2013 European Control Conference (ECC)
July 17-19, 2013, Zürich, Switzerland.

What to do when hybrid systems "freeze" due to an interconnection?

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Abstract— **The paper addresses the question of a composition and decomposition of hybrid systems. Motivated by a simple example, we propose an extended definition of a hybrid system that allows for a natural and simple way to consider an interconnection of several hybrid systems as one hybrid system and to decompose one large hybrid system in a composition of several ones. A stability result and existence of solutions for the new framework are shown.**

I. INTRODUCTION

Hybrid systems represent an important class of dynamical systems from theoretical and practical point of view [28]. Many papers related to mathematical description and foundations related to this class of systems appeared recently [2], [15], [19], [5], [10], [16]. Another currently active research field is related to interconnected large-scale systems [3], [8], [9], [12], [13] and [20]. Interconnections of hybrid systems were considered in [7], [14], [17], [21] and [27]. However it turns out, that a description of an interconnection in case of hybrid systems is not a trivial issue. For example, stability results for interconnections are possible only under some restrictive and physically unnatural conditions, see [6], [21]. In particular a very natural way to consider such interconnections leads to the existence of solutions that are physically meaningless. This will be illustrated with a simple example of an interconnection of two bouncing balls that are connected by an elastic spring. The aim of our paper is to discuss these kind of problems that occur for interconnections of hybrid systems and to suggest a possible way out to solve them. In particular a more general framework for a definition of a hybrid system is introduced that takes into account the cases when one of the subsystems jumps while another one flows continuously at the same time instant. This extension allows to cover more general class of interconnected hybrid systems in view of stability properties. The existence of solutions is also discussed in this framework.

The structure of the paper is as follows. In Section II, we recall a standard definition and an example of hybrid systems. The motivation of this paper is illustrated by an example in Section III. In Section IV, our modified description of hybrid systems is introduced. A stability result and the existence of solutions are provided on Section V and Section VI respectively.

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II. PRELIMINARIES

Throughout this paper we use the following notation. Let us denote $\mathbb{N}_+ := \{0, 1, 2, \ldots\}$. By $\mathbb{R}_+$ we denote the set of nonnegative real numbers, $\mathbb{R}^n_+$ is then the positive orthant $\{x \in \mathbb{R}^n : x \geq 0\}$. By x^T we denote the transposition of a vector $x \in \mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^n$. The Euclidean norm is denoted by $|\cdot|$. The *tangent cone* to a set $C \subset \mathbb{R}^n$ at $x \in C$, denoted by $T_C(x)$, is the set of all $v \in \mathbb{R}^n$ for which there exist real numbers $\alpha_i \searrow 0$ and vectors $v_i \to v$ such that for $i = 1, 2, \ldots, x + \alpha_i v_i \in C$, see [11], [4]. Define the set $\Omega(\delta, \Delta) := \{x \in \mathbb{R}^n : \delta < |x| < \Delta\}$.

With the aim to handle interconnections we consider systems with inputs. A hybrid system with input can be defined as follows [5], [10]:

Consider a system with state $x \in \chi \subset \mathbb{R}^N$, input $u \in U \subset \mathbb{R}^M$ and dynamics given by

$$\begin{array}{rcll} \dot{x} & = & f(x,u), & (x,u) \in C, \\ x^+ & = & g(x,u), & (x,u) \in D, \end{array} \tag{1}$$

where $f : C \to \mathbb{R}^N, g : D \to \chi$ with $C, D \subset \chi \times U$.

This hybrid system is described by (f, g, C, D, χ, U). Function f describes continuous dynamics defined on the set C, function g describes instantaneous jumps defined on the set D. For the existence of solutions we assume that the following basic regularity conditions [5], [11] hold throughout the paper:

1) χ is open, U is closed, and $C, D \subset \chi \times U$ are relatively closed in $\chi \times U$;
2) f, g are continuous.

Note that these conditions do not guarantee the uniqueness of solutions. For discussion of uniqueness of solutions and their continuous dependence on initial conditions for hybrid systems we refer to [11] and [15]. The solutions to hybrid systems are defined as follows, cf. [23], [10], [17].

Definition 2.1 (Hybrid Time Domain): A subset of $\mathbb{R}_+ \cup \mathbb{N}_+$ is called a hybrid time domain, denoted by dom, if it is given as a union of finitely or infinitely many intervals $[t_k, t_{k+1}] \times \{k\}$, where the numbers $0 = t_0, t_1, \ldots$ form a finite or infinite, nondecreasing sequence of real numbers. The "last" interval is allowed to be of the form $[t_K, T) \times \{k\}$ with T finite or $T = +\infty$.

Definition 2.2: A hybrid signal is a function defined on a hybrid time domain. The hybrid input is a hybrid signal $u : \mathrm{dom}\ u \to U \subset \mathbb{R}^M$ such that $u(\cdot, k)$ is Lebesgue measurable and locally essentially bounded for each k. We call u *external input*.

Definition 2.3: A hybrid arc is such a hybrid signal $x : \mathrm{dom}\ x \to \chi$, that $x(\cdot, k)$ is locally absolutely continuous for each k.

Definition 2.4 (Solution to a Hybrid System): A hybrid arc and a hybrid input is a solution pair (x, u) to the hybrid system (1) if

(i) $\text{dom } x = \text{dom } u$ and $(x(0,0), u(0,0)) \in C \cup D$,

(ii) for all $k \in N_+$ and almost all $(t,k) \in \text{dom } x$, for $(x(t,k), u(t,k)) \in C$, holds

$$\dot{x}(t,k) = f(x(t,k), u(t,k)) \tag{2}$$

(iii) for all $(t,k) \in \text{dom } x$ such that $(t, k+1) \in \text{dom } x$, for $(x(t,k), u(t,k)) \in D$ holds

$$x(t, k+1) = g(x(t,k), u(t,k)). \tag{3}$$

System (1) without input, i.e. with $u \equiv 0$, looks as follows:

$$\begin{array}{rcll} \dot{x} & = & f(x), & x \in C, \\ x^+ & = & g(x), & x \in D. \end{array} \tag{4}$$

We consider the following types of stability for hybrid systems without inputs, see [26], [5].

Definition 2.5 (GS): System (4) is called *globally stable (GS)*, if there exists $\sigma \in \mathcal{K}_\infty$ such that any solution x satisfies

$$|x(t,k)| \leq \sigma(|x_0|), \forall (t,k) \in \text{dom } x. \tag{5}$$

Definition 2.6 (GA): System (4) is called *globally attractive (GA)*, if for each $\epsilon > 0$ and $r > 0$ there exists $T > 0$ such that, for any solution x, $|x(0,0)| \leq r$, $(t,k) \in \text{dom}, x$, and $t + k \geq T$ imply $|x(t,k)| \leq \epsilon$.

Definition 2.7 (GAS): System (4) is called *globally asymptotically stable (GAS)*, if it is both GS and GA.

One of the standard examples for a hybrid system is the bouncing ball [10], [23]:

$$\dot{x} = \begin{pmatrix} x_2 \\ -\gamma \end{pmatrix} =: f(x), \quad x \in C, \tag{6}$$

$$x^+ = \begin{pmatrix} x_1 \\ -\lambda x_2 \end{pmatrix} =: g(x), \quad x \in D, \tag{7}$$

where x_1 denotes the vertical position and x_2 the velocity of the ball, $C := \{x \in \mathbb{R}^2 \times U : x_1 \geq 0\}$ and $D := \{x \in \mathbb{R}^2 \times U : x_1 = 0, x_2 \leq 0\}$, γ represents the gravitation force, $\lambda \in (0,1)$ called restitution coefficient, is the number describing the share of the energy that remains after each inelastic collision with the floor.

It is known that the resting state of this hybrid system is globally asymptotically stable [25]. Let us consider some solutions of this system.

For a given initial condition, say $x_1(t_0) = x_1(0) = h$ and $x_2(t_0) = x_2(0) = 0$ in case there is no external input $u \equiv 0$ the solution can be written as follows. The first arc until the first jump (continuous flow between t_0 and t_1) is given by

$$x_1(t) = -\frac{1}{2}\gamma t^2 + h, \tag{8}$$

$$x_2(t) = -\gamma t, \tag{9}$$

where $t_1 = \sqrt{\frac{2h}{\gamma}}$ is the time of the first touch with the ground. The further arcs (between t_j and t_{j+1}) are given by

$$x_1(t) = -\frac{1}{2}\gamma t^2 + (x_2^j + \gamma t_j)t - (\frac{1}{2}\gamma t_j^2 + x_2^j t_j), \tag{10}$$

$$x_2(t) = -\gamma t + (x_2^j + \gamma t_j), \tag{11}$$

and the states x_1^j, x_2^j after the jump at t_j are given by

$$x_1^j := x_1^+(t_j) = 0 \tag{12}$$

$$x_2^j := x_2^+(t_j) = -\lambda(-\gamma t_j + (x_2^{j-1} + \gamma t_{j-1})), \tag{13}$$

where $t_j := \frac{2x_2^{j-1} + \gamma t_{j-1}}{2\gamma}$ is the time of the jth jump. Then the total amount of time that the system spends in C is

$$t_{\max} = \sqrt{\frac{2h}{\gamma}} + \frac{2\lambda}{1-\lambda}\sqrt{\frac{2h}{\gamma}}. \tag{14}$$

Note that $t_{\max}$ is finite due to $\lambda \in (0,1)$ and there are infinitely many jumps until $t_{\max}$. Sometimes such $t_{\max}$ is called Zeno-time, see [1], [18] and [29]. There is no more continuous flow after $t_{\max}$. Each interval of the hybrid time domain is of the from $[\cdot, t] \times \{\cdot\}$ such that $t < t_{\max}$, i.e., the solutions never, in mathematical sense, reach to $t_{\max}$.

As mentioned above, the solution of this system corresponding to the resting state $(0,0)$ is GAS. Let us see what happens with stability if we take two such bouncing balls and consider them as one hybrid system?

III. Motivation

A. Two Bouncing Balls

Consider two bouncing balls with states $x^1 \in \mathbb{R}^2$ and $x^2 \in \mathbb{R}^2$ respectively. The upper index indicates the number of a ball. Let they be interconnected by an elastic spring with elastic coefficient $k \geq 0$. The case $k = 0$ means that the balls are disconnected and move independently.

The mass of the spring is ignored. The motion of balls is along different lines, so that a collision is not possible. In this case we have additional force due to the elastic spring that is an interaction force between the bouncing balls. By the Hooke's law this force is proportional to the strain of the spring and is given by $\pm k(x_1^1 - x_1^2)$. Hence the dynamics of each ball is influenced by the other one, and it is given by the following equations, where the upper index denotes the number of the ball:

$$\dot{x}^1 = \begin{pmatrix} x_2^1 \\ -\gamma - k(x_1^1 - x_1^2) \end{pmatrix} := f^1(x^1, x^2), (x^1, x^2) \in C^1, \tag{15}$$

$$x^{1+} = \begin{pmatrix} x_1^1 \\ -\lambda x_2^1 \end{pmatrix} =: g^1(x^1), \quad (x^1, x^2) \in D^1, \tag{16}$$

and

$$\dot{x}^2 = \begin{pmatrix} x_2^2 \\ -\gamma + k(x_1^1 - x_1^2) \end{pmatrix} := f^2(x^1, x^2), (x^1, x^2) \in C^2, \tag{17}$$

$$x^{2+} = \begin{pmatrix} x_1^2 \\ -\lambda x_2^2 \end{pmatrix} =: g^2(x^2), \quad (x^1, x^2) \in D^2, \tag{18}$$

where

$$\begin{aligned} C^1 &= \{(x^1, x^2) \in \mathbb{R}^4 : x_1^1 \geq 0\}, \\ C^2 &= \{(x^1, x^2) \in \mathbb{R}^4 : x_1^2 \geq 0\}, \\ D^1 &= \{(x^1, x^2) \in \mathbb{R}^4 : x_1^1 = 0, x_2^1 \leq 0\}, \\ D^2 &= \{(x^1, x^2) \in \mathbb{R}^4 : x_1^2 = 0, x_2^2 \leq 0\}. \end{aligned} \tag{19}$$

Let us now consider this interconnection of two bouncing balls as one hybrid system (1). For this purpose we need to

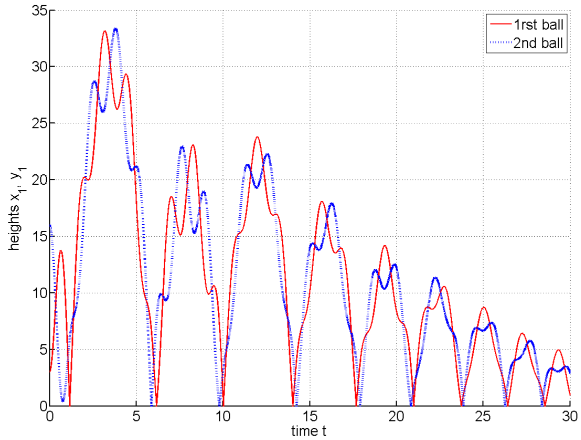

Fig. 1. Trajectories two bouncing balls connected by a spring.

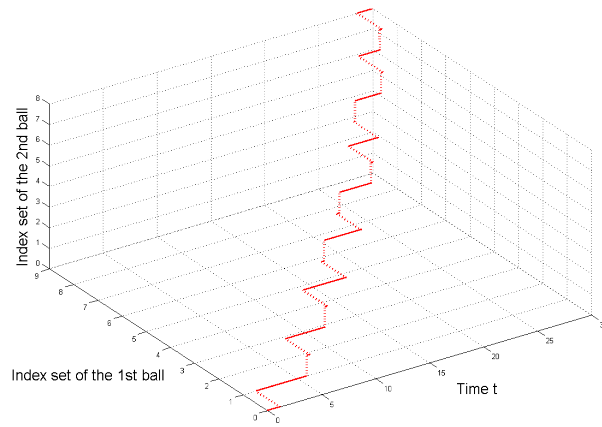

Fig. 2. Time domain of two interconnected balls.

define the new state and new sets C and D for the whole interconnection. It is natural to define $z := (x^1, x^2) \in \mathbb{R}^4$ as the state of the whole system. Now we have to define C and D as well as the functions f and g describing the flow and jumps respectively for the whole system.

Any time one of the balls jumps, the state $z \in \mathbb{R}^4$ jumps, i.e., the whole interconnection undergoes a jump. Hence it is natural to define $D := D^1 \cup D^2 \in \mathbb{R}^4$ and $C := C^1 \cap C^2$. A natural choice for f and g is as follows

$$f(z) := (f^{1T}, f^{2T})^T, \tag{20}$$

$$g(z) := (\widetilde{g}^{1T}, \widetilde{g}^{2T})^T, \tag{21}$$

where for $i = 1, 2$

$$\widetilde{g}^i(z) = \widetilde{g}^i(x^1, x^2) := \begin{cases} g^i(x^i), & \text{if } (x^1, x^2) \in D^i, \\ x^i, & \text{otherwise.} \end{cases}$$

With this notation the interconnection can be written as one hybrid system without inputs in form (4)

$$\dot{z} = f(z), \quad z \in C, \tag{22}$$

$$z^+ = g(z), \quad z \in D. \tag{23}$$

The same approach was also used in [6], [22] to describe an interconnection of hybrid systems.

B. Solution and stability problems

We do not see any other reasonable definition for C, D, f, g in the given setting then written above. However, this choice leads to the following problems illustrated by the example. Consider the following initial condition $x_1^1(0) = x_2^1(0) = 0$ and $x_1^2(0) = h > 0$, $x_2^2(0) = v \in \mathbb{R}$. Then observe that the following hybrid arc

$$x_1^1(t, j) = x_2^1(t, j) = 0, \quad x_1^2(t, j) = h, \quad x_2^2(t, j) = v \tag{24}$$

is a solution for (22)-(23) with the hybrid time domain given by $\{(0, j)\}_{j=0}^{\infty}$. This can be checked by a direct substitution of (24) into (22)-(23) and taking into account that in an intersection $C^i \cap D^i$ both jumps and continuous flow are allowed. In this case $t_{\max} = 0$ and the system jumps infinitely many times from a non-zero state to the same state. This "frozen" solution appears due to the interconnection and leads to the following problems that are relevant not only for the considered example but for interconnections of other hybrid systems with a stable equilibrium point:

- The above particular solution has no physical meaning.
- This solution shows that the resting state is not GAS any more.

This artificial loss of stability is contra intuitive, it happens due to the physically meaningless solution that needs to be ruled out by a suitable improvement of the notion of hybrid system. This is the main motivation of the paper and we provide such a generalization of hybrid systems below.

Apart of the mentioned problems there is another one: In general there is a solution corresponding to the case when one ball reaches its resting state in a finite time (after infinite number of jumps its continuous motion stops for while) and then it is pulled out from this state by the second ball (and moves again continuously for a while after that). This problem is very interesting but it will not be considered in this paper, because it appears not necessarily due to the interconnection but can happen with only one ball with external input. This problem is related to the issue of extension of solutions over the Zeno behavior [1], [29] and [18]. We give some comments later about that issue but in the rest of the paper for simplicity we assume that each subsystem can have at most one accumulation time point of jumps, i.e., at most one Zeno type motion can happen with each subsystem.

To solve the problem of artificial solutions and related stability loss we propose a slightly extended definition of a hybrid system in the next section.

IV. Generalized Hybrid Systems

Throughout this section we fix numbers $N_i, n \in \mathbb{N}$ such that $\sum_{i=1}^{n} N_i = N$. Let $x \in \mathbb{R}^N$ be partitioned into n parts: $x = (x^1, \ldots, x^n)$ with $x^i \in \mathbb{R}^{N_i}$, where N_i are given fixed numbers and $u \in \mathbb{R}^M$ be an external input. Let $C_i \subset \mathbb{R}^N \times U$ and $D_i \subset \mathbb{R}^N \times U$ be given. Let $f_i : C_i \times U \to \mathbb{R}_i^N$ and $g_i : D_i \times U \to \chi_i$ be given continuous functions. For any given (x, u) consider the following index sets

$$I_C(x,u) := \{i : (x,u) \in C_i\},\ I_D(x,u) := \{i : (x,u) \in D_i\}. \tag{25}$$

For any (x, u) such that $(x, u) \in C_i \cup D_i,\ \forall i = 1, \dots, n$ it holds that $I_C \cup I_D = \{1, \dots, n\}$. Note that if $C_i \cap D_i \neq \emptyset$ for some i, it holds that $I_C \cap I_D \neq \emptyset$.

A. New Definitions

A generalized hybrid system is then given by

$$\dot{x}^i = f_i(x,u), \quad i \in I_C(x,u); \tag{26}$$

$$(x^i)^+ = g_i(x,u), \quad i \in I_D(x,u). \tag{27}$$

In case for a given x and u there are some $i \in I_C \cap I_D \neq \emptyset$, it is allowed for x^i that it can flow or jump. This is similar to the case when $C \cap D \neq \emptyset$ for system (22)-(23), where the system may flow or jump. In the special case $D_i = D$ and $C_i = C$ for all $i = 1, \dots, n$ we arrive to the same definition of a hybrid system as in [10], whose trajectories can flow in continuous time and also jump at discrete instants. However our definition is more general due to the possibility to have continuous flows for some parts of the state also at those instants when other parts can jump. This definition allows to consider one large hybrid system as an interconnection of several ones or vice versa to consider several interconnected hybrid systems as one larger hybrid system. The idea is to partition the state of a system in several parts that are allowed to jump separately while other parts are allowed to flow. In this case we have to take into account such situations when one part of the state "stops" while another part "moves".

Definition 4.1 (Generalized Hybrid Time Domain): Let $t^1_{\max}, \dots, t^n_{\max}$ be fixed positive real numbers and $0 = t_0 \leq t_1 \leq t_2 \leq \dots$ be a nondecreasing sequence of numbers. These are Zeno times for $x^1, x^2, \dots, x^n$. The union of finitely or infinitely many intervals $[t_k, t_{k+1}] \times \{k^1\} \times \dots \times \{k^n\} \subset \mathbb{R}_+ \times \mathbb{N}^n_+$ with $k = k^1 + \dots + k^n$ with $k^i \in \mathbb{N}_+$, where the "last" interval is allowed to be of the form $[t_k, T) \times \{k^1\} \times \dots \times \{k^n\}$ with T finite or $T = +\infty$, is called hybrid time domain for system (26)-(27) and is denoted by $\mathrm{dom}_{k^1,\dots,k^n}$.

Here the upper index numerates the parts of the state x, i.e., i corresponds to x^i and k^i counts the jumps of this part of the state. Number k corresponds to the total number of jumps of all parts of the state. For brevity we introduce multi-indices

$$\overline{k} = (k^1, \dots, k^n) \in \mathbb{N}^n_+, \quad \overline{p} = (p_1, \dots, p_n)$$

where p_i is either 0 or 1. Denote $p = p_1 + \dots + p_n$.

The definitions of hybrid signal, hybrid input and hybrid arc are defined similar to Section II using $\mathrm{dom}_{\overline{k}}$.

Definition 4.2 (Solutions to a generalized Hybrid System): A hybrid arc x with a given partition and a hybrid input u are a solution pair (x, u) to (26), (27) , if

(i) $(x(0,0), u(0,0)) \in C_i \cup D_i,\ \forall i$

(ii) for any multi-index $\overline{k} = (k^1, \dots, k^n) \in \mathbb{N}^n_+$ and almost all $t \in \mathbb{R}_+$ with $(t, \overline{k}) \in \mathrm{dom}_{k^1,\dots,k^n}\ x$ it holds that for $i \in I_C(x^1(t, \overline{k}), \dots, x^n(t, \overline{k}), u(t, \overline{k}))$,

$$\begin{aligned}\dot{x}^i(t,\overline{k}) = \quad & f_i(x^1(\min\{t, t^1_{\max}\}, \overline{k}), \dots, \\ & x^n(\min\{t, t^n_{\max}\}, \overline{k}), u(t, \overline{k}));\end{aligned} \tag{28}$$

(iii) for all $(t, \overline{k}) \in \mathrm{dom}_{\overline{k}}\ x$ such that $(t, \overline{k} + \overline{p}) \in \mathrm{dom}_{\overline{k}}\ x$ with $p \geq 1$ for $i \in I_D(x^1(t, \overline{k}), \dots, x^n(t, \overline{k}), u(t, \overline{k})$,

$$\begin{aligned}(x^i)^+(\min\{t, t^k_{\max}\}, \overline{k} + \overline{p}) = g_i(x^1(\min\{t, t^1_{\max}\}, \overline{k}), \\ \dots, x^n(\min\{t, t^n_{\max}\}, \overline{k}), u(t, \overline{k})).\end{aligned} \tag{29}$$

where I_C, I_D is a partition of the index sets defined in (25).

Remark 4.3: The numbers $t^i_{\max}$ must not be known in advance. They should be found as for example in (14) and should be considered as a part of solution or, more precisely, of its hybrid time domain. $t^i_{\max}$ is the total time during which the i-th part of the state flows.

We say that a solution is maximal if it cannot be extended and it is called complete if its domain is unbounded.

B. On additional artificial solutions

The advantage of such definition is, in particular, that we can avoid the meaningless "frozen" solutions shown above in the example with two bouncing balls connected by a spring. To see this let us again consider the interconnection (15)-(18) as one hybrid system of the form (26)-(27) with $n = 2$, $N_1 = N_2 = 2$, the same sets C_i and D_i and $U = \emptyset$. The functions f_i and g_i remain also unchanged. The sets $I_C(x)$ and $I_D(x)$ for the interconnection are given by

$$\begin{aligned}
I_C(x) = \{1,2\}, I_D(x) = \emptyset, &\quad \text{if } x^1_1 > 0,\ x^2_1 > 0,\\
I_C(x) = \{1,2\}, I_D(x) = \{1\}, &\quad \text{if } x^1_1 = 0, x^2_1 > 0,\\
I_C(x) = \{1,2\}, I_D(x) = \{2\}, &\quad \text{if } x^1_1 > 0,\ x^2_1 = 0,\\
I_C(x) = \{1,2\}, I_D(x) = \{1,2\}, &\quad \text{if } x^1_1 = 0, x^2_1 = 0.
\end{aligned}$$

Consider the same initial conditions $x^1_1(0) = x^1_2(0) = 0$, $x^2_1(0) = h$, $x^2_2(0) = v$.

Now observe that the hybrid arc (24) is not a solution of (26)-(27) with these data, because it corresponds to $I_C = \{1, 2\}$, $I_D = \{1\}$, i.e., the second subsystem is not allowed to jump. From this we see that our approach naturally avoids the additional artificial solutions discussed above.

The first arcs of solution to our example corresponding to the continuous flow up to the first jump with initial conditions $x^1_1(0) = x^1_2(0) = 0$, $x^2_1(0) = h$, $x^2_2(0) = v$ is given by

$$\begin{aligned}
x^1_1(t,0) &= -\frac{1}{2k}e^{-2kt} - \frac{1}{12}\gamma k t^4 + \frac{1}{6}kvt^3 - \frac{1}{2}\gamma t^2 - t + \frac{1}{2k},\\
x^1_2(t,0) &= e^{-2kt} - \frac{1}{3}\gamma k t^3 + \frac{1}{2}kvt^2 - \gamma t - 1,\\
x^2_1(t,0) &= \frac{1}{2k}e^{-2kt} + \frac{1}{12}\gamma k t^4 - \frac{1}{6}kvt^3 - \frac{1}{2}\gamma t^2 + (v+1)t\\
&\quad + h - \frac{1}{2k},\\
x^2_2(t,0) &= -e^{-2kt} + \frac{1}{3}\gamma k t^3 - \frac{1}{2}kvt^2 - \gamma t + (v+1).
\end{aligned}$$

Further arcs can be calculated iteratively. A simulated solution and the corresponding generalized hybrid time domain are shown on figures 1 and 2 . Let us see whether the system with two balls is GAS in the new setting.

V. On stability properties

First we state a general stability result in our new framework that can be obtained by adapting a nested Matrosov theorem from [25] to system (26)-(27).

Proposition 5.1: Let system (26)-(27) be GS. Then, it is GAS if there exist $m \in \mathbb{N}$ and for each $0 < \delta < \Delta$,

- a number $\mu > 0$,
- continuous functions $w_{c,j} : (\cap_i \bar{C}_i) \cap \Omega(\delta, \Delta) \to \mathbb{R}$, $w_{d,j} : (\cup_i \bar{D}_i) \cap \Omega(\delta, \Delta) \to \mathbb{R}$, $j \in \{1, \dots, m\}$,
- functions $V_j : \mathbb{R}^N \setminus 0 \to \mathbb{R}$, $j \in \{1, \dots, m\}$ are C^1 on an open set containing $(\cap_i \bar{C}_i) \cap \Omega(\delta, \Delta)$,

such that, for each $j \in \{1, \dots, m\}$,

$$|V_j(x)| \le \mu \qquad \forall x \in (\cup_i \bar{C}_i) \cup (\cup_i D_i) \cup (\cup_i g_i(D_i)) \cap \Omega(\delta, \Delta), \tag{30}$$

$$\left\langle \nabla V_j(x), (f_1^T, \dots, f_n^T)^T \right\rangle \le w_{c,j}(x),\ \forall x \in (\cap_i C_i) \cap \Omega(\delta, \Delta), \tag{31}$$

$$V_j((\widetilde{g}_1^T, \dots, \widetilde{g}_n^T)^T(x)) - V_j(x) \le w_{d,j}(x), \forall x \in (\cup_i D_i) \cap \Omega(\delta, \Delta), \tag{32}$$

$$\widetilde{g}_i(x) := \begin{cases} x^i & , x \in C_i \setminus D_i, \\ g_i(x) & , \text{ otherwise}, \end{cases}$$

and with the definitions $w_{c,0}, w_{d,0} : \mathbb{R}^N \to \{0\}$ and $w_{c,m+1}, w_{d,m+1} : \mathbb{R}^N \to \{1\}$, we have, for each $l \in \{0, \dots, m\}$,
1) if $x \in (\cap_i \bar{C}_i) \cap \Omega(\delta, \Delta)$ and $w_{c,j}(x) = 0$ for all $j \in \{0, \dots, l\}$ then $w_{c,l+1}(x) \le 0$,
2) if $x \in (\cup_i \bar{D}_i) \cap \Omega(\delta, \Delta)$ and $w_{d,j}(x) = 0$ for all $j \in \{0, \dots, l\}$ then $w_{d,l+1}(x) \le 0$.

Proof: The proof follows in a similar way as the proof of Theorem 3.2 in [25] and is omitted due to space reasons. ■

Return now to the interconnection of two bouncing balls. Define $z := (x^1, x^2)$. Using $V(z) = V(x^1, x^2) := \frac{1}{2}\left((x_2^1)^2 + (x_2^2)^2\right) + \gamma x_1^1 + \gamma x_1^2 + \frac{k}{2}(x_1^2 - x_1^1)^2$ as a Lyapunov function, it can be shown that the system is GS, see [25], [24]. It remains to apply Proposition 5.1 to prove GAS of the interconnection. For this purpose let us define

$$\begin{aligned} V_1(z) &:= V_1(x^1, x^2) \\ &= \frac{1}{2}\left((x_2^1)^2 + (x_2^2)^2\right) + \gamma x_1^1 + \gamma x_1^2 + \frac{k}{2}(x_1^2 - x_1^1)^2, \\ V_2(z) &:= V_2(x^1, x^2) = \gamma x_2^1 + \gamma x_2^2. \end{aligned}$$

Consider the following four cases:

- both components of the state flow continuously:
$$\begin{aligned} \dot{V}_1(z) =& x_2^1(-kx_1^1 + kx_1^2 - \gamma) + x_2^2(kx_1^1 - kx_1^2 - \gamma) \\ &+ \gamma x_2^1 + \gamma x_2^2 + k(x_1^2 - x_1^1)(x_2^2 - x_2^1) = 0, \\ \dot{V}_2(z) =& -2\gamma^2. \end{aligned}$$
- both components of the state jump:
$V_1(z^+) - V_1(z) = -\frac{1}{2}(1-\mu^2)(x_2^1)^2 - \frac{1}{2}(1-\mu^2)(x_2^2)^2$,
$V_2(z^+) - V_2(z) = -(1+\mu)\gamma(x_2^1 + x_2^2)$
- the first component of the state jumps and the second flows continuously:
$V_1(z^+) - V_1(z) = -\frac{1}{2}(1-\mu^2)(x_2^1)^2$,
$V_2(z^+) - V_2(z) = -(1+\mu)\gamma x_2^1$
- the first component of the state flows continuously and the second jumps:
$V_1(z^+) - V_1(z) = -\frac{1}{2}(1-\mu^2)(x_2^2)^2$,
$V_2(z^+) - V_2(z) = -(1+\mu)\gamma x_2^2$

From Proposition 5.1 with such V_1 and V_2, the interconnected system (15)-(18) is GAS, see also Example 4.1 in [25].

VI. Existence of solutions

Here we provide conditions that guarantee the existence of solutions to system (26)-(27).

To this end we define basic regularity conditions for this system:

1) χ_i is open, U is closed, and $C_i, D_i \subset \chi_1 \times \cdots \times \chi_n \times U$ are relatively closed in $\chi_1 \times \cdots \times \chi_n \times U$;
2) f_i, g_i are continuous.

Theorem 6.1 (Existence of solutions): Assume basic regularity conditions for system (26)-(27) hold.

If one of the following conditions holds:
(i) $(x^0, u^0) \in D_i$ for all $i \in \{1, \dots, n\}$;
(ii) $(x^0, u^0) \in C_i$ and for some neighborhood P of (x^0, u^0), for all $(x', u^0) \in P \cap C_i$, $T_{C_i}(x', u^0) \cap f_i(x', u^0) \ne \emptyset$, for all $i \in \{1, \dots, n\}$;
(iii) $1 \le |I_C(x^0, u^0)| < n$, $1 \le |I_D(x^0, u^0)| < n$ and for some neighborhood P of (x^0, u^0), for all $i \in I_{C_i}(x^0, u^0)$, $(x', u^0) \in P \cap C_i$, $T_{C_i}(x', u^0) \cap f_i(x', u^0) \ne \emptyset$,
then there exists a solution (x, u) for hybrid system (26)-(27) with $(t, \bar{k}) \in \operatorname{dom} x$ for some $t > 0$ and $\bar{k} \in \mathbb{N}_+^n$.

Furthermore, if for all $i \in \{1, \dots, n\}$, $g_i(D_i) \in C_i \cup D_i$, then there exist $t > 0$, $\bar{k} \in \mathbb{N}_+^n$ such that $(x(t, \bar{k}), u(t, \bar{k})) \in C_i \cup D_i$.

Proof: For cases (i)-(ii) the proof goes along the lines of the proof of Proposition 2.4 in [11]. Consider the case (iii). A solution can be constructed by concatenating all the states after the jumps at $t = 0$ and the first continuous arc after these jumps. Unfortunately, the proof of the last case is omitted due to space reasons. ■

VII. Remarks on Further Problems

Here we would like to address some interesting problems that we have not discussed so far. One of them is the third kind of problems mentioned in Section III-B. In the previous sections we have assumed for simplicity that each part x^i of the whole state x can undergo at most one Zeno behavior. But in general it may easily happens that a (sub)system undergoes several Zeno-type motions. As a simple example we consider one bouncing ball (6)-(7) and introduce an external input there:

$$\dot{x} = \begin{pmatrix} x_2 \\ -\gamma + u \end{pmatrix} =: f(x, u), \quad x \in C, \tag{33}$$

$$x^+ = \begin{pmatrix} x_1 \\ -\lambda x_2 \end{pmatrix} =: g(x, u), \quad x \in D. \tag{34}$$

Let us take the same initial conditions as we used above $x(0) = h$, $x_2 = 0$ and define the input signal u so that it takes value 1 for $2t_{\max} \leq t \leq 3t_{\max}$ and is zero otherwise. Here $t_{\max}$ is given by (14). Then after being at rest for a while after $t_{\max}$ the ball will be elevated to some finite height and dropped again. The second Zeno-type behavior will follow. This kind of behavior can be modeled by introducing multiple Zeno-time $t_{\max,1}, t_{\max,2}, \dots$ and appropriate extension of the framework presented above.

There is also another interesting issue that we do not consider here. A hybrid behavior can appear due to interconnection of systems that are not hybrid by their nature. For example a movement of a mass-point in a free space under some forces is usually modeled by ordinary differential equations and is not hybrid by its nature. However if we consider two such mass-points in the same space, so that they can collide, then the resulting systems exhibits hybrid behavior. If we like to model it as a hybrid systems we will need to define the corresponding flow and jump sets due to interconnection.

VIII. Conclusions

In this paper we have discussed some problems that appear in stability analysis of interconnections of hybrid systems. As well we have proposed possible solutions to these kind of problems. We have proposed a modified definition of a hybrid system that slightly generalizes the one of the recently developed definitions. This allows for a simple composition/decomposition of a hybrid system into several ones. This is motivated and illustrated by standard examples. Under basic regularity conditions existence of solution for such hybrid system is shown. In a future work we are going to handle both multiple Zeno times and extension of hybrid time domain after Zeno following some recent results, e.g. [1], [18], and [29] related to this issue.

IX. ACKNOWLEDGMENTS

We are pleased to acknowledge the Royal Thai Government for full financial support to the last author.

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