

Solving discrete-time game theoretic periodic Riccati equations: An iterative procedure

V. Dragan¹, S. Aberkane² and I.G. Ivanov³

Abstract—This paper addresses the problem of solving a class of periodic discrete-time Riccati equation with an indefinite sign of its quadratic term. More precisely, we focus on the computation of the stabilizing solutions of discrete-time game theoretic periodic Riccati equations. A convergent iterative algorithm is proposed for this purpose.

I. INTRODUCTION

Periodic systems represent an important sub-class of time-varying systems that received much attention during the past several decades. This is partly due to the fact that a variety of dynamic systems possess periodic behavior: networked control systems with communication constraints, celestial mechanics, multirate sampled-data control, *etc.* The theory of stability, optimal and robust control, as well as important applications of such systems, can be found in several references in the current literature. One can refer to the recent monograph [1] and the references therein.

In this paper, we will address the problem of solving a class of periodic discrete-time Riccati equation with an non-definite sign of its quadratic term. The motivation behind this problem is that such equation is closely related to the so called full information H_∞ control of discrete-time periodic systems. More specifically, we will propose a reliable procedure for numerical computation of the stabilizing solution of such an equation. The proposed solution will be given in terms of a convergent iterative algorithm. Note that as pointed out by [9], even in the time invariant case, the problem addressed here is challenging. This is due to non definiteness (of the sign) of the quadratic term of the considered Riccati equations, making most of the existing methods in the literature (which are built on the assumption of definite sign of the quadratic term) useless. Our main result can be viewed as an extension of the results in [9] for the deterministic continuous-time invariant case and in [2], [3] for the stochastic continuous-time invariant case to the discrete-time time-varying case. This paper is organized as follows: Section 2 describes the problem setting. Section 3 gives some auxiliary results. The main results are given in Section 4. A numerical example is given in Section 5.

This work was supported by the Grant 145/2011, CNCS Romania

¹V. Dragan is with "Simion Stoilow" Institute of Mathematics of the Romanian Academy, P.O.Box 1-764, RO-014700, Bucharest, Romania Vasile.Dragan@imar.ro

²S. Aberkane is with Université de Lorraine, CRAN, UMR 7039, Campus Sciences, BP 70239, Vandoeuvre-les-Nancy Cedex, 54506, and with CNRS, CRAN, UMR 7039, France samir.aberkane@univ-lorraine.fr

³I. G. Ivanov is with Faculty of Economics and Business Administration, Sofia University, Sofia 1113, Bulgaria ivanov@feb.uni-sofia.bg

Notations. A^T stands for the transpose of the matrix A . \mathbb{I} is identity matrix of appropriate dimension. The expression $MN\star$ is equivalent to MNM^T .

II. THE PROBLEM SETTING

Consider the discrete-time Riccati equation (DTRE):

$$\begin{aligned} X(t) = & A^T(t)X(t+1)A(t) \\ & - [A^T(t)X(t+1)B(t) + C^T(t)D(t)] \\ & \times [R_\gamma(t) + B^T(t)X(t+1)B(t)]^{-1} \star \\ & + C^T(t)C(t), t \in \mathbb{Z} \end{aligned} \quad (\text{II.1})$$

where $A(t) \in \mathbb{R}^{n \times n}$, $B(t) = \begin{pmatrix} B_1(t) & B_2(t) \end{pmatrix}$, $B_k(t) \in \mathbb{R}^{n \times m_k}$, $k = 1, 2$, $C(t) \in \mathbb{R}^{p \times n}$, $D(t) = \begin{pmatrix} D_1(t) & D_2(t) \end{pmatrix}$, $D_k(t) \in \mathbb{R}^{p \times m_k}$, $k = 1, 2$, $R_\gamma(t) = D^T(t)D(t) + \begin{pmatrix} -\gamma^2 \mathbb{I} & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{m \times m}$, $m = m_1 + m_2$, $\gamma > 0$ is a given scalar. Throughout the paper we assume:

H1: There exists an integer $\theta \geq 1$ such that $A(t + \theta) = A(t)$, $B(t + \theta) = B(t)$, $C(t + \theta) = C(t)$, $D(t + \theta) = D(t)$ for $t \in \mathbb{Z}$.

In our approach the class of admissible solutions consists of all bounded sequences $\{X(t)\}_{t \in \mathbb{Z}} \subset \mathcal{S}_n$ satisfying (II.1) and the following two sign conditions:

$$\begin{aligned} & D_1^T(t)D_1(t) + B_1^T(t)X(t+1)B_1(t) \\ & - (B_1^T(t)X(t+1)B_2(t) + D_1^T(t)D_2(t)) \\ & \times (B_2^T(t)X(t+1)B_2(t) + D_2^T(t)D_2(t))^{-1} \star - \gamma^2 \mathbb{I} < 0 \end{aligned} \quad (\text{II.2})$$

$$D_2^T(t)D_2(t) + B_2^T(t)X(t+1)B_2(t) > 0, \quad (\text{II.3})$$

$t \in \mathbb{Z}$. Therefore we are interested by the global and bounded on \mathbb{Z} solutions for which the matrices $R_\gamma(t) + B^T(t)X(t+1)B(t)$, $t \in \mathbb{Z}$ are non-definite. Throughout the paper \mathcal{S}_d stands for the linear space of $d \times d$ real symmetric matrices.

Definition 2.1: An admissible solution $\{\tilde{X}(t)\}_{t \in \mathbb{Z}}$ is called **stabilizing solution**, if the zero state equilibrium of the discrete - time linear system on \mathbb{R}^n :

$$x(t+1) = [A(t) + B(t)\tilde{F}(t)]x(t) \quad (\text{II.4})$$

is exponentially stable, where

$$\begin{aligned} \tilde{F}(t) = & - [R_\gamma(t) + B^T(t)\tilde{X}(t+1)B(t)]^{-1} \\ & \times [B^T(t)\tilde{X}(t+1)A(t) + D^T(t)C(t)] \end{aligned} \quad (\text{II.5})$$

The stabilizing solution of DTRE (II.1) is involved in the designing of the solution of H_∞ control problem with

level of attenuation γ associated to the discrete - time linear system:

$$x(t+1) = A(t)x(t) + B_1(t)w(t) + B_2(t)u(t) \quad (\text{II.6})$$

and the cost functional

$$J(w(\cdot), u(\cdot)) = \sum_{t=-\infty}^{\infty} (\|C(t)x(t) + D_1(t)w(t) + D_2(t)u(t)\|^2 - \gamma^2\|w(t)\|^2) \quad (\text{II.7})$$

where $u(\cdot) \in \mathbb{R}^{m_2}$ are the control parameters and $w(\cdot) \in \ell_2(\mathbb{Z}, \mathbb{R}^{m_1})$ model the exogenous disturbances whose effect should be attenuated. For details see e.g. [1], [5], [7] and the references therein. We recall that $\ell_2(\mathbb{Z}, \mathbb{R}^{m_1}) = \{\{w(t)\}_{t \in \mathbb{Z}} : \sum_{t=-\infty}^{\infty} w^T(t)w(t) < \infty\}$.

The iterative procedures of Kleinman type (see e.g. [8]) cannot be directly applied for the numerical computation of the stabilizing solution of DTRE (II.1) because its quadratic term is non-definite sign.

In this paper we propose an iterative procedure for the numerical computation of the stabilizing solution $\tilde{X}(\cdot)$ of (II.1). This method extends to the discrete-time time-varying case the method developed in [9] for the deterministic continuous-time time - invariant case and in [2], [3], [4] for the stochastic continuous-time case.

Each iteration consists in the computation of the stabilizing solution of a suitable DTRE with defined sign of its quadratic part. The proposed algorithm may be described in the following steps:

Step 0. Take

$$X^{(0)}(t) = 0, \quad t \in \mathbb{Z}, \quad (\text{II.8})$$

and compute $Z^{(0)}(\cdot)$ as the stabilizing solution of the DTRE:

$$\begin{aligned} Z^{(0)}(t) &= A^T(t)Z^{(0)}(t+1)A(t) \\ &\quad - (A^T(t)Z^{(0)}(t+1)B_2(t) + C^T(t)D_2(t)) \\ &\quad \times \left(B_2^T(t)Z^{(0)}(t+1)B_2(t) + D_2^T(t)D_2(t) \right)^{-1} \star \\ &\quad + C^T(t)C(t) \end{aligned} \quad (\text{II.9})$$

Step k, $k \geq 1$. Take $X^{(k)}(t) = Z^{(k-1)}(t) + X^{(k-1)}(t)$ and compute $Z^{(k)}(\cdot)$ as the stabilizing solution of the DTRE with defined sign:

$$\begin{aligned} Z^{(k)}(t) &= (A(t) + B(t)F^{(k)}(t))^T Z^{(k)}(t+1) \star \\ &\quad - (A(t) + B(t)F^{(k)}(t))^T Z^{(k)}(t+1)B_2(t) \\ &\quad \times \left(R_2^{(k)}(t) + B_2(t)Z^{(k)}(t+1)B_2(t) \right)^{-1} \star \\ &\quad + M^{(k)}(t) \end{aligned} \quad (\text{II.10})$$

where

$$\begin{aligned} F^{(k)}(t) &= - \left(R_\gamma(t) + B^T(t)X^{(k)}(t+1)B(t) \right)^{-1} \\ &\quad \times (B^T(t)X^{(k)}(t+1)A(t) + D^T(t)C(t)) \end{aligned} \quad (\text{II.11})$$

$$R_2^{(k)}(t) = D_2^T(t)D_2(t) + B_2^T(t)X^{(k)}(t+1)B_2(t), \quad (\text{II.12})$$

$$\begin{aligned} M^{(k)}(t) &= A^T(t)X^{(k)}(t+1)A(t) \\ &\quad - (A^T(t)X^{(k)}(t+1)B(t) + C^T(t)D(t)) \\ &\quad \times \left(R_\gamma(t) + B(t)X^{(k)}(t+1)B(t) \right)^{-1} \star \\ &\quad + C^T(t)C(t) - X^{(k)}(t) \end{aligned} \quad (\text{II.13})$$

In section 4 we shall provide a set of assumptions which guarantee the fact that the sequences $\{X^{(k)}(t)\}_{k \geq 0}$, $\{Z^{(k)}(t)\}_{k \geq 0}$, $t \in \mathbb{Z}$ are well defined and convergente.

III. SEVERAL PRELIMINARY RESULTS

In this section, we will introduce some preliminary results that will be used in the proof of the main result of the paper. The proofs of these auxiliary results are not given due to page limitation.

A. Let us consider the so called **full information** control problem described by the controlled system (II.6) and the cost functional (II.7). This means that at each time instance t both the state vector $x(t)$ as well as the exogenous disturbance $w(t)$ are available for measurements. The class of admissible controls consists of the memoryless control laws $u_{KW}(t) = K(t)x(t) + W(t)w(t)$. Thus, for a given scalar $\gamma > 0$, \mathcal{A}_γ stands for the set of the pairs of the θ -periodic sequences $(\{K(t)\}_{t \in \mathbb{Z}}, \{W(t)\}_{t \in \mathbb{Z}}) \subset \mathbb{R}^{m_2 \times n} \times \mathbb{R}^{m_2 \times m_1}$ with the properties:

- (α) The closed-loop system: $x(t+1) = (A(t) + B_2(t)K(t))x(t)$, is exponentially stable.
- (β) $J(u_{KW}(\cdot), w(\cdot)) < 0$ for all $0 \neq w(\cdot) \in \ell_2(\mathbb{Z}, \mathbb{R}^{m_1})$.

Often we shall write $(K(\cdot), W(\cdot)) \in \mathcal{A}_\gamma$ instead of $(\{K(t)\}_{t \in \mathbb{Z}}, \{W(t)\}_{t \in \mathbb{Z}}) \in \mathcal{A}_\gamma$. Applying the discrete-time time-varying version of Bounded Real Lemma (see [5], [7]) to the system:

$$\begin{cases} x(t+1) = \bar{A}(t)x(t) + \bar{B}(t)w(t) \\ z(t+1) = \bar{C}(t)x(t) + \bar{D}(t)w(t) \end{cases} \quad (\text{III.1})$$

where

$$\begin{cases} \bar{A}(t) = A(t) + B_2(t)K(t) \\ \bar{B}(t) = B_1(t) + B_2(t)W(t) \\ \bar{C}(t) = C(t) + D_2(t)K(t) \\ \bar{D}(t) = D_1(t) + D_2(t)W(t) \end{cases}$$

one obtains:

Proposition 3.1: Under the assumption H1) the following are equivalent:

- (i) $(K(\cdot), W(\cdot)) \in \mathcal{A}_\gamma$,
- (ii) the discrete-time Riccati type equation

$$\begin{aligned} X(t) &= \bar{A}(t)^T X(t+1) \star - [\bar{A}(t)^T X(t+1)\bar{B}(t) \\ &\quad \bar{C}(t)^T \bar{D}(t)] \times [\gamma^2 \mathbb{I} - \bar{D}(t)^T \bar{D}(t) \\ &\quad - \bar{B}(t)^T X(t+1)\bar{B}(t)]^{-1} \star + \bar{C}(t)^T \bar{C}(t) \end{aligned} \quad (\text{III.2})$$

has a global solution $\{\tilde{X}_{KW}(t)\}_{t \in \mathbb{Z}}$ with the properties:

- a) $\tilde{X}_{KW}(t+\theta) = \tilde{X}_{KW}(t)$, $t \in \mathbb{Z}$;

- b) $\gamma^2\mathbb{I} - (D_1(t) + D_2(t)W(t))^T \star -(B_1(t) + B_2(t)W(t))^T \tilde{X}_{KW}(t+1) \star > 0, t \in \mathbb{Z};$
c) $\tilde{X}_{KW}(\cdot)$ is a stabilizing solution.

The next result follows from the developments in [5] or [7].

Theorem 3.2: Assume

- a) the assumption $H1)$ is fulfilled;
b) $D_2^T(t)D_2(t) > 0, t \in \mathbb{Z}$ and the pair $(\tilde{C}(\cdot), \tilde{A}(\cdot))$ is detectable, where $\tilde{C}(t) = [\mathbb{I} - D_2(t)(D_2^T(t)D_2(t))^{-1}D_2^T(t)]C(t)$ and $\tilde{A}(t) = A(t) - B_2(t)(D_2^T(t)D_2(t))^{-1}D_2^T(t)C(t)$.

Under these conditions the following are equivalent:

- i) \mathcal{A}_γ is not empty;
ii) The DTRE (II.1) has a bounded and stabilizing solution $\{\tilde{X}(t)\}_{t \in \mathbb{Z}}$. Moreover, this solution has the properties:
 $\alpha)$ $\tilde{X}(t+\theta) = \tilde{X}(t), t \in \mathbb{Z};$
 $\beta)$ $0 \leq \tilde{X}(t) \leq \tilde{X}(t), t \in \mathbb{Z}$ for arbitrary global and positive semidefinite solution $\{\tilde{X}(t)\}_{t \in \mathbb{Z}}$ of (II.1) satisfying the sign conditions (II.2) and (II.3).

Remark 3.1: Under the assumptions of Theorem 3.2 if $\{\tilde{X}(t)\}_{t \in \mathbb{Z}}$ is the stabilizing solution of DTRE (II.1) we define

$$\tilde{K}(t) = - \left[D_2^T(t)D_2(t) + B_2^T(t)\tilde{X}(t+1)B_2(t) \right]^{-1} \\ \times \left[B_2^T(t)\tilde{X}(t+1)A(t) + D_2^T(t)C(t) \right]$$

$$\tilde{W}(t) = - \left[D_2^T(t)D_2(t) + B_2^T(t)\tilde{X}(t+1)B_2(t) \right]^{-1} \\ \times \left[D_2^T(t)D_1(t) + B_2^T(t)\tilde{X}(t+1)B_1(t) \right]$$

$t \in \mathbb{Z}$.

Since the equation (II.1) satisfied by $\tilde{X}(\cdot)$ may be rewritten in the form of a Riccati equation of type (III.2) and $\tilde{X}_{\tilde{K}, \tilde{W}}(\cdot) \equiv \tilde{X}(\cdot)$ is the stabilizing solution of that Riccati equation, we may conclude that the pair $(\tilde{K}(\cdot), \tilde{W}(\cdot))$ lies in \mathcal{A}_γ . For more details see for example [1], [5].

B. The DTRE (II.1) may be written in a compact form of a nonlinear backward equation on \mathcal{S}_n :

$$X(t) = \mathcal{G}(t, X(t+1)) \quad (\text{III.3})$$

where $\mathcal{G}(\cdot, \cdot) : \text{Dom } \mathcal{G} \rightarrow \mathcal{S}_n$ is defined by the right hand side of (II.1) and $\text{Dom } \mathcal{G}$ consists of all pairs $(t, X) \in \mathbb{Z} \times \mathcal{S}_n$ satisfying the sign conditions (II.2) and (II.3) with $X(t+1)$ replaced by X .

Lemma 3.3: If $(t, Y), (t, Y+Z)$ are in $\text{Dom } \mathcal{G}$, then

$$F(t, Y+Z) - F(t, Y) = -[R_\gamma(t) + B^T(t)(Y+Z)B(t)] \\ \times B^T(t)Z(A(t) + B(t)F(t, Y)) \quad (\text{III.4})$$

where $F(t, Y)$ and $F(t, Y+Z)$ are designed via (III.5) for $X = Y$ and $X = Y+Z$, respectively.

$$F(t, X) = -[R_\gamma(t) + B^T(t)XB(t)]^{-1} \\ \times [B^T(t)XA(t) + D^T(t)C(t)] \quad (\text{III.5})$$

The proof can be done by direct calculation. The details are omitted.

In our developments an important role is played by the factorization

$$R_\gamma(t, X) + B^T(t)ZB(t) = \\ \left(\begin{array}{cc} V_{11}(t, X+Z) & 0 \\ V_{21}(t, X+Z) & V_{22}(t, X+Z) \end{array} \right)^T \left(\begin{array}{cc} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{array} \right) \star \quad (\text{III.6})$$

under the assumptions that $(t, X+Z) \in \text{Dom } \mathcal{G}$. Detailing (III.6) we have

$$-V_{11}^T(t, X+Z)V_{11}(t, X+Z) + V_{21}^T(t, X+Z) \\ \times V_{21}(t, X+Z) \\ = -\gamma^2\mathbb{I} + D_1^T(t)D_1(t) + B_1^T(t)(X+Z)B_1(t)$$

$$V_{22}^T(t, X+Z)V_{21}(t, X+Z) = D_2^T(t)D_1(t) \\ + B_2^T(t)(X+Z)B_1(t)$$

$$V_{22}^T(t, X+Z)V_{22}(t, X+Z) = D_2^T(t)D_2(t) \\ + B_2^T(t)(X+Z)B_2(t) \quad (\text{III.7})$$

We take

$$V_{22}(t, X+Z) = [D_2^T(t)D_2(t) + B_2^T(t)(X+Z)B_2(t)]^{\frac{1}{2}} \quad (\text{III.8})$$

$$V_{21}(t, X+Z) = [D_2^T(t)D_2(t) + B_2^T(t)(X+Z)B_2(t)]^{-\frac{1}{2}} \\ \times [D_2^T(t)D_1(t) + B_2^T(t)(X+Z)B_1(t)] \quad (\text{III.9})$$

$$V_{11}(t, X+Z) = [\gamma^2\mathbb{I} - D_1^T(t)D_1(t) - B_1^T(t)(X+Z)B_1(t) \\ + [D_1^T(t)D_2(t) + B_1^T(t)(X+Z)B_2(t)] \\ \times [D_2^T(t)D_2(t) + B_2^T(t)(X+Z)B_2(t)]^{-1} \star]^{\frac{1}{2}} \quad (\text{III.10})$$

Lemma 3.4: Let $\{X(t)\}_{t \in \mathbb{Z}}, \{Z(t)\}_{t \in \mathbb{Z}}$ be two sequences from \mathcal{S}_n . Assume:

- a) $(t, X(t+1)), (t, X(t+1) + Z(t+1))$ are in $\text{Dom } \mathcal{G}$ for all $t \in \mathbb{Z}$;
b) the sequences under consideration are interconnected via the equality:

$$Z(t) = [A(t) + B(t)F(t, X(t+1))]^T Z(t+1) \star \\ - [A(t) + B(t)F(t, X(t+1))]^T Z(t+1)B_2(t) \\ \times [R_2(t, X(t+1)) + B_2^T(t)Z(t+1)B_2(t)]^{-1} \star \\ + \mathcal{G}(t, X(t+1)) - X(t), \quad t \in \mathbb{Z} \quad (\text{III.11})$$

Under these conditions we have

$$\mathcal{G}(t, X(t+1) + Z(t+1)) - [X(t) + Z(t)] \\ = [(A(t) + B(t)F(t, X(t+1)))^T Z(t+1) \\ \times \hat{B}(t, X(t+1) + Z(t+1))] V_{11}^{-2}(t, X(t+1) \\ + Z(t+1)) \star, \quad t \in \mathbb{Z}. \quad (\text{III.12})$$

The next result play a crucial role in the proof of the convergence of the iterative algorithm described by (II.8) -

(II.13). It is the discrete-time time-varying counterpart of Lemma 2 in [9]. (δ_k)

Lemma 3.5: Let $\{X(t)\}_{t \in \mathbb{Z}} \subset \mathcal{S}_n$ and $\{Z(t)\}_{t \in \mathbb{Z}} \subset \mathcal{S}_n$ be bounded sequences interconnected via the equality:

$$\begin{aligned} Z(t) = & [A(t) + B(t)F(t, X(t+1))]^T Z(t+1) [A(t) + \\ & B(t)F(t, X(t+1))] \\ & - [A(t) + B(t)F(t, X(t+1))]^T Z(t+1) B_2(t) \\ & \times [R_2(t, X(t+1)) + B_2^T(t)Z(t+1)B_2(t)]^{-1} \star \\ & + \mathcal{G}(t, X(t+1)) - X(t) \end{aligned} \quad (\text{III.13})$$

Let $(K(\cdot), W(\cdot))$ be an arbitrary pair of gain matrices belonging to $\mathcal{A}(\gamma)$ and $\{\tilde{X}_{KW}(t)\}_{t \in \mathbb{Z}}$ be the bounded and stabilizing solution of the corresponding DTRE (III.2). Set $\hat{A}(t) = A(t) + [B_1(t) + B_2(t)W(t)]F_1(t, X(t+1)) + B_2(t)K(t)$ and $\check{A}(t) = A(t) + [B_1(t) + B_2(t)W(t)]F_1(t, X(t+1) + Z(t+1)) + B_2(t)K(t)$, $t \in \mathbb{Z}$, with $F_1(t, X(t+1)) = \begin{pmatrix} \mathbb{I} & 0 \end{pmatrix} F(t, X(t+1))$ and $F_1(t, X(t+1) + Z(t+1)) = \begin{pmatrix} \mathbb{I} & 0 \end{pmatrix} F(t, X(t+1) + Z(t+1))$. Under the considered assumptions the following statements are true:

- i) $X(t) + Z(t) \leq \tilde{X}_{KW}(t)$, $t \in \mathbb{Z}$ if the zero solution of the discrete-time linear system $x(t+1) = \hat{A}(t)x(t)$ is exponentially stable;
- ii) the zero solution $x(t) \equiv 0$ of the discrete-time linear system $x(t+1) = \check{A}(t)x(t)$ is exponentially stable if $X(t) + Z(t) \leq \tilde{X}_{KW}(t)$, $t \in \mathbb{Z}$

Lemma 3.6: Assume

- a) $D_2^T(t)D_2(t) > 0$, $\forall t \in \mathbb{Z}$;
- b) for a scalar $\gamma > 0$ the set $\mathcal{A}(\gamma)$ is not empty.

Under these conditions the following hold:

- i) $(t, 0) \in \text{Dom } \mathcal{G}$ for all $t \in \mathbb{Z}$.
- ii) For any $(K(\cdot), W(\cdot)) \in \mathcal{A}(\gamma)$, we have $(t, \tilde{X}_{KW}(t+1)) \in \text{Dom } \mathcal{G}$, $\forall t \in \mathbb{Z}$, $\tilde{X}_{KW}(\cdot)$ being the bounded and stabilizing solution of the corresponding DTRE of type (III.2).

IV. CONVERGENCE OF ALGORITHM

In this section we shall use the results of the previous section to show that the sequences $\{X^{(k)}(t)\}_{k \geq 0}$ and $\{Z^{(k)}(t)\}_{k \geq 0}$, $t \in \mathbb{Z}$ are well defined via (II.9) to (II.13) and they are convergent.

Theorem 4.1: Assume

- a) the assumptions of Theorem 3.2 are fulfilled;
- b) the set $\mathcal{A}(\gamma)$ is not empty.

Under these conditions, the sequences $\{X^{(k)}(t)\}_{k \geq 0}$ and $\{Z^{(k)}(t)\}_{k \geq 0}$, $t \in \mathbb{Z}$ introduced via (II.9) - (II.13) are well defined and for each $k \geq 0$ have the properties:

- (α_k) If $(K(\cdot), W(\cdot)) \in \mathcal{A}(\gamma)$ and $\tilde{X}_{KW}(\cdot)$ is the stabilizing solution of the corresponding DTRE (III.2), then $X^{(k)}(t) + Z^{(k)}(t) \leq \tilde{X}_{KW}(t)$, $t \in \mathbb{Z}$.
- (β_k) $(t, X^{(k+1)}(t+1)) \in \text{Dom } \mathcal{G}$ for all $t \in \mathbb{Z}$.
- (γ_k) The zero solution of system (IV.1) is exponentially stable

$$\begin{aligned} x(t+1) = & [A(t) + (B_1(t) + B_2(t)W(t))F_1^{(k+1)}(t) \\ & + B_2(t)K(t)]x(t) \end{aligned} \quad (\text{IV.1})$$

$$\begin{aligned} & \mathcal{G}(t, X^{(k+1)}(t+1)) - X^{(k+1)}(t) \\ & = [A(t) + B(t)F^{(k)}(t)]^T Z^{(k)}(t+1) \hat{B}^{(k+1)}(t) \\ & \times \left(V_{11}^{(k+1)}(t) \right)^{-2} \star \end{aligned} \quad (\text{IV.2})$$

for $k \geq 1$, while:

$$\begin{aligned} & \mathcal{G}(t, X^{(1)}(t+1)) - X^{(1)}(t) \\ & = [A(t)Z^{(0)}(t+1) \hat{B}^{(1)}(t) + C^T(t) \hat{D}^{(1)}(t)] \\ & \times \left(V_{11}^{(1)}(t) \right)^{-2} \end{aligned} \quad (\text{IV.3})$$

with:

$$\hat{D}^{(1)}(t) = D_1(t) - D_2(t) \left(V_{22}^{(1)}(t) \right)^{-1} V_{21}^{(1)}(t) \quad (\text{IV.4})$$

$$\begin{aligned} \hat{B}^{(k+1)}(t) = & B_1(t) - B_2(t) \left(V_{22}(t, X^{(k+1)}(t+1)) \right)^{-1} \\ & \times V_{21}(t, X^{(k+1)}(t+1)) \end{aligned} \quad (\text{IV.5})$$

$$V_{11}^{(k+1)}(t) = V_{11}(t, X^{(k+1)}(t+1)) \quad (\text{IV.6})$$

The matrix $V_{ij}^{(k)}(t)$ being defined as in (III.8)-(III.10) for $X + Z = X^{(k)}(t+1) + Z^{(k)}(t+1) = X^{(k+1)}(t+1)$.

Furthermore, the sequences $\{X^{(k)}(t)\}_{k \geq 0}$ and $\{Z^{(k)}(t)\}_{k \geq 0}$, $t \in \mathbb{Z}$ are convergent and we have $\lim_{k \rightarrow \infty} X^{(k)}(t) = \tilde{X}(t)$, $\lim_{k \rightarrow \infty} Z^{(k)}(t) = 0$, $t \in \mathbb{Z}$, $\tilde{X}(\cdot)$ being the stabilizing and θ -periodic solution of (II.1).

Proof: The proof is done by recurrence.

Step 1. In this step, we show that properties $(\alpha_k) - (\delta_k)$ are fulfilled for $k = 0$. Let $k = 0$. The fact that $D_2^T(t)D_2(t) > 0$ allows us to rewrite (II.9) in the form:

$$\begin{aligned} Z(t) = & \check{A}^T(t)Z(t+1)\check{A}(t) - [\check{A}^T(t)Z(t+1)B_2(t)] \\ & \times [D_2^T(t)D_2(t) + B_2^T(t)Z(t+1)B_2(t)]^{-1} \star \\ & + \check{C}^T(t)\check{C}(t) \end{aligned} \quad (\text{IV.7})$$

where

$$\check{A}(t) = A(t) - B_2(t)(D_2^T(t)D_2(t))^{-1}D_2^T(t)C(t)$$

$$\check{C}(t) = [I_p - D_2(t)(D_2^T(t)D_2(t))^{-1}D_2^T(t)]C(t).$$

Based on the fact that \mathcal{A}_γ is assumed to be nonempty we deduce that $(\check{A}(\cdot), B_2(\cdot))$ is stabilizable. Having in mind the detectability of the pair $(\check{C}(\cdot), \check{A}(\cdot))$ one obtains a set of sufficient conditions which guarantee the existence of the stabilizing solution $Z^{(0)}(\cdot)$ of the DTRE (IV.7) or equivalently (II.9).

Additionally, this solution is θ -periodic and positive semidefinite. The fact that $Z^{(0)}(\cdot)$ is the stabilizing solution of (II.9) means that the zero solution of the linear system

$$x(t+1) = [A(t) + B_2(t)\Phi_2^{(0)}(t)]x(t)$$

is exponentially stable, where

$$\begin{aligned} \Phi_2^{(0)}(t) &= - \left[D_2^T(t) D_2(t) + B_2^T(t) Z^{(0)}(t+1) B_2(t) \right]^{-1} \\ &\quad \times \left[B_2^T(t) Z^{(0)}(t+1) A(t) + D_2^T(t) C(t) \right] \end{aligned} \quad (\text{IV.8})$$

By direct calculation one obtains that $\tilde{X}_{KW}(\cdot) - Z^{(0)}(\cdot)$ is the θ -periodic solution of the discrete-time backward affine equation

$$\begin{aligned} Y(t) &= [A(t) + B_2(t)K(t)]^T Y(t+1) [A(t) + B_2(t)K(t)] \\ &\quad + M^{(0)}(t) \end{aligned} \quad (\text{IV.9})$$

where

$$\begin{aligned} M^{(0)}(t) &= \tilde{F}_{KW}^T(t) \Pi_{KW}(t) \tilde{F}_{KW}(t) + [K(t) - \Phi_2^{(0)}(t)]^T \\ &\quad \times \left(D_2^T(t) D_2(t) + B_2^T(t) Z^{(0)}(t+1) B_2(t) \right) \star \end{aligned} \quad (\text{IV.10})$$

Therefore, $M^{(0)}(t) \geq 0$ for all $t \in \mathbb{Z}$. Since $A(\cdot) - B_2(\cdot)K(\cdot)$ defines an exponentially stable evolution, we may deduce that the equation (IV.9) has a unique θ -periodic solution and additionally, that solution is positive semidefinite. So, we may conclude that

$$\tilde{X}_{KW}(t) \geq Z^{(0)}(t) = Z^{(0)}(t) + X^{(0)}(t), \quad t \in \mathbb{Z}. \quad (\text{IV.11})$$

Thus, we have shown that (α_k) is true for $k = 0$. Further on, (IV.11) yields

$$\begin{aligned} R_\gamma(t) + B^T(t) (Z^{(0)}(t+1) + X^{(0)}(t+1)) B(t) \\ \leq R_\gamma(t) + B^T(t) \tilde{X}_{KW}(t+1) B(t) \end{aligned} \quad (\text{IV.12})$$

for all $t \in \mathbb{Z}$. Combining Corollary 4.5 in [6] and Lemma 3.6 from above, we obtain that (IV.12) leads to

$$\begin{aligned} D_1^T(t) D_1(t) + B_1^T(t) X^{(1)}(t+1) B_1(t) \\ - (D_1^T(t) D_2(t) + B_1^T(t) X^{(1)}(t+1) B_2(t)) \\ \times \left[D_2^T(t) D_2(t) + B_2^T(t) X^{(1)}(t+1) B_2(t) \right]^{-1} \star - \gamma^2 I_{m_1} < 0 \end{aligned} \quad (\text{IV.13})$$

for all $t \in \mathbb{Z}$. Since $D_2^T(t) D_2(t) + B_2^T(t) X^{(1)}(t+1) B_2(t) > 0$ we may conclude via (IV.13) that (β_k) is true for $k = 0$. As a consequence one obtains that $F^{(1)}(t)$ is well defined via (II.11) with $k = 1$. Set $F_1^{(1)}(t) = \begin{bmatrix} I_{m_1} & 0 \end{bmatrix} F^{(1)}(t)$. To check that (γ_k) is true for $k = 0$, we have to show that the zero solution of the discrete-time linear equation

$$x(t+1) = [A(t) + (B_1(t) + B_2(t)W(t))F_1^{(1)}(t) + B_2(t)K(t)]x(t)$$

is exponentially stable. This fact follows immediately applying Lemma 3.5 (ii) for $X(t) = X^{(0)}(t) = 0$, $Z(t) = Z^{(0)}(t)$ for all $t \in \mathbb{Z}$. It remains to check that (δ_k) is true for $k = 0$. To this end, we apply Lemma 3.4 and obtain

$$\begin{aligned} \mathcal{G}(t, X^{(1)}(t+1) - X^{(1)}(t)) &= \\ \left[(A(t) + B(t)F^{(0)}(t))^T Z^{(0)}(t+1) \hat{B}^{(1)}(t) \right] &\left(V_{11}^{(1)}(t) \right)^{-2} \star \end{aligned} \quad (\text{IV.14})$$

where $F^{(0)}(t) = -R_\gamma^{-1}(t)D^T(t)C(t)$ while $\hat{B}^{(1)}(t)$ and $V_{11}^{(1)}(t)$ are defined via (IV.5) and (IV.6), respectively for $k = 1$.

Step 2. Let us now show that if the properties $(\alpha_0) - (\delta_0)$ hold $Z^{(1)}$ is well defined as the stabilizing solution of the DTRE (II.10) for $k = 1$. Using (IV.14) we write (II.10) for $k = 1$ in the form:

$$\begin{aligned} Z^{(1)}(t) &= (A^{(1)}(t))^T Z^{(1)}(t+1) A^{(1)}(t) \\ &\quad - \left[(A^{(1)}(t))^T Z^{(1)}(t+1) B_2(t) \right] \\ &\quad \times \left[R^{(1)}(t) + B_2^T(t) Z^{(1)}(t+1) B_2(t) \right]^{-1} \star \\ &\quad + (C^{(1)}(t))^T C^{(1)}(t) \end{aligned} \quad (\text{IV.15})$$

where we denoted $A^{(1)}(t) = A(t) + B(t)F^{(1)}(t)$, and

$$\begin{aligned} C^{(1)}(t) &= \\ (V_{11}^{(1)}(t))^{-1} \left(\hat{B}^{(1)}(t) Z^{(0)}(t+1) A(t) + (\hat{D}^{(1)}(t))^T C(t) \right) \end{aligned} \quad (\text{IV.16})$$

To show that (IV.15) has a θ -periodic and stabilizing solution it is sufficient to show that $(A^{(1)}(\cdot), B_2(\cdot))$ is stabilizable and $(C^{(1)}(\cdot), A^{(1)}(\cdot))$ is detectable.

Choose $K^{(1)}(t) = \begin{bmatrix} W(t) & -I_{m_2} \end{bmatrix} F^{(1)}(t) + K(t)$. By direct calculations one obtains $A^{(1)}(t) + B_2(t)K^{(1)}(t) = A(t) + (B_1(t) + B_2(t)W(t))F^{(1)}(t) + B_2(t)K(t)$ which defines an exponentially stable evolution because (γ_0) is true. So, we have shown that $(A^{(1)}(\cdot), B_2(\cdot))$ is stabilizable.

Further on, we remark that if we use again the factorization (III.6) - (III.7) we obtain $A^{(1)}(t) = A(t) + B_1(t)(V_{11}^{(1)}(t))^{-1} C^{(1)}(t) + B_2(t)\Phi_2^{(0)}(t)$. If we take $L^{(1)}(t) = -B_1(t)(V_{11}^{(1)}(t))^{-1}$ we obtain that $A^{(1)}(t) + L^{(1)}(t)C^{(1)}(t) = A(t) + B_2(t)\Phi_2^{(0)}(t)$. Therefore, the zero solution of the discrete-time linear equation

$$x(t+1) = (A^{(1)}(t) + L^{(1)}(t)C^{(1)}(t))x(t)$$

is exponentially stable.

Thus we have shown that $(C^{(1)}(\cdot), A^{(1)}(\cdot))$ is detectable. So, we may conclude that (IV.15) has a bounded and stabilizing solution $\{Z^{(1)}(t)\}_{t \in \mathbb{Z}}$. Moreover, this solution is θ -periodic and $Z^{(1)}(t) \geq 0$ for all $t \in \mathbb{Z}$. Set $X^{(2)}(t) = X^{(1)}(t) + Z^{(1)}(t)$ for all $t \in \mathbb{Z}$. We have

$$0 \leq X^{(1)}(t) \leq X^{(2)}(t). \quad (\text{IV.17})$$

Step 3. Let $k \geq 1$ and assume that the θ -periodic sequences $\{X^{(s)}(t)\}_{t \in \mathbb{Z}}, \{Z^{(s)}(t)\}_{t \in \mathbb{Z}}$ are well defined for any $0 \leq s \leq k$; $\{Z^{(s)}(t)\}$ being the θ -periodic stabilizing solution of the equation (II.10) for k replaced by s . Let us assume that the $(\alpha_s) - (\delta_s)$ are true for $0 \leq s \leq k-1$. Using similar arguments as in **Step 1**, we show in this step that $(\alpha_s) - (\delta_s)$ hold for $s = k$. The details of the derivation are omitted due to page limitation.

Step 4. In this step, using similar arguments as in **Step 2**, we show that the Riccati equation (II.10) written for k

replaced by $k + 1$, has a bounded and stabilizing solution $\{Z^{(k+1)}(\cdot)\}$. The details are omitted.

Step 5. Using the results in Steps 1-4, we conclude that the sequences $\{X^{(k)}(t)\}_{k \geq 0}, \{Z^{(k)}(t)\}_{k \geq 0}, t \in \mathbb{Z}$ are well defined and

$$0 \leq X^{(k)}(t) \leq X^{(k+1)}(t) \leq \tilde{X}_{KW}(t) \quad (IV.18)$$

for all $t \in \mathbb{Z}$. This means that for each $t \in \mathbb{Z}$ the sequence $\{X^{(k)}(t)\}_{k \geq 0}$ is convergent.

Let

$$\tilde{Y}(t) = \lim_{k \rightarrow \infty} X^{(k)}(t). \quad (IV.19)$$

Taking the limit for $k \rightarrow \infty$ in (IV.18) we obtain that

$$0 \leq \tilde{Y}(t) \leq \tilde{X}_{KW}(t), \quad t \in \mathbb{Z} \quad (IV.20)$$

for every pair $(K(\cdot), W(\cdot)) \in \mathcal{A}(\gamma)$. Based on Remark 3.1 and (IV.20) we have

$$0 \leq \tilde{Y}(t) \leq \tilde{X}(t), \quad t \in \mathbb{Z}, \quad (IV.21)$$

$\tilde{X}(t)$ being the stabilizing solution of (II.1).

Taking the limit for $k \rightarrow \infty$ in (II.10) - (II.13) we obtain that $\{\tilde{Y}(t)\}_{t \in \mathbb{Z}}$ is a positive semidefinite and θ -periodic solution of (II.1). Based on the minimality property of the stabilizing solution of the Riccati equation (II.1) we deduce that $\tilde{X}(t) \leq \tilde{Y}(t), t \in \mathbb{Z}$. Hence

$$\tilde{Y}(t) = \tilde{X}(t). \quad (IV.22)$$

Finally, let us remark that from $Z^{(k)}(t) = X^{(k+1)}(t) - X^{(k)}(t)$ one gets that $\lim_{k \rightarrow \infty} Z^{(k)}(t) = 0$. Thus, the proof is complete. ■

V. NUMERICAL EXAMPLE

Consider a discrete-time 3-periodic linear system described by

$$A(0) = \begin{bmatrix} -0.3 & 0 & 0.1 \\ 1 & 0.5 & 0 \\ 3 & -2 & 0.4 \end{bmatrix}, A(1) = \begin{bmatrix} -0.5 & 0.5 & 0 \\ 1.3 & 1.6 & 0 \\ 3 & -2.5 & 0.7 \end{bmatrix},$$

$$A(2) = \begin{bmatrix} -0.3 & 0.2 & 0 \\ 2 & 0.3 & 0.1 \\ 3 & -2 & 1.2 \end{bmatrix}, B_1(0) = \begin{bmatrix} -1 \\ 0.1 \\ 0 \end{bmatrix},$$

$$B_1(1) = \begin{bmatrix} 1 \\ 0.5 \\ 1 \end{bmatrix}, B_1(2) = \begin{bmatrix} 1 \\ 1 \\ 0.8 \end{bmatrix}, B_2(0) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

$$B_2(1) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, B_2(2) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, C(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T,$$

$$C(1) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^T, C(2) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^T, D_1(0) = 0.3, D_1(1) = 0.1,$$

$$D_1(2) = 0.5, D_2(0) = 0.1, D_2(1) = 0.2, D_2(2) = 0.1$$

Our experiments are executed using MATLAB on a 2, 53 GHz Intel Core 2 Duo computer. In our computations we use a tolerance (for the stopping criteria) $\text{tol} = 1e - 10$ and $\gamma^2 = 6$. The algorithm converges to the stabilizing solution after 3 main iterations leading to:

$$\tilde{X}(0) = \begin{bmatrix} 6.4450 & 5.8143 & -1.183 \\ 5.8143 & 5.8770 & -1.187 \\ -1.183 & -1.187 & 0.2401 \end{bmatrix},$$

$$\tilde{X}(1) = \begin{bmatrix} 9.9175 & -8.067 & 2.2583 \\ -8.067 & 6.6908 & -1.813 \\ 2.2583 & -1.813 & 0.5185 \end{bmatrix},$$

$$\tilde{X}(2) = \begin{bmatrix} 0.1025 & -0.043 & -0.149 \\ -0.043 & 0.0551 & -0.110 \\ -0.149 & -0.110 & 1.0468 \end{bmatrix}.$$

VI. CONCLUSION

In this paper, we have addressed the problem of computing the stabilizing solutions of a class of periodic discrete-time Riccati equation with an non-definite sign of its quadratic term. We have proposed a convergent iterative algorithm and showed its efficiency on some numerical examples.

REFERENCES

- [1] S. Bittanti, P.Colaneri, *Periodic Systems, Filtering and Control*, Springer- Verlag, London, 2009.
- [2] V. Dragan, I. Ivanov: A numerical procedure to compute the stabilizing solution of game theoretic Riccati equations of stochastic control, *International Journal of Control*, 84,4 (2011), pag. 783- 800.
- [3] V. Dragan, I. Ivanov : Computation of the stabilizing solution of game theoretic Ric- cati equation arising in stochastic H_∞ control problems, *Numerical Algorithms*, 57,3,(2011), pag. 357- 375.
- [4] Y. Feng, B.D.O. Anderson, An iterative algorithm to solve state- perturbed stochastic algebraic Riccati equations in LQ zero-sum games, *Systems and Control Letters*, 59, (2010), 50-56.
- [5] A. Halanay, V. Ionescu, *Time varying discrete linear systems*, Berlin, Birkhauser, 1994.
- [6] G. Freiling, A. Hochhaus, Properties of the solutions of rational matrix difference equations, *Advances in difference equations, IV, Comput. Math. Appl.*, 45 (2003), 1137-1154.
- [7] A. Ichikawa, H. Katayama, *Linear Time Varying Systems and Sampled- Data*, Lecture Notes in control and Information Sciences 265, Springer, New York, 2001.
- [8] D. L. Kleinman, On an iterative technique for Riccati equation computation, *IEEE Transactions on Automatic Control*, 13, 1968, 114- 115.
- [9] A. Lanzon, Y. Feng, B. D. O. Anderson and M. Rotkowitz, Computing the Positive Stabilizing Solution to Algebraic Riccati Equations with an Indefinite Quadratic Term via a Recursive Method, *IEEE Trans. Automat. Contr.* , 53, 10, 2280-2291, 2008.