(3)$	**$A_2(1)$**	$A_2(2)$	…
σ=3	**$A_3(1)$**	**$A_3(2)$**	$A_3(3)$	$A_3(1)$	$A_3(2)$	$A_3(3)$	$A_3(1)$	**$A_3(2)$**	…

Fig. 2. Interplay between a periodic switching signal $\gamma(\cdot) \in \mathscr{H} = \{1,2,3\}$ and a jumping signal $\sigma(\cdot) \in \mathscr{S}_1$. The grey boxes correspond to the activated subsystems.



	γ=1	γ=2	γ=3	γ=1	γ=2	γ=3	γ=1	γ=2	…
σ=1	$A_1(1)$	$A_1(2)$	**$A_1(3)$**	**$A_1(1)$**	**$A_1(2)$**	**$A_1(3)$**	$A_1(1)$	$A_1(2)$	…
σ=2	**$A_2(1)$**	**$A_2(2)$**	$A_2(3)$	$A_2(1)$	$A_2(2)$	$A_2(3)$	$A_2(1)$	$A_2(2)$	…
σ=3	$A_3(1)$	$A_3(2)$	$A_3(3)$	$A_3(1)$	$A_3(2)$	$A_3(3)$	**$A_3(1)$**	**$A_3(2)$**	…

Fig. 3. Interplay between a periodic switching signal $\gamma(\cdot) \in \mathscr{H} = \{1,2,3\}$ and a jumping signal $\sigma(\cdot) \in \mathscr{S}_2$. The grey boxes correspond to the activated subsystems.

norm. Finally, an original example inspired by the celebrated "Belgian chocolate problem" of [2] is discussed in Section V. The paper ends with some concluding remarks.

II. PRELIMINARIES

In this paper we consider systems defined in discrete-time ($k \in Z$) and described by the difference equations

$$x(k+1) \quad = \quad A^{\gamma(k)}_{\sigma(k)}x(k) + B^{\gamma(k)}_{\sigma(k)}u(k) \tag{1}$$

$$y(k) \quad = \quad C^{\gamma(k)}_{\sigma(k)}x(k) + D^{\gamma(k)}_{\sigma(k)}u(k) \tag{2}$$

where $x(k) \in R^n$, $u(k) \in R^m$, $y(k) \in R^p$, $\sigma(\cdot)$ is an exogenous jumping signal, taking values in the finite set $\mathscr{M} = \{1,2,\ldots,M\}$, and $\gamma(\cdot)$ is a periodic signal, with period $T \geq 1$, taking values in the finite set $\mathscr{H} = \{1,2,\ldots,H\}$. This implies that, for any given $i \in \mathscr{M}$, it results $A_i^{\gamma(k+T)} = A_i^{\gamma(k)}$ for each $k \in Z$, and the same holds true for the other system matrices. Hereafter, for ease of notation, we put

$$A_i(k) := A_i^{\gamma(k)},\ B_i(k) := B_i^{\gamma(k)},\ C_i(k) := C_i^{\gamma(k)},\ D_i(k) := D_i^{\gamma(k)}$$

Then, the overall system can be considered as a network of T-periodic subsystems, sharing the same state vector, with a signal $\sigma(\cdot)$ that dictates the jumps between the subsystems. The i-th periodic subsystem is said to be *active* at time k if $\sigma(k) = i$. We will consider two sets of exogenous jumping signals. The first is called $\mathscr{S}$ and consists of all possible signals $\sigma(\cdot)$. The second, that will be denoted as $\mathscr{S}_\Delta$, is constituted by all jumping signals with the *dwell-time* constraint $t_{h+1} - t_h \geq \Delta \geq 1$, where t_h denotes the h-th jumping instant. Of course $\mathscr{S}_1 = \mathscr{S}$.

The interplay between the switching signal $\sigma(\cdot)$ and the periodicity of the system parameters is schematically illustrated in Figures 2 and 3, focusing only on the dynamic matrix $A_{\sigma(k)}(k)$. Precisely, in these figures, as two examples, it is shown how the subsystems are activated along the first periods of length $T = 3$ according to either the jumping signal $\sigma(k) = \{3,3,1,2,1,1,2,3,\ldots\}$, or the jumping signal $\sigma(k) = \{2,2,1,1,1,1,3,3,\ldots\}$. Notice that $\sigma(\cdot)$ of Figure 2 belongs to $\mathscr{S}_1$ whereas $\sigma(\cdot)$ of Figure 3 belongs to $\mathscr{S}_2$ (provided that the dwell-time constraint is met with even afterwards).

In order to assess stability and compute the H_2 norm of the system, it is necessary to briefly summarize a few known facts, see [1], on the theory of linear periodic systems in discrete-time, i.e. system (1), (2) when $\sigma(k) = i$ is constant for all $k \in Z$.

It is well known that the stability of a linear periodic system depends only on the matrix function $A_i(\cdot)$. The i-th subsystem is (asymptotically) stable if and only if the eigenvalues (also called *characteristic multipliers*) of the monodromy matrix $\Phi_i(T,0)$ lie in the open unit disc of the complex plane. Letting $\Phi_i(k,j) = A_i(k-1)A_i(k-2)\ldots A_i(j)$, $k > j$, $\Phi_i(k,k) = I$, be the transition matrix of the i-th subsystem, the monodromy matrix is the transition matrix over one period. An alternative condition is formulated in terms of the periodic Lyapunov inequality

$$A_i(k)'P_i(k+1)A_i(k) < P_i(k) \tag{3}$$

The i-th subsystem is asymptotically stable if and only if there exists a T-periodic solution $P_i(k) > 0$, $\forall k$, of (3).

Let us now turn to the H_2 norm of the i-th subsystem. Let δ_k be the Kronecker symbol and e_r the r-th column of the identity matrix. Assuming stability, one can associate with any impulsive input $u(k) = e_r\delta_{k-s}$, $r = 1,2,\ldots,m$, $s = -T, -T+1, \ldots, -1$ and initial state $x(s) = 0$, the output response $y^{r,s}(k)$, $k \geq s$. The H_2 norm of the the i-th subsystem is defined as

$$J_{2i} = \frac{1}{T}\sum_{s=-T}^{-1}\sum_{r=1}^{m}\sum_{k=s}^{\infty} y^{r,s}(k)'y^{r,s}(k) \tag{4}$$

It turns out that (see Section 9.1.1 of [1])

$$J_{2i} = \frac{1}{T}\sum_{k=0}^{T-1}\operatorname{trace}(B_i(k)'\bar{P}_i(k+1)B_i(k) + D_i(k)'D_i(k))$$

where $\bar{P}_i(\cdot)$ is such that $\bar{P}_i(k) \geq 0$, $\bar{P}_i(k+T) = \bar{P}_i(k)$, $\forall k$ and solves the periodic Lyapunov equation

$$A_i(k)'\bar{P}_i(k+1)A_i(k) + C_i(k)'C_i(k) = \bar{P}_i(k) \tag{5}$$

Notice that, thanks to periodicity of the coefficients, J_{2i} does not depend on the initial time instant.

III. STABILITY

In this section we study the stability of system (1) assuming that all subsystems are stable. First we consider the case when $\sigma(\cdot) \in \mathscr{S}_1$, i.e. the system undergoes arbitrary jumps.

Theorem 3.1: System (1) is stable in $\mathscr{S}_1$ if there exist T-periodic positive definite solutions $P_i(\cdot)$ of the Lyapunov inequalities

$$A_i(k)'P_j(k+1)A_i(k) < P_i(k), \quad \forall i,j \tag{6}$$

■

A necessary condition for stability under arbitrary jumps is that all subsystems are stable. Notice that the conditions expressed by (6) are easily testable as they consist of LMI's in the unknowns $P_i(0)$, $P_i(1)$, $\ldots$, $P_i(T-1)$, $i = 1,2,\ldots,M$. The proof of Theorem 3.1 consists in showing that the *piecewise* quadratic function $V(x,k) = x'P_{\sigma(k)}(k)x$ is a Lyapunov function for the switched system, i.e. it decreases along the trajectories of the system, irrespectively of the jumps. Notice also that a more conservative sufficient condition can be found by looking for a *common* quadratic Lyapunov function, i.e. for a positive definite T-periodic matrix function $P(\cdot)$ satisfying

$$A_i(k)'P(k+1)A_i(k) < P(k), \quad \forall i \tag{7}$$

Conversely, a less conservative sufficient LMI condition based on a parameter-dependent Lyapunov function could be worked out along the lines traced in [9] for linear polytopic systems.

Consider now the set $\mathscr{S}_\Delta$ consisting of all jumping signals whose value can change after an interval of $\Delta > 0$ steps at least, i.e. $t_{k+1} \geq t_k + \Delta$. It is intuitive to infer that, if all the subsystems are stable and Δ is large enough, the overall system (1) is stable in $\mathscr{S}_\Delta$. However, for a given $\Delta > 0$, the following sufficient condition can be derived.

Theorem 3.2: System (1) is stable in $\mathscr{S}_\Delta$ if there exist T-periodic positive definite solutions $P_i(\cdot)$ satisfying the following inequalities

$$\begin{aligned} A_i(k)'P_i(k+1)A_i(k) &< P_i(k), \quad \forall i && (8)\\ \Phi_i(k,k-\Delta)'P_j(k)\Phi_i(k,k-\Delta) &< P_i(k-\Delta), \forall i \neq j && (9) \end{aligned}$$

■

The proof is again based on the function $V(x,k) = x'P_{\sigma(k)}(k)x$, in that, as it can be easily verified, the conditions (8), (9) imply that $V(x,k)$ decreases along all admissible system trajectories over any interval of length not less than Δ.

Remark 3.1: Theorems 3.1, 3.2 state sufficient conditions for asymptotic stability. Notice however that also exponential stability with a certain decay rate ρ, with $|\rho| < 1$, can be proven. To this end, it suffices to multiply $P_i(k)$ by ρ and $P_i(k+1-\Delta)$ by ρ^Δ, so that $V(x(k+l),k+l) < \rho^l V(x(k),k)$.

Remark 3.2: Obviously, a necessary condition for stability in $\mathscr{S}_\Delta$ is that all subsystems are stable. Notice that the inequalities (8), (9) consist of LMIs in the unknowns $P_i(0), P_i(1), \ldots P_i(T-1)$, $i = 1,2,\ldots,M$ for any fixed Δ. Stability under arbitrary jumps (Theorem 3.1) is recovered for $\Delta = 1$. The case $\Delta = 0$ is meaningless, but (7) can be considered as the limit case (e.g. $P_i(k) = P_j(k)$) of (8), (9) when $\Delta \to 0$. It is interesting to find the minimum $\Delta^\star$ for which (8), (9) are feasible. This can be obtained by decreasing Δ starting from a large value. In fact, notice that $\lim_{\Delta\to\infty}\Phi_i(k,k-\Delta) = 0$, so that the inequalities (9) are feasible for Δ sufficiently large. Notice however that $\Delta^\star$ is just an upper bound of the *minimum dwell-time*, i.e. the minimum Δ, say Δ_{min}, for which the system is stable in $\mathscr{S}_\Delta$. Indeed the use of piecewise quadratic Lyapunov functions gives a sufficient condition that can be conservative, in the sense that $\Delta^\star$ is generally stricly greater than Δ_{min}. However, the powerfulness and computational appeal of piecewise quadratic functions can be pushed forward by state augmentation and use of Gram matrices, as done in [6] for continuous-time switched systems. With such an extension, it is possible to find a sequence of $\Delta^\star$ associated to piecewise quadratic functions in the extended space eventually converging to the minimum dwell-time.

Remark 3.3: The sufficient conditions of Theorem 3.2 are nested in Δ, namely if they hold for a certain Δ, they also hold for larger values of Δ. To see this, premultiply both members of (9) by $A_i(k+1-\Delta)'$ and postmultiply them by its transpose, and use (8). The resulting conditions are sufficient to prove stability in $\mathscr{S}_{\Delta+1}$.

IV. H_2 NORM

Let us now turn to the H_2 norm of system (1), (2). As already seen, let δ_k be the Kronecker symbol and e_r the r-th column of the identity matrix. One can define the impulse response $y^{r,s}(k)$, $k \geq s$ associated with $u(k) = e_r\delta(k-s)$, $r = 1,2,\ldots,m$, $s = -T,-T+1,\ldots,-1$ and initial state $x(s) = 0$. For each fixed $\sigma(\cdot) \in \mathscr{S}_\Delta$ it is possible to define the output energy as

$$J_{2\sigma(\cdot)} = \frac{1}{T}\sum_{s=-T}^{-1}\sum_{r=1}^{m}\sum_{k=s}^{\infty} y^{r,s}(k)'y^{r,s}(k) \tag{10}$$

It is interesting to compute the worst value of this performance index as $\sigma(\cdot)$ varies in $\mathscr{S}_\Delta$ (this value does not depend on the initial time, as obvious). This computation is quite difficult, but the next results will provide an upper bound $\bar{J}_2(\Delta)$ to this value. We consider first the set $\mathscr{S}_1$, i.e. arbitrary jumps.

Theorem 4.1: Assume that there exist positive definite and T-periodic matrices $P_i(\cdot)$ such that

$$A_i(k)'P_j(k+1)A_i(k) + C_i(k)'C_i(k) < P_i(k), \quad \forall i,j \tag{11}$$

Then, the system (1), (2) is stable in $\mathscr{S}_1$ and

$$\sup_{\sigma(\cdot)\in\mathscr{S}_1} J_{2\sigma(\cdot)} < \bar{J}_2(1)$$

$$\bar{J}_2(1) = \frac{1}{T}\sum_{k=0}^{T-1}\max_{i,j}\operatorname{trace}(B_i(k)'P_j(k+1)B_i(k) + D_i(k)'D_i(k))$$

■

The above result is derived by observing that the function $V(x,k) = x'P_{\sigma(k)}(k)x$ can be interpreted as the storage function associated with the H_2 problem, as done in [7] for switched time-invariant systems.

We now consider the class $\mathscr{S}_\Delta$ with $\Delta > 1$. The definition of the output energy given in (10) requires the computation of the impulse responses $y^{r,s}(k)$ with $-T \leq s < 0$. We start with the computation of an upper bound of the energy of the impulse response when the impulses are applied at time $s=t_0-q$, where t_0 is a jumping instant and $0<q\le\Delta$, so that $\sigma(t_0-q)=\sigma(t_0-q+1)=\cdots=\sigma(t_0-1)$. The next result is based on the backward propagation of the solution of the difference Lyapunov equation

$$X_i(k)=A_i(k)'X_i(k+1)A_i(k)+C_i(k)'C_i(k) \qquad (12)$$

We define as $X_i(\tau,k,Z)$, $\tau\le k$, the solution of eq. (12) with final condition $X_i(k,k,Z)=Z$. Equation (12) is useful to cope with the impulse response computation in the interval from $s\in[-T,-1]$ to the first switching instant t_0, whose length may be less than Δ.

Lemma 4.1: Let $s=t_0-q$, $q\le\Delta$. Assume that there exist positive definite T-periodic matrices $P_i(\cdot)$, $i\in\mathscr{M}$ satisfying

$$A_i(k)'P_i(k+1)A_i(k)+C_i(k)'C_i(k)<P_i(k), \quad \forall i \qquad (13)$$

and, $\forall i\neq j$,

$$\Phi_i(k,k-\Delta)'P_j(k)\Phi_i(k,k-\Delta)+\Omega_i(k-1,\Delta)<P_i(k-\Delta) \qquad (14)$$

with

$$\Omega_i(k,\Delta)=\sum_{q=k-\Delta+1}^{k}\Phi_i(q,k-\Delta+1)'C_i(q)'C_i(q)\Phi_i(q,k-\Delta+1)$$

Then, the system (1), (2) is stable in $\mathscr{S}_\Delta$ and

$$\begin{aligned}E_{y\sigma(\cdot)}(s) &:= \sum_{r=1}^{m}\sum_{k=s}^{\infty}y^{r,s}(k)'y^{r,s}(k)\\ &\le \operatorname{trace}\big(B_{\sigma(s)}(s)'X_{\sigma(s)}\left(s+1,t_0,P_{\sigma(t_0)}(t_0)\right)B_{\sigma(s)}(s)\\ &\quad +D_{\sigma(s)}(s)'D_{\sigma(s)}(s)\big) \qquad (15)\end{aligned}$$

■

Notice that the upper bound (15) is valid (with the equal sign) also for constant switching signals. In that case, t_0 can be assumed equal to infinity and s is an arbitrary time instant. The backward integration of (12) over an infinite horizon is such that $X_i(k)=\bar{P}_i(k)$, where $\bar{P}_i(k)$ is the periodic solution of (5).

We finally consider the impulses applied in any point $s\in[-T,-1]$. Let us fix a given $\sigma(\cdot)\in\mathscr{S}_\Delta$ and define $t_0\ge 0$ be the first jumping instant after $k=-1$ and call $t_{-1},t_{-2},\cdots,t_{-N}$ the possible jumping instants in the period $[-T,-1]$. We can apply Lemma 4.1 in the subintervals $s\in[t_{-r-1},t_{-r}-1]$, $r=0,1,\cdots,N-1$, observing that the upper bound (15) also holds when $q>\Delta$ provided that $t_0-q\ge t_{-1}$, i.e when the impulse is applied in any point between two successive jumps.

Lemma 4.2: Assume that there exist positive definite T-periodic matrices $P_i(\cdot)$ satisfying (13) and (14). Then, the system (1), (2) is stable in $\mathscr{S}_\Delta$ and

$$J_{2\sigma(\cdot)}=\frac{1}{T}\sum_{s=-T}^{-1}\sum_{r=1}^{m}\sum_{k=s}^{\infty}y^{r,s}(k)'y^{r,s}(k)\le\frac{1}{T}\sum_{s=-T}^{-1}\operatorname{trace}(\Theta(s))$$

with

$$\begin{aligned}\Theta(s)&=B_{\sigma(s)}(s)'X_{\sigma(s)}\left(s+1,t(s),P_{\sigma(t(s))}(t(s))\right)B_{\sigma(s)}(s)\\ &\quad +D_{\sigma(s)}(s)'D_{\sigma(s)}(s)\end{aligned}$$

where $t(s)$ is the first jumping instant after $s\in[-T,-1]$.

■

Again, notice that this upper bound is valid (with the equal sign) also when $\sigma(\cdot)$ is constant. In that case, $t_0=\infty$ and there are no jumps in the period $[-T,-1]$, so that the backward solution $X_i(k)$ coincides with $\bar{P}_i(k)$ solution of (5).

The final result hinges on the relationship between the positive definite T-periodic matrices $P_i(\cdot)$ satisfying (13) and the solution $X_i(\tau,k,Z)$, $\tau\le k$, of eq. (12). As a matter of fact it results that

$$X_i(k-\Delta,k,P_j(k))<P_i(k-\Delta), \quad i\neq j$$

and, if $\tau\le k-\Delta$

$$X_i(\tau,k,P_j(k))\le X_i(\tau,k-\Delta,P_i(k-\Delta)), \quad i\neq j$$

We are now in a position to work out an upper bound for the worst H_2 norm of the switched periodic system in $\mathscr{S}_\Delta$.

Theorem 4.2: Assume that there exist positive definite T-periodic matrices $P_i(\cdot)$ satisfying (13) and (14). Then, the system (1), (2) is stable in $\mathscr{S}_\Delta$ and

$$\sup_{\sigma(\cdot)\in\mathscr{S}_\Delta}J_{2\sigma(\cdot)}<\bar{J}_2(\Delta)=\frac{1}{T}\sum_{s=-T}^{-1}\max_{i,j}\max_{q\in[1,\Delta]}\operatorname{trace}(\Psi_{ij}(s,q)) \qquad (16)$$

with

$$\Psi_{ij}(s,q)=B_i(s)'X_i(s+1,s+q,P_j(s+q))B_i(s)+D_i(s)'D_i(s) \qquad (17)$$

■

Remark 4.1: The case of arbitrary jumps (Theorem 4.1) is recovered by letting $\Delta=1$ in (14). In this case, (13), (14) coincide with (11). Moreover, $X_i(s+1,s+1,P_j(s+1))=P_j(s+1)$, so that the upper bound $\bar{J}_2(1)$ of Theorem 4.1 coincides with $\bar{J}_2(\Delta)$ of Theorem 4.2.

V. A NUMERICAL EXAMPLE

This example is inspired by the challenging "Belgian chocolate problem" proposed in [2] and studied in a periodic framework in [8].

Consider the feedback control system depicted in Figure 4, where

$$G(s)=\frac{s^2-1}{s^2-2\xi s+1}$$

and

$$F(t)=\begin{cases}F_1 & t\in[kT_c,kT_c+T_c/2)\\ F_2 & t\in[kT_c+T_c/2,(k+1)T_c)\end{cases}$$

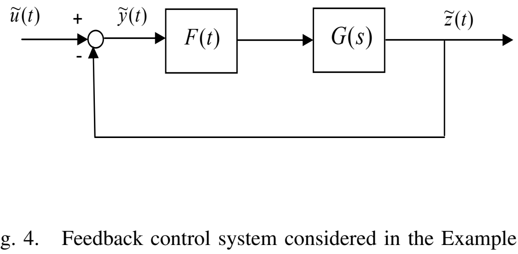


Fig. 4. Feedback control system considered in the Example.

The parameter ξ plays an important role in the problem of stabilization through a stable minimum phase controller, making the design very hard when ξ approaches 1. Here we assume that the parameter ξ can assume only two values, namely $\xi_1 = 0.88$ and $\xi_2 = 0.92$ and may jump between them. Accordingly, we are well advised to write a state space realization for the resulting closed-loop jumping system as

$$\begin{aligned}\dot{\tilde{x}}(t) &= \tilde{A}_{\tilde{\sigma}(t)}\tilde{x}(t) + \tilde{B}F(t)(\tilde{u}(t) - \tilde{z}(t))\\ \tilde{z}(t) &= \tilde{C}_{\tilde{\sigma}(t)}\tilde{x}(t) + \tilde{D}(\tilde{u}(t) - \tilde{z}(t))\\ \tilde{y}(t) &= \frac{1}{1+\tilde{D}F(t)}\left(\tilde{C}_{\tilde{\sigma}(t)}\tilde{x}(t) + \tilde{D}F(t)\tilde{u}(t)\right)\end{aligned}$$

where $\tilde{\sigma}(t) = i$ if $\xi = \xi_i$, $i = 1,2$ and

$$\begin{aligned}\tilde{A}_i &= \begin{bmatrix} 0 & 1 \\ -1 & 2\xi_i \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\\ \tilde{C}_i &= \begin{bmatrix} -2 & 2\xi_i \end{bmatrix}, \quad \tilde{D} = 1\end{aligned}$$

We now discretize the system via ZOH sampling. To this end define $x(k) = \tilde{x}(kT_c/2)$, $\tilde{u}(t) = u(k), t \in [kT_c/2, (k+1)T_c/2)$, $y(k) = \tilde{y}(kT_c/2)$, $\sigma(k) = \tilde{\sigma}(kT_c/2)$ and let $T_c = 2$, $F_1 = -0.2778$, $F_2 = 0.3226$. The discrete-time system takes the form (1), (2) with $T = 2$, $\mathscr{M} = \{1,2\}$, $\mathscr{H} = \{1,2\}$ and

$$\begin{aligned}A_1^1 &= \begin{bmatrix} 2.623 & 0.353 \\ 2.891 & 0.389 \end{bmatrix}, \quad A_1^2 = \begin{bmatrix} -23.278 & 21.696 \\ -112.821 & 95.098 \end{bmatrix}\\ A_2^1 &= \begin{bmatrix} 2.551 & 0.332 \\ 2.720 & 0.354 \end{bmatrix}, \quad A_2^2 = \begin{bmatrix} -27.776 & 27.126 \\ -141.054 & 126.950 \end{bmatrix}\\ B_1^1 &= \begin{bmatrix} 0.910 \\ 1.622 \end{bmatrix}, \quad B_1^2 = \begin{bmatrix} -9.805 \\ -45.563 \end{bmatrix}\\ B_2^1 &= \begin{bmatrix} 0.870 \\ 1.526 \end{bmatrix}, \quad B_2^2 = \begin{bmatrix} -11.621 \\ -56.964 \end{bmatrix}\\ C_1^1 &= \begin{bmatrix} -7.200 & 6.336 \end{bmatrix}, \quad C_1^2 = \begin{bmatrix} 6.200 & -5.456 \end{bmatrix}\\ C_2^1 &= \begin{bmatrix} -7.200 & 6.624 \end{bmatrix}, \quad C_2^2 = \begin{bmatrix} 6.200 & -5.704 \end{bmatrix}\\ D_1^1 &= D_2^1 = -3.600, \quad D_1^2 = D_2^2 = 3.100\end{aligned}$$

We are interested in analyzing stability and H_2 performance (in terms of the sensitivity function from the reference u to the error y) of the closed-loop system for different classes of jumping signals. It is easy to see that both periodic subsystems with $\sigma(k) = 1, \forall k$ and $\sigma(k) = 2, \forall k$ are stable. The impulse responses starting from $s \in [-2,-1]$ for both subsystems are shown in Figures 5 and 6.

However the overall system is not stable under arbitrary jumps. Indeed, the periodic signal $\sigma(\cdot) \in \mathscr{S}_1$ defined as

$$\sigma(k) = \begin{cases} 2, & k \quad \text{even} \\ 1, & k \quad \text{odd} \end{cases}$$

is such that the associated monodromy matrix $A_2^2 A_1^1$ has an eigenvalue outside the unit circle, so that the jumping system is unstable. As a matter of fact, the LMI's (6) of Theorem 3.1 are not feasible. Figure 7 reports the impulse responses starting from $s \in [-2,-1]$ associated with the jumping signal defined above.

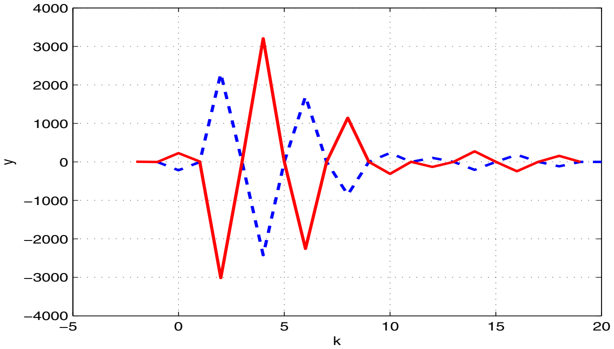


Fig. 5. Impulse responses for $\sigma(k) = 1$ starting from $s = -2$ (red solid) and $s = -1$ (blue dashed).

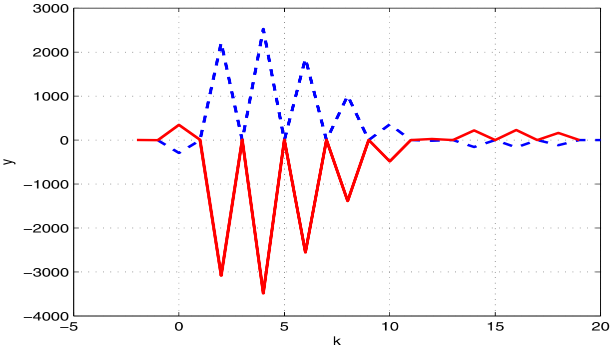


Fig. 6. Impulse responses for $\sigma(k) = 2$ starting from $s = -2$ (red solid) and $s = -1$ (blue dashed).

The LMI's (8), (9) of Theorem 3.2 are not feasible for $\Delta < 10$. However they become feasible for $\Delta = 10$, thus implying that the system is stable in $\mathscr{S}_{10}$. Figure 8 shows the impulse responses starting from $s \in [-2,-1]$ associated with a jumping signal $\sigma(\cdot)$ that commutes every $\Delta = 10$ steps. Apparently, both responses asymptotically converge to zero.

Consider now the performance in terms of the H_2 norm from u to y. Compute first the norm when the parameter ξ is not jumping, using the definition (4). If $\sigma(k) = 1, \forall k$, it turns out that the H_2 norm is $J_{21} = 2.058 \cdot 10^7$. If $\sigma(k) = 2, \forall k$, it turns out that the H_2 norm is $J_{22} = 2.333 \cdot 10^7$. Since the jumping system is not stable in $\mathscr{S}_1$, an upper bound of the H_2 norm cannot be obtained when the switching is arbitrary. However, Theorem 4.2 can be applied to compute an upper bound for $\Delta = 10$, i.e. when the dwell-time is not less than 10. Feasible T-periodic positive definite solutions $P_i(\cdot)$ of inequalities (13) and (14) with $\Delta = 10$ do exist and a minimal upper bound of the worst-case $J_{2\sigma(\cdot)}$ is calculated on the basis of eqs. (16), (17). It turns out that $\bar{J}_2(10) = 4.95 \cdot 10^8$. By further increasing Δ the bound is slightly reduced. However, it is not true that the limit of the bound for Δ going to infinity coincides with the maximum between J_{21} and J_{22}. This is due to the fact that, even for Δ arbitrarily large, one jump may occur anywhere on the time axis.

VI. CONCLUDING REMARKS

In this paper, we have investigated some properties of the so-called dual switching systems, i.e. systems characterized

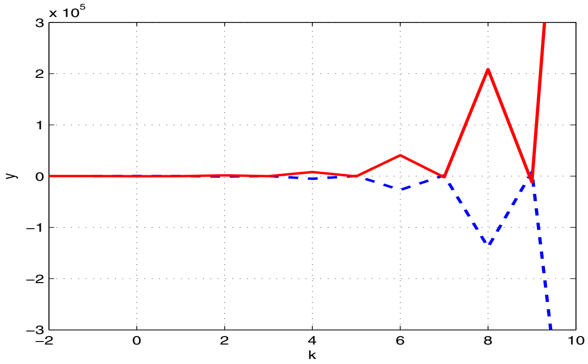


Fig. 7. Impulse responses for $\sigma(k) = 2$, k even, and $\sigma(k) = 1$, k odd, starting from $s = -2$ (red solid) and $s = -1$ (blue dashed).

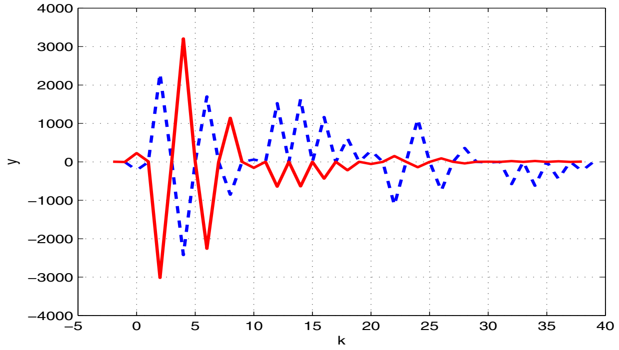


Fig. 8. Impulse responses for a periodic $\sigma(\cdot) \in \mathscr{S}_{10}$ starting from $s = -2$ (red solid) and $s = -1$ (blue dashed).

by two independent sources of switching. In particular, we have considered the case when the first signal is periodic in time and the second is arbitrary in the set of jumping signals with dwell-time constraints. For such systems, the stability and H_2 performance properties have been studied, clarifying the interplay between periodic dynamics and uncontrolled jumps. We have worked out LMI-based sufficient conditions to check stability and to compute upper bounds for the H_2 norm. The results generalize those available for switched time-invariant discrete-time systems and are amenable to be extended to more complicate situations, where the switching signals are not periodic and/or not deterministic. Even more important is the development of appropriate design techniques when the switching is a controlled variable.

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