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On Controller Design for Delay-independent Stability of Linear Time-invariant Systems with Multiple Delays*

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Abstract— **In this paper, we present a new control-design procedure for linear time-invariant (LTI) systems with "multiple" delays. These procedures are based on algebraic tools and allow designing controllers that can stabilize such systems regardless of how large/small the delays are. That is, with these controllers, the system at hand can be rendered delay-independent stable (DIS). The essence of the control design is based on the Rekasius transformation, algebraic tools, elimination techniques, and Sturm sequences. The advantages of the design procedure are that it simplifies the control design to managing the roots of some single-variable polynomials while also preserving the controller structure and complying with the necessary and sufficient conditions of stability.**

I. INTRODUCTION

The presence of delays in control systems is often times a source of instability [1], [2]. A careful control design is therefore critical in order to assure that a controlled system performs properly and remains stable despite the delays.

Even for linear time-invariant (LTI) systems, control design in the presence of delays can be challenging, especially within a non-conservative framework. Frequency domain technique (FDT) is one promising direction for obtaining non-conservative results as demonstrated for analysis and synthesis problems of LTI systems [3]–[6]. However, in general, finding all the stabilizing controller gains for the system can be a major challenge since the corresponding eigenvalue problems are infinite dimensional and nonlinear.

When the delays are uncertain, design of delay-independent stable (DIS) controllers is desirable with the expectation that the system's functionality, e.g., output feedback control, can still be maintained. Based on FDT, several approaches are proposed for *testing* delay-independent stability of LTI systems with constant delays [6]–[9]. However some problems related to *control design* in order to render such systems DIS have not been thoroughly investigated for the "multiple" delay case, to the best of the authors' knowledge.

This paper is on the control synthesis for achieving delay-independent stability in LTI systems. We extend here our previous results on DIS problem [8], [10]–[12], with the objective to reveal the regions in controller-gain space, $\{k_i\}_{i=1}^{z} \in \mathbb{R}$, where the multiple-delay system at hand can be made DIS while still respecting the controller structure. For this, an algebraic approach is proposed for revealing these regions, as well as some of their boundaries in closed forms, under certain conditions. In this approach, no restriction is imposed on the system order and/or the ranks of control matrices. Specifically, it combines the Rekasius substitution [9], [13], Resultant elimination method and discriminant operations [14] to reveal the DIS regions. This paper also considers the class of Metzlerian delay systems, which simplifies the control design for achieving DIS.

As a note of formalism, we use boldface characters for vectors, matrices, and sets. $\mathbb{N}$, $\mathbb{R}_+$, and $\mathbb{R}_{0+}$, denote respectively the set of natural numbers, positive real numbers, and nonnegative real numbers. We refer to $\mathbb{C}$ as the entire complex plane, and represent right (left) half open complex plane with $\mathbb{C}_+(\mathbb{C}_-)$, while $\mathbb{C}_0$ denotes the imaginary axis and $\mathbb{C}_{0+} = \mathbb{C}_0 \bigcup \mathbb{C}_+$. The Laplace variable is given by $s \in \mathbb{C}$. $\{\tau_\ell\}_{\ell=1}^{L} = (\tau_1, \tau_2, \ldots, \tau_L)$ is the delay vector, and $\{T_\ell\}_{\ell=1}^{L} = (T_1, T_2, \ldots, T_L)$ is the pseudo-delay vector. $\Re(\cdot)$ is the real and $\Im(\cdot)$ is the imaginary part of $(\cdot)$.

II. PRELIMINARIES

A. Problem Formulation

The focus of this paper is on the control design for the following LTI system based on static output feedback,

$$\left.\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \bar{\mathbf{B}}\mathbf{u}(t), \\ \mathbf{y}(t) &= \textstyle\sum_{\ell=1}^{L} \mathbf{C}_\ell \mathbf{x}(t-\tau_\ell), \\ \mathbf{u}(t) &= \mathbf{K}\mathbf{y}(t), \end{aligned}\right\} \quad (1)$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\bar{\mathbf{B}} \in \mathbb{R}^{n\times m}$, and $\mathbf{C}_\ell \in \mathbb{R}^{r\times n}$ are given output constant matrices, $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector, $\mathbf{u}(t) \in \mathbb{R}^m$ is the control input, $\{\tau_\ell\}_{\ell=1}^{L}$ are the unknown delays that arise when measuring system states, $\mathbf{y}(t) \in \mathbb{R}^r$ is the plant output, $L \in \mathbb{N}$ is the maximum number of delays in the system, and $\mathbf{K} \in \mathbb{R}^{m\times r}$ represents all possible controller gains with entries $\{k_i\}_{i=1}^{z} = \{k_i | k_i \in \mathbb{R}, i = 1, \ldots, z\}$. We define $\mathbf{B}_\ell = \bar{\mathbf{B}}\mathbf{K}\mathbf{C}_\ell \neq \mathbf{0}$ whose entries are parameterized by the *unknown controller gains*, $\{k_i\}_{i=1}^{z}$.

Remark 1: For the particular model in (1), delay-independent stability requires that the uncontrolled system and the delay-free ($\{\tau_\ell\}_{\ell=1}^{L} = 0$) controlled system to be Hurwitz stable [3]. Moreover, under Hurwitz stability conditions, system (1) does not have a pole on the origin for $\{\tau_\ell\}_{\ell=1}^{L} = 0$. This automatically guarantees that $s = 0$ cannot be a feasible solution of the corresponding characteristic equation for finite $\tau_\ell \geq 0$, see [3] for details.

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Since stability of (1) needs to be guaranteed for successful feedback control, we focus on stability in the rest of the text. For studying stability, we start with the characteristic equation of (1), which, in Laplace domain s, reads

$$\begin{aligned} f(s,\{\tau_\ell\}_{\ell=1}^{L},\{k_i\}_{i=1}^{z}) &= \left| s\mathbf{I}-\mathbf{A}-\sum_{\ell=1}^{L}\mathbf{B}_\ell\, e^{-\tau_\ell\, s} \right| \\ &= \sum_{q=0}^{Q} P_q(s,\{k_i\}_{i=1}^{z})\, e^{-\sum_{\ell=1}^{L}\xi_{q\ell}\tau_\ell\, s} = 0, \end{aligned} \tag{2}$$

where P_q are polynomials in terms of s and the controller gains $\{k_i\}_{i=1}^{z}$, with $Q \in \mathbb{Z}_+$, and $\xi_{q\ell} \in \mathbb{N}$.

Property 1 (DIS Property [15]): Given $\{k_i\}_{i=1}^{z}$, system in (1) is DIS if and only if (2) has no roots on $\mathbb{C}_{0+}$ for all delay values. That is,

$$f(s,\{\tau_\ell\}_{\ell=1}^{L},\{k_i\}_{i=1}^{z}) \neq 0, \tag{3}$$

in $\{\tau_\ell\}_{\ell=1}^{L} \in \mathbb{R}_{0+}^{L}$ for $\forall s \in \mathbb{C}_{0+}$.

Finding $\{k_i\}_{i=1}^{z}$ controller gains satisfying the DIS condition given above can be difficult. This is because (2) has infinitely many roots, and verifying (3) is computationally involved. Using the continuity property of the roots of (2) on $\mathbb{C}$ and the τ-decomposition theorem are useful [5], [16], [17]. In summary, since there exists no delay in the highest-order derivative of the states in (1), the characteristic equation (2) represents a *retarded* class LTI system with multiple delays [2] and due to this reason, stability properties of (1) is preserved as $\tau_\ell : 0 \to 0^+$, and we have that stability transition of (1) in the parameter space occurs at some critical delay values τ_ℓ for which at least one root of (2) lies on $\mathbb{C}_0$, at $s = j\omega$, where $\omega \in \mathbb{R}_{0+}$ without loss of generality. *Complimentary* to this scenario is the case when the roots of (2) never lie on $\mathbb{C}_{0+}$ for all delay values. This is the delay-independent stability scenario, formulated in (3).

One enabling manipulation introduced earlier [4], [9], [13] can be utilized here. Thanks to the Rekasius transformation, one can convert (2) to an algebraic polynomial without sacrificing the stability switching properties of (2). The transformation leads to

$$\begin{aligned} &g(j\omega,\{T_\ell\}_{\ell=1}^{L},\{k_i\}_{i=1}^{z}) \\ &\quad = \left(f \,\Bigg|_{\substack{e^{-j\omega\tau_\ell} = \frac{1-j\omega T_\ell}{1+j\omega T_\ell},\\ \ell=1,\dots,L.}} \right) \prod_{\ell=1}^{L}(1+j\omega T_\ell)^{c_\ell}, \end{aligned} \tag{4}$$

where $\{T_\ell\}_{\ell=1}^{L} \in \mathbb{R}$, c_ℓ denotes the *order of commensuracy* of delay τ_ℓ, where $c_\ell = \max\{\xi_{q\ell} | \forall q = 0 \dots Q\}$, and g is a polynomial with complex coefficients[1].

B. On Algebraic Tools

Definition 1 (Sturm Sequences of a Polynomial [18]): Let $P(x)$ be a polynomial with real coefficients. The Sturm sequence associated with $P(x)$ is the sequence of polynomials $P_0 = P(x), P_1(x), P_2(x), \dots, P_\lambda(x)$.

[1]It must be noted that all the $s = j\omega$ roots of (2) and of (4) are identical [4], [13]. Hence guaranteeing (1) to become DIS is analogous to requiring that no $s = j\omega$ solutions of (4) exist for all $\{T_\ell\}_{\ell=1}^{L} \in \mathbb{R}$.

Definition 2: Let $\alpha \in \mathbb{R}$ and $P(\alpha) \neq 0$. The sign variations $var_S(P(\alpha))$ of the Sturm sequence associated with $P(x)$ is the number of sign variations in the sequence $P(\alpha) = P_0(\alpha), P_1(\alpha), \dots, P_\lambda(\alpha)$.

Theorem 1: (Number of Distinct Real Zeros of a Polynomial [18]) Let a polynomial $P(x)$ has no repeated roots. In the interval $[a,b]$, $a,b \in \mathbb{R}$, $a < b$, the number of distinct real roots of $P(x)$ is given by $var_S(P(a)) - var_S(P(b))$. Note that with Sturm sequences, one can always correctly count the number of distinct positive real roots of $P(x)$ [19]. See the cited references for the repeated-root case.

Definition 3: Let F_1 and F_2 be two polynomials both in terms of μ and ν. *Resultant* R_μ (or R_ν) of F_1 and F_2 is computed via the determinant of Sylvester matrix obtained by eliminating μ (or ν). The resultant of F_1 and $\partial F_1/\partial\mu$ (or $\partial F_1/\partial\nu$) by eliminating μ (or ν) is called the *discriminant* of the polynomial F_1 with respect to μ (or ν) [20]. The discriminant function is defined as,

$$Discrim_\mu(F_1(\mu,\nu)) = \frac{(-1)^{\frac{d(d-1)}{2}}}{\alpha_d} R_\mu(F_1, \partial F_1/\partial\mu), \tag{5}$$

where the order of F_1 is d with respect to the elimination variable μ, and $\alpha_d(\nu)$ is its leading coefficient.

Property 2: The discriminant $Discrim_\mu(F_1(\mu,\nu))$ of the polynomial $F_1(\mu,\nu)$ vanishes if and only if F_1 has a multiple root in $\mathbb{C}$ [21].

III. Main Results

In this section the main goal is to implement the DIS analysis on a control design scheme as a synthesis problem. We first show that a feasible controller to render the closed-loop system DIS exists under certain conditions.

Theorem 2 (Existence of DIS Gain Regions [22]): For a given set of delays $\{\tau_\ell\}_{\ell=1}^{L}$, there exists at least one region in controller gain parameter space $\{k_i\}_{i=1}^{z}$, where the system in (1) is DIS, if and only if $\mathbf{A} + \sum_{\ell=1}^{L}\mathbf{B}_\ell e^{-\tau_\ell s}$ is DIS for $\{k_i\}_{i=1}^{z} = \{\tilde{k}_i\}_{i=1}^{z}$.

A. DIS Control Design

We now develop an algebraic approach to find the controller gains $\{k_i\}_{i=1}^{z}$ that make the control system (1) DIS. Recall from the previous section that DIS analysis of (1) can be converted to analyzing $s = j\omega$ roots of (4). This is where we start. We rewrite g as follows,

$$\begin{aligned} g(j\omega,\, \{T_\ell\}_{\ell=1}^{L},\, \{k_i\}_{i=1}^{z}) &= g_{\Re}(\omega,\, \{T_\ell\}_{\ell=1}^{L},\, \{k_i\}_{i=1}^{z}) \\ &\quad + j\, g_{\Im}(\omega,\, \{T_\ell\}_{\ell=1}^{L},\, \{k_i\}_{i=1}^{z}) = 0, \end{aligned} \tag{6}$$

where $g_{\Re}$ and $g_{\Im}$ are respectively the real and imaginary parts of g. Given $\{k_i\}_{i=1}^{z}$, in order to solve $(\omega, \{T_\ell\}_{\ell=1}^{L})$ pairs from (6), one should guarantee that $g_{\Re} = 0$ and $g_{\Im} = 0$, concurrently. At this step, in order to reduce the number of unknowns, we can eliminate T_L from these coupled equations using the Resultant Theory [14], which leads to a $2c_L$-order Sylvester matrix denoted by S, where c_L is the commensurate degree of τ_L. When $g_{\Re} = 0$ and $g_{\Im} = 0$ have common solutions, then S is singular (but the converse is not always true). That is, the resultant $R_{T_L}(g_{\Re}, g_{\Im}) = \det(S)$, which is a function of ω, $\{T_\ell\}_{\ell=1}^{L-1}$, and $\{k_i\}_{i=1}^{z}$, should be zero as a necessary condition for (6) to be satisfied.

DIS analysis for the multiple time-delay system at hand can actually be constructed as an optimization problem using the principles of resultant theory and starting with R_{T_L}, see the Appendix and [8]. From the Appendix, we obtain the polynomial $D(\omega)$ while incorporating the controller gains $\{k_i\}_{i=1}^{z}$, which becomes $D(\omega, \{k_i\}_{i=1}^{z})$. Next, we use $D(\omega, \{k_i\}_{i=1}^{z})$ to develop an algebraic approach to select $\{k_i\}_{i=1}^{z}$ such that the system is DIS, as described in the Appendix based on the roots of $D(\omega) = 0$. For DIS, either one of the following conditions should hold [8],

- $D(\omega, \{k_i\}_{i=1}^{z})$ has no $\omega \in \mathbb{R}$ roots.
- $D(\omega, \{k_i\}_{i=1}^{z})$ has some $\omega \in \mathbb{R}$ roots but the corresponding T_ℓ values satisfying (4) are not all in $\mathbb{R}$.

B. Detection of DIS Gain Regions

We know that the polynomial $V_1 := D(\omega, \{k_i\}_{i=1}^{z})$ is even with respect to ω, excluding its $\omega = 0$ zeros as per Remark 1, [8], [10]. Thus a variable change as $y = \omega^2$ becomes convenient, leading to

$$\Phi(y, \{k_i\}_{i=1}^{z}) = V_1(\omega^2, \{k_i\}_{i=1}^{z}), \tag{7}$$

where the number of real zeros of V_1 is twice the number of positive real zeros of $\Phi(y, \{k_i\}_{i=1}^{z})$. We note that working with $\Phi(y, \{k_i\}_{i=1}^{z})$ is more advantageous for two reasons; one being for the sake of computational efficiency due to reduced order, and the other for converting the problem into the investigation of positive real roots of $\Phi(y, \{k_i\}_{i=1}^{z})$. This is exactly when Sturm sequences become instrumental. Instead of studying the real roots of $V_1 = 0$, one can actually inspect the existence of positive real roots of $\Phi(y, \{k_i\}_{i=1}^{z}) = 0$, for a given set of controller gains $\{k_i\}_{i=1}^{z}$. This can be done as follows:

- If no roots of $\Phi(y, \{k_i\}_{i=1}^{z}) = 0$ are positive, this implies that no admissible $y = \omega^2 > 0$ roots exist, hence a stability switch is impossible. If, by construction, the conditions in Remark 1 also hold, then the system is stable independent of the amount of each delay τ_ℓ.
- Presence of at least one positive real zero of $\Phi(y, \{k_i\}_{i=1}^{z})$, however, requires checking whether a solution in $\{T_\ell\}_{\ell=1}^{L} \in \mathbb{R}$ exists satisfying (6). If such a set of $\{T_\ell\}_{\ell=1}^{L}$ does not exist, and the conditions in Remark 1 are satisfied, then the system is DIS.

The DIS conditions above can be checked via point-wise inspection of controller gains $\{k_i\}_{i=1}^{z}$. In the sequel, however, we use these conditions to explore if we can analytically identify the DIS boundaries in the controller gain space without parametric sweeping, and based on necessary and sufficient conditions.

C. Control Design for Delay-independent Stability using Sturm Sequences (point-wise sweeping of control gains)

Using Sturm sequences, we can find the DIS regions with necessary and sufficient conditions. Application of Sturm sequences requires care, since $V_1 = 0$ is only a necessary condition for $g_{\Re} = 0$ and $g_{\Im} = 0$ to have common roots.

Theorem 3 (Controller Gain Detection for DIS [22]): The system in (1) is DIS in the delay parameter space $\{\tau_\ell\}_{\ell=1}^{L}$ with given $\{k_i\}_{i=1}^{z} = \{k_i^*\}_{i=1}^{z}$ gains in $\mathbf{B}_\ell$, if and only if

i) $\left|s\mathbf{I} - \mathbf{A} - \sum_{\ell=1}^{L} \mathbf{B}_\ell\right|$ is Hurwitz stable,

ii)-(a) $var_S(\Phi(\omega = 0)) - var_S(\Phi(\omega \to \infty)) = 0$,

ii)-(b) $var_S(\Phi(\omega = 0)) - var_S(\Phi(\omega \to \infty)) \neq 0$ for some $\omega = \omega^* \in \mathbb{R}_+$ but at least one condition in (B)-(C) of Theorem 8 in the Appendix is not satisfied.

Note that the Sturm sequences guarantee the necessary and sufficient conditions for $\Phi(y) = 0$ *not to have positive real roots*, but this alone does not alone guarantee DIS with necessary and sufficient conditions. Therefore, one should also consider condition (ii)-(b) in Theorem 3. This is because there may exist controller gains for which $\Phi(y)$ possesses positive real zeros that correspond to inadmissable $T_\ell \in \mathbb{C}$ values. In such cases, the system with the selected controller gains can still be DIS. Note that condition (ii)-(b) can be easily checked by calculating $T_1^* = T_1(\omega^*)$, $T_2^* = T_2(T_1^*, \omega^*)$,... from (B) of Theorem 8, see Appendix, and back substituting all $T_1^* \ldots T_\ell^*, \omega^*$ in $g_{\Re} = 0$ and $g_{\Im} = 0$ to check if these solutions satisfy (6).

D. Derivation of Fundamental Closed-form DIS Boundaries

In this section, we show under certain conditions that DIS boundaries in controller gain space $\{k_i\}_{i=1}^{z}$ can be identified. We start with the following Lemma.

Lemma 1: Let (7) be rewritten as,

$$\Phi(y, \{k_i\}_{i=1}^{z}) = \sum_{\ell=0}^{L} \gamma_\ell(\{k_i\}_{i=1}^{z}) y^\ell = 0, \tag{8}$$

where $\gamma_\ell \in \mathbb{R}$. If, on the complex y plane and with respect to the parameter $\{k_i\}_{i=1}^{z}$, there exists any crossing of the root loci of $\Phi(y, \{k_i\}_{i=1}^{z}) = 0$ with the positive real axis, this crossing occurs either *(i)* with at least one double *positive* real root of $\Phi(y, \{k_i\}_{i=1}^{z}) = 0$, or *(ii)* with at least one zero root of $\Phi(y, \{k_i\}_{i=1}^{z}) = 0$ at the origin of the y-plane.

The proof follows from the behavior of the roots of polynomials with real coefficients.

According to Property 2, Definition 3, and in light of Lemma 1, we are now able to state that if double roots of $\Phi(y, \{k_i\}_{i=1}^{z})$ exist for some $\{k_i\}_{i=1}^{z} \in \mathbb{R}$, then the discriminant of $\Phi(y, \{k_i\}_{i=1}^{z}) = 0$ with eliminating y should vanish,

$$\varphi(\{k_i\}_{i=1}^{z}) = Discrim_y(\Phi(y, \{k_i\}_{i=1}^{z})) = 0. \tag{9}$$

This formulation offers a way to investigate case *(i)* of Lemma 1. In other words, for a given $\{k_i\}_{i=1}^{z}$, the roots of $\varphi(\{k_i\}_{i=1}^{z}) = 0$ can be double negative/possitive real and/or double complex conjugate.

As per Lemma 1, the cases that yield a non-DIS system are when there exist either positive real roots $y^* > 0$ of (9) (for case *(i)* of Lemma 1), or $\gamma_0(\{k_i\}_{i=1}^{z}) = 0$ (for case *(ii)* of Lemma 1), and if, in each case, the corresponding $\omega = \sqrt{y^*} > 0$ and $\{T_\ell^*\}_{\ell=1}^{L}$ satisfy the characteristic equation in (6). Otherwise, for a given $\{k_i\}_{i=1}^{z}$ we *may* have a DIS property inside the regions bordered by the hypersurfaces defined by $\varphi(\{k_i\}_{i=1}^z) = 0$. Hence, one has to inspect all these enclosed regions arising due to the boundaries $\varphi(\{k_i\}_{i=1}^z) = 0$ and $\gamma_0(\{k_i\}_{i=1}^z) = 0$, and identify the segments of those boundaries encapsulating the DIS regions. Let us denote these *refined* segments with $\Lambda(\{k_i\}_{i=1}^z)$, which is obviously a subset of $\Gamma(\{k_i\}_{i=1}^z) = \{\{k_i\}_{i=1}^z \in \mathbb{R}| \varphi(\{k_i\}_{i=1}^z) = 0 \cup \gamma_0(\{k_i\}_{i=1}^z) = 0\}$, which is the union of all the hypersurfaces along which the system can be *potentially* switching from DIS to non-DIS property in the controller gain space. To identify all the DIS regions, one has to inspect all the arising enclosed regions in k_i, denoted by $\mathcal{S}_\chi$, $\chi = 1, \ldots, \kappa$, and encapsulated by the boundaries of $\Gamma(\{k_i\}_{i=1}^z)$.

Theorem 4: System in (1) is DIS in the region $\mathcal{S}_\chi$, $\chi = 1, 2, \ldots, \theta$, $\theta < \kappa$, if for a test point $\mathbf{K} =: \{k_i\}_{i=1}^z \in \mathcal{S}_\chi$, the following two conditions simultaneously hold:
i) $|s\mathbf{I} - \mathbf{A} - \sum_{\ell=1}^L \mathbf{B}_\ell|$ is Hurwitz stable,
ii) Single point inspection in $\{\mathcal{S}_\chi\}_{\chi=1}^\theta$ when $\{y > 0 | \Phi(y, \{k_i\}_{i=1}^z) = 0\} = \emptyset$,
Proof. Condition (i) is obvious from Theorem 3. Furthermore, system (1) can never change its stability/instability behavior with respect to delay τ_ℓ if (2) does not possess any roots on $\mathbb{C}_0$. That is, $s = j\omega$ should not be a root of (6). Based on the features of the polynomial in (8), this can be guaranteed if the polynomial $\Phi(y, k_i) = 0$ does not have $y = \omega^2 \in \mathbb{R}_+$ roots for a single test point in each identified closed region $\mathcal{S}_\chi$, $\chi = 1, 2, \ldots, \theta$. By continuity, this property is guaranteed to hold for all the points in the region enclosed by some parts of $\Gamma(\{k_i\}_{i=1}^z)$. The condition (ii) of theorem thus reveals the fundamental DIS regions $\{\mathcal{S}_\chi\}_{\chi=1}^\theta$ in the controller gain space.□

Theorem 5: System in (1) is DIS in $\{k_i\}_{i=1}^z \in \mathbb{R}^z \backslash \{\mathcal{S}_\chi\}_{\chi=1}^\theta$, if and only if the conditions below are satisfied:
i) $|s\mathbf{I} - \mathbf{A} - \sum_{\ell=1}^L \mathbf{B}_\ell|$ is Hurwitz stable,
ii) Point-wise sweep of controller gains in $\{k_i\}_{i=1}^z \in \mathbb{R}^z \backslash \{\mathcal{S}_\chi\}_{\chi=1}^\theta$ whenever $\{y > 0 | \Phi(y, \{k_i\}_{i=1}^z) = 0\} \neq \emptyset$ for some $y^* = (\omega^*)^2 \in \mathbb{R}_+$, and when at least one of the conditions in (B)-(C) of Theorem 8 in the Appendix does not hold.
Proof. The condition provided in this theorem guarantees the only case for which $\Phi(y, k_i) = 0$ have roots $y = (\omega^*)^2 \in \mathbb{R}_+$, but these roots do not correspond to a root in $T_\ell \in \mathbb{R} \bigcup \{\mp\infty\}$ that vanishes $g_\Re$ and $g_\Im$ simultaneously. This inspection however requires point-wise sweep in all the remaining regions in $\{k_i\}_{i=1}^z \in \mathbb{R}^z \backslash \{\mathcal{S}_\chi\}_{\chi=1}^\theta$. Hence, along with the conditions (i)-(ii) of Theorem 4, all the DIS regions of system (1) can be identified.□

Remark 2: Checking the DIS property in the regions $\mathcal{S}_\chi$ requires a single-point test in $\mathcal{S}_\chi$, see Theorem 4. With this, some of the DIS regions and their boundaries $\Lambda(\{k_i\}_{i=1}^z)$ can be revealed. When a positive real root y of $\Phi(y, k_i) = 0$ exists in $\mathcal{S}_\chi$, then it is not possible to claim that all the T_ℓ corresponding to all k_i in $\mathcal{S}_\chi$ can remain in complex domain. Hence, the point-wise scanning in such regions is necessary, see Theorem 5.

IV. A Special Case

Among the special types of systems, the class of positive systems play an important role for continuous-time and discrete-time systems [23], [24]. The positive continuous-time systems also known as Metzlerian systems can also be defined even in the presence of delays.

Definition 4: The system (1) without the feedback control $\mathbf{K}$ is internally positive or Metzlerian delay system if for every initial condition and all input $\mathbf{u}(t) \in \mathbb{R}_+^m$, $t \geqslant 0$, we have $\mathbf{x}(t) \in \mathbb{R}_+^n$ and $\mathbf{y}(t) \in \mathbb{R}_+^r$ for $t \geqslant 0$.

Theorem 6: The system (1) without the feedback gain $\mathbf{K}$ is internally positive if and only if $\mathbf{A}$ is a Metzler matrix and the matrices $\bar{\mathbf{B}}$, $\mathbf{C}_\ell$ for all $\ell = 1, \ldots, L$ are nonnegative matrices, i.e., $\bar{\mathbf{B}} \in \mathbb{R}_+^{n \times m}$, and $\mathbf{C}_\ell \in \mathbb{R}_+^{r \times n}$.
Incorporating the feedback control $\mathbf{K}$ into (1), we obtain

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \sum_{\ell=1}^{L} \mathbf{B}_\ell \mathbf{x}(t - \tau_\ell). \tag{10}$$

Thus, based on Definition 4 and Theorem 6, one can state that (10) is a Metzlerian delay system if and only if $\mathbf{A}$ is a Metzler matrix, and the matrices $\mathbf{B}_\ell \in \mathbb{R}_+^{n \times n}$, $\ell = 1, \ldots, L$ are nonnegative.

Lemma 2: The Metzlerian delay system (10) without delay ($\tau_\ell = 0, \ell = 1, \ldots, L$) is asymptotically stable if and only if one of the following equivalent conditions are satisfied:
1) Eigenvalues of $\bar{\mathbf{A}} = \mathbf{A} + \sum_{\ell=1}^L \mathbf{B}_\ell$ have negative real parts.
2) All coefficients of the characteristic equation $\det(\lambda\mathbf{I} - \bar{\mathbf{A}}) = \lambda^p + \bar{a}_{p-1}\lambda^{p-1} + \ldots + \bar{a}_1\lambda + \bar{a}_0$ are positive, i.e., $\bar{a}_i > 0$ for $i = 1, \ldots, p-1$.
3) All principal minors of the matrix $-\bar{\mathbf{A}}$ are positive.
4) The matrix $\bar{\mathbf{A}}$ is nonsingular and $-(\bar{\mathbf{A}})^{-1} > 0$.
5) There exists a positive definite diagonal matrix $\mathbf{P}$ such that $\bar{\mathbf{A}}^T\mathbf{P} + \mathbf{P}\bar{\mathbf{A}}$ is negative definite.
6) There exists a positive vector $\mathbf{v} \in \mathbb{R}_+^n$ such that $\bar{\mathbf{A}}\mathbf{v} < 0$.
Now, assuming that (10) is a Metzlerian delay system, we can state the following result:

Theorem 7: The Metzlerian delay system (10) is DIS if and only if the Metzlerian system without delay $\dot{\mathbf{x}}(t) = \bar{\mathbf{A}}\mathbf{x}(t)$ is asymptotically stable.
The proof of this result can be established by writing the solution expression of (10) and show that if (10) is asymptotically stable then there exists a strictly positive vector $\mathbf{v} \in \mathbb{R}_+^n$ satisfying $\bar{\mathbf{A}}\mathbf{v} < 0$, which is precisely the part (6) of Lemma 2 associated with the Metzlerian system without delay. See [25], [26] for the details of the proof.
With the aid of Theorem 7, the Property 1 in Section II can further be simplified. Given $\{k_i\}_{i=1}^z$, the Metzlerian delay system (10) is DIS if and only if,

$$f(s, \{\tau_\ell\}_{\ell=1}^L, \{k_i\}_{i=1}^z)\Big|_{\{\tau_\ell\}_{\ell=1}^L = 0} = \left|\lambda\mathbf{I} - \bar{\mathbf{A}}\right|, \tag{11}$$

has no roots on $\mathbb{C}_{0+}$. Consequently, the existence of DIS gain regions of Theorem 2 of Section III can also be simplified as follows.

Corollary 1: For a given set of delays $\{\tau_\ell\}_{\ell=1}^L$, there exists at least one region in controller gain parameter space

$\{k_i\}_{i=1}^{z}$ where the Metzlerian delay system (10) is DIS if and only if $\bar{\mathbf{A}} = \mathbf{A} + \sum_{\ell=1}^{L} \mathbf{B}_\ell$ is asymptotically stable for $\{k_i\}_{i=1}^{z} = \{\tilde{k_i}\}_{i=1}^{z}$.

In a similar fashion Theorem 3 simplifies as follows.

Corollary 2: The Metzlerian delay system (10) is DIS in the delay parameter space $\{\tau_\ell\}_{\ell=1}^{L}$ with given gains $\{k_i\}_{i=1}^{z} = \{k_i^*\}_{i=1}^{z}$ in $\mathbf{B}_\ell$ if and only if $\bar{\mathbf{A}} = \mathbf{A} + \sum_{\ell=1}^{L} \mathbf{B}_\ell$ is asymptotically stable.

The derivation of fundamental closed-form DIS boundaries remain the same as described in Section III-D for the Metzlerian delay systems.

V. Case Study

For the sake of plotting and concise illustration of the results, we use two unknown controller gains k_1 and k_2, however this is not a limitation in the approach presented in Section III. Consider the characteristic equation in (2) as a second order system with two delays τ_1 and τ_2,

$$CE(s, \tau_1, \tau_2, k_1, k_2) = s^2 + 5\,s + 9 + (k_1\,s + 2)e^{-\tau_1\,s} \\ -(k_2\,s + 5)e^{-\tau_2\,s} + (k_1 + k_2)e^{-(\tau_1+\tau_2)\,s} = 0, \quad (12)$$

where feedback is affected by delays τ_1 and τ_2.

Recall from Remark 1 that for the system to be DIS, it is required that the open-loop system, which is the system without the delay influence as per (1), is stable. This is the case since $s^2 + 5\,s + 9 = 0$ has stable roots. Next, we analyze the stability of the delay-free controlled system as per condition (i) of Theorem 3, by studying (12) for $\tau_1 = 0$ and $\tau_2 = 0$, which reads,

$$s^2 + (k_1 - k_2 + 5)\,s + k_1 + k_2 + 6 = 0, \quad (13)$$

which is stable if and only if all the coefficients of s are positive. The positivity conditions result in the light-gray region in the controller gain plane as shown in Figure 1. We then implement our proposed approach, explained in the previous section, to obtain the function $D(\omega, k_1, k_2)$, and by change of variable $\omega^2 = y$, we inspect the positive real roots of $\Phi(y, k_1, k_2) = 0$, which is in the following form

$$\Phi(y, k_1, k_2) = \sum_{i=0}^{6} \gamma_i(k_1, k_2) y^i = 0, \quad (14)$$

where γ_i are polynomials in terms of k_1 and k_2. Using Theorem 3, we confirm that $\Phi(y, 0, 0) = 0$ has no $y \in \mathbb{R}_+$ roots, hence we have the point $(\tilde{k}_1, \tilde{k}_2) = (0, 0)$ in the DIS region as per Theorem 2, and we expect to find a DIS region at least around this point. We next apply the Sturm sequences on (14) point-by-point on $k_1 - k_2$ plane, and identify the points for which (ii)-(a) and (ii)-(b) of Theorem 3 are satisfied. The dotted points illustrated in Figure 1 represent delay-independent stability region. For comparison, we also present the DIS regions detected by Descartes Rule of Signs in dark gray (following from [10]) which is superimposed with the DIS region found by Theorem 3, see Figure 1.

We next investigate the conditions in Theorem 4 to reveal the fundamental analytic DIS boundaries in controller gain space. For this, we first calculate $\varphi(k_1, k_2) = 0$ and identify

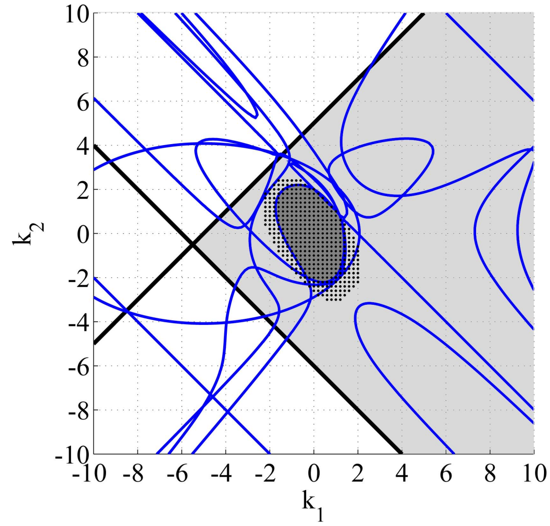

Fig. 1. Comparison of DIS region found with the application of Theorem 3 (dotted) and of Descartes Rule of Signs (dark gray) following from [10]. Black: Boundaries of stability of the delay-free controlled system in (13). Blue: Boundaries found by following [10].

$\gamma_0(k_1, k_2) = 0$. Suppressing the details, we provide the plot of $\Lambda(k_1, k_2)$ which is identified by checking the DIS property at only one point in each arising region as described in Theorem 4. Controller gains that render the system in (12) DIS are shown by dark-gray shaded regions in Figure 2, and they are bordered by the boundaries $\Lambda(k_1, k_2)$. Application

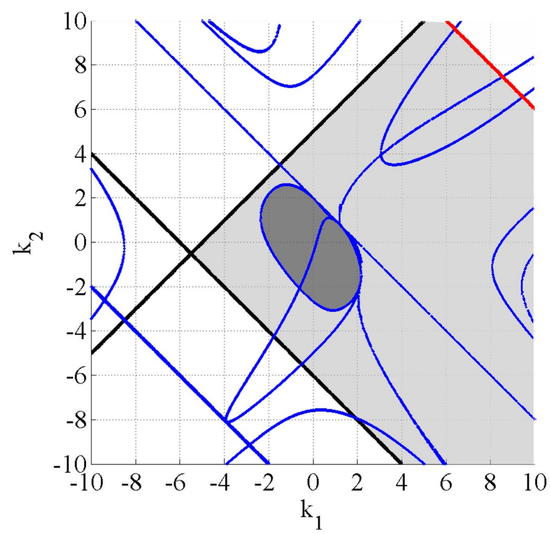

Fig. 2. Fundamental DIS boundaries in (k_1, k_2) plane found by using Theorem 4. Black: Boundaries of stability of the delay-free controlled system in (13). Blue: $\varphi(k_1, k_2) = 0$. Red: $\gamma_0(k_1, k_2) = 0$.

of Theorem 5 point-wise in the remaining light gray regions in Figure 2 does not return any other admissible points in $k_1 - k_2$ plane in the scale considered and the grid size taken, (grid size equal to 0.2 units) for rendering the system DIS. Therefore, based on these, we conclude that the only DIS region in this scale is given by the dark-gray region in Figure 2. Finally, we note that the boundaries $\Lambda(k_1, k_2)$ precisely encapsulate the dotted DIS gains in Figure 1, confirming the validity of Theorem 4.

Remark 3: Consistent with practical engineering problems, here the DIS control design assumes that delays τ_ℓ can be large but are finite, and hence the study here focuses on the *weak* DIS of (1). See [3] for details.

VI. Conclusion

We propose an approach for controller design for achieving delay-independent stability in linear time-invariant (LTI) systems with multiple delays including Metzlerian systems as a special case, based on algebra and polynomial theory. The approach can also admit various controller structures, while complying with the necessary and sufficient conditions of delay-independent stability. A case study is presented to demonstrate the design approach.

VII. Appendix

We start with converting the DIS analysis to an optimization problem in T_ℓ domain. Since it is necessary that $R_{T_L} = 0$ for $g_\Re = 0$ and $g_\Im = 0$ to have common zeros, then for ω to exhibit an extremum, it is necessary that $\frac{\partial \omega}{\partial T_{L-1}} = 0$, which can be found from

$$\frac{\partial R_{T_L}(g_\Re, g_\Im)}{\partial T_{L-1}} + \frac{\partial R_{T_L}(g_\Re, g_\Im)}{\partial \omega}\frac{\partial \omega}{\partial T_{L-1}} = 0. \qquad (15)$$

Under regular point assumption, we have $\partial R_{T_L}(g_\Re, g_\Im)/\partial\omega \neq 0$ [27], and the following holds[2]

Definition 5: Let $V_L = R_{T_L}(g_\Re, g_\Im)$ and $W_L = \frac{\partial V_L}{\partial T_{L-1}}$, and for d in descending order from $L-1$ to 1, $d = L - 1 \ldots 1$, calculate the following equations sequentially, $V_d = R_{T_d}(V_{d+1}, W_{d+1})$, and $W_d = \frac{\partial V_d}{\partial T_{d-1}}$, $d \neq 1$.

Theorem 8: [6], [8] For a given set of controller gains $\{k_i\}_{i=1}^{z}$, the system in (1) possesses at least one pole on $\mathbb{C}_0$ for some delays $\{\tau_\ell\}_{\ell=1}^{L}$ if and only if the following conditions are satisfied for some $\omega = \omega^* \in \mathbb{R}$ and $\{T_\ell^*\}_{\ell=1}^{L} \in \mathbb{R}^L$:

A) The repeated resultant, which is defined as $D(\omega) := R_{T_1} = V_1$ and only a function of ω, is satisfied $D(\omega^*) = 0$.

B) For ω^*, all the polynomials V_d, W_d, $d = 2 \ldots L$, used to construct each one of the resultants in Definition 5 are satisfied at $\{T_\ell^*\}_{\ell=1}^{L}$.

C) The transformed characteristic equation (4) for ω^* and $\{T_\ell^*\}_{\ell=1}^{L}$ is satisfied, $g_\Re = 0$, $g_\Im = 0$.

[2]The essence of Theorem 8 does not change for singular points [6], [8].

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