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Stability Concerns for Indirect Consumer Control in Smart Grids

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Abstract— **Demand side management will be an important tool for maintaining a balanced electrical grid in the future, when the penetration of volatile resources, such as wind and solar energy increases. Recent research focuses on two different management approaches, namely direct consumer control by an external third party, and indirect consumer control through incentives and price signals. In this work we present a simple formulation of indirect control, where the behavior of each consumer, is governed by local optimization of energy consumption. The local optimization accounts for both cost of energy and distribution losses, as well as any discomfort incurred by consumers from any shift in energy consumption. Our work will illustrate that in the simplest formulation of indirect control, the stability is greatly affected of both the behavior of consumers, and the number of consumers to include. We will show how instability is related to the local optimization problem of the consumer, and the information made available to him.**

I. INTRODUCTION

Current scientific and political interests are directed towards increasing the use of renewable energy and reduce power production from fossil fuels. A major concern related to this, is how to maintain a stable and balanced electrical grid, when a large part of the power comes from volatile resources, such as wind and sun. A suggested approach to this problem, is load shifting by use of different energy storages [1], for instance actively controlled consumption. The possibility of adjusting consumption is based on prognoses that a large part of the future consumption will be for transportation, *i.e*, electric vehicles (EVs), and heating systems in form of electric heat pumps (EHPs). Load shifting by active control of such consumption, is feasible for instance if a private house is to be maintained at a certain temperature, then shutting off the EHP would not be noticeable for some period of time. Similar considerations can be made for EVs, whereby these and similar consumption types, are usually called flexible consumption. It is expected that grid balancing in the future could be handled to some extend by adjusting the flexible consumption, rather than adjusting production. On this basis, recent research has focused on how to employ different types of consumers for grid balancing [2], [3].

Two different approaches are investigated for including consumers: direct and indirect control. By direct control is understood that consumers sign off control rights for part of their consumption to some third party, *e.g.*, a power company, power retailer or similar. The third party would then be able to control for instance the EHP of a number of households, honoring some prior agreement on for instance temperature bounds or other discomfort constraints. By aggregating the flexible consumption from a large number of consumers, a large energy storage can be maintained. Direct control is considered in [2], [3] among others.

The other approach is indirect control, where each consumer is in full control of his own consumption, however, a third party would present an incentive to act in a certain way. This could for instance be as a price signal indicating a high cost of energy when volatile resources are scarce, and low cost of energy when resources are ample. Based on such a price signal, consumers can plan their energy consumption as a trade-off on cost of energy, and the discomfort experienced for instance by lowering the indoor temperature. Indirect control and concepts relating to demand side management in general, have been outlined by [4], [5], [6], among others. However, the stability properties of indirect control schemes have not yet been fully analyzed. As the stability is affected by the specific behavior of the consumers, and further as the behavior of consumers cannot be decided directly, but only affected by incentives, it is important to consider how different types of consumers affect stability. Therefore the focus of this work is to suggest a basic mathematical formulation of indirect control, and show how the stability of this is affected by different consumer strategies.

In this work, the price of energy will not only be affected by the available amount of renewable energy. In our model, the cost of electricity will also encompass the cost of grid-losses. Grid-losses are included for two reasons. Firstly, in the Danish system, the cost of grid losses are covered by the distribution system operator [7], [8], and passed on to consumers through tariffs [9]. On this basis, consumers will have an incentive to minimize losses, not individually but rather as a society of consumers. Further, from a socio-economic point of view, as energy generation becomes more volatile by the increased penetration of renewable resources, it is beneficial to reduce losses and increase the efficiency of energy utilization.

This paper is organized as follows: Section II outlines the modeling of consumers, grid losses and energy prices, whereafter Section III outlines our indirect control framework. In Section IV, we analyze the stability of the presented framework, followed by numerical examples in Section V. Concluding remarks are presented in Section VI. A final Appendix elaborates part of the stability analysis.

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II. MODELING

This section first presents the modeling of both the cost of energy and cost of losses for each consumer. Subsequently, the consumer behavior is modeled.

A. Cost of energy

We consider a time horizon of length m, starting from the current time t_c. Without loss of generality, we set $t_c = 1$. We consider hourly measurements from consumers, meaning that the interval $[1,\ m]$ is divided into m discrete samples at hourly instances,

$$\mathcal{T} \equiv \{1, 2, \ldots, m\}.$$

Consider n households, and let $\mathbf{x}_i = (x_i(1), \ldots, x_i(m)) \in \mathbf{R}^m, i \in \{1, \ldots, n\}$ be the energy consumption in units of Watt-hours (Wh), for each household, during each hour $t \in \mathcal{T}$. Further, let $w(t) \in \mathbf{R}_+$, $t \in \mathcal{T}$ be a known price signal, providing the price of energy during each hour t, in units of ¤/Wh, where ¤ represents an arbitrary currency. As indicated, we assume that $w(t) > 0, \forall t$. Strictly speaking, at times it is possible that $w(t) < 0$, especially when introducing a significant amount of wind and solar power. We shall however, leave the analysis of these cases for future study. We let

$$\mathbf{W} = \mathrm{diag}(w(1), \ldots, w(m) \in \mathbf{R}_+^{m \times m},$$

be the diagonal, fixed, price matrix with the main diagonal consisting of the price of energy. Given the price signal, the cost of energy $c_{e,i} : \mathbf{R}^m \to \mathbf{R}$ for each household, is expressed by

$$c_{e,i}(\mathbf{x}_i) = \mathbf{1}^T \mathbf{W} \mathbf{x}_i, \quad i = 1, \ldots, n, \tag{1}$$

in units of currency (¤), where $\mathbf{1} = (1, \ldots, 1) \in \mathbf{R}^m$.

B. Cost of losses

Let all households be located closely together, for instance as a single street or small suburban town. We refer collectively to such a group of households, as a community. The power feed to the community from the remaining grid, is introduced through a lossy tie-connection. These losses represents ohmic losses, transformer losses, *etc*.

The community can conceptually be illustrated as in Fig. 1, as a number of radials connected to the remaining grid.

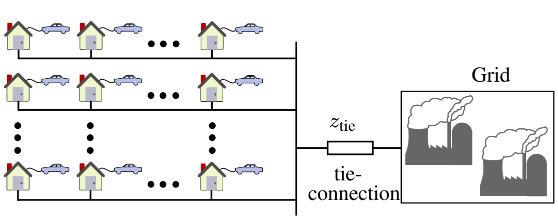

Fig. 1. Conceptual schematic outline of the community.

We assume that the households of the community are located sufficiently close, so the losses within the community are minor compared to losses in the tie-connection, and may be disregarded. The loss of energy in the tie-line is modeled in the following, where we assume that the grid is balanced, allowing us to conduct the analysis for a single phase equivalent system [10].

Remember that $x_i(t)$ in units of kWh, is the total energy consumption through the hour $t \in \mathcal{T}$. Let $p_i(t) \in \mathbf{R}$ in units of kW, denote the corresponding average power consumption through the hour t, *i.e.*;

$$x_i(t) = T_s p_i(t) \quad \forall i, t,$$

where $T_s = 1$ hrs is the length of the interval $t \in \mathcal{T}$. Conservation of power entails that the accumulated average power consumption of the community $p(t) \in \mathbf{R}$, is expressed as $p(t) = \sum_{i=1}^n p_i(t)$. It is important to notice that $p(t)$ *is not an instantaneous power, but instead, the average power during the hour* t.

Let $\gamma \in [0,\ 1]$ denote the power factor of the community, when aggregating consumption from all individual households. We assume γ constant, *i.e.*, $\gamma(t) = \gamma$, $\forall t \in \mathcal{T}$. As a consequence, the apparent power of the community, during hour t, is then $s(t) = p(t)/\gamma$ [11]. Since we have disregarded losses within the community, all households will consume power at the same voltage v, so the magnitude of the current is given by

$$|i(t)| = \frac{|s(t)|}{v} = \frac{|p(t)|}{\gamma v}.$$

The average tie-line losses, $x_{\text{tie}}(t)$ during hour t, depends on the squared current magnitude, and is thus represented as

$$x_{\text{tie}}(t) = \frac{p(t)^2}{(\gamma v)^2} r_T = \beta \left(\sum_{i=1}^n x_i(t) \right)^2 \quad \forall t \in \mathcal{T}, \tag{2}$$

where $r_T > 0$ is the tie-line resistance, and $\beta = r_T/(T_s \gamma v)^2$ is a loss parameter.

The cost of losses are expressed as $c_l(t) = w(t) x_{\text{tie}}(t)$, The total cost of losses is distributed among the individual consumers, by use of their individual share factor, *i.e*, their share of the total consumption. That is

$$c_{l,i}(t) = c_l(t) \frac{x_i(t)}{\sum_{j=1}^n x_j(t)} = \beta w(t) x_i(t) \left(\sum_{j=1}^n x_j(t) \right),$$

for $i \in \{1, \ldots, n\}$. The above is in units of currency (¤). With a slight misuse of notation we write the total cost of losses over the horizon, of a single household as

$$c_{l,i}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \sum_{t \in \mathcal{T}} c_{l,i}(t) = \beta \mathbf{x}_i^T \mathbf{W} \left(\sum_{j=1}^n \mathbf{x}_j \right). \tag{3}$$

Adding (3) to the cost of energy in (1), the total cost of energy and losses across the horizon for a single household, becomes

$$\begin{aligned} c_i(\mathbf{x}_1, \ldots, \mathbf{x}_n) &= c_{e,i}(\mathbf{x}_i) + c_{l,i}(\mathbf{x}_1, \ldots, \mathbf{x}_n) \\ &= \mathbf{1}^T \mathbf{W} \mathbf{x}_i + \beta \mathbf{x}_i^T \mathbf{W} \left(\sum_{j=1}^n \mathbf{x}_j \right), \end{aligned} \tag{4}$$

from which it is clear that the cost incurred by single household, depends of the consumption of the entire community, due to the added cost of losses.

C. Consumer flexibility

As mentioned in Section I, the electricity consumption will involve some level of flexible consumption. It will also contain amounts of inflexible consumption, which cannot be temporally shifted, *i.e.*, lights, television, *etc*. We will therefore model the energy consumption as

$$x_i(t) = \overline{x}_i(t) + \tilde{x}_i(t), \quad \forall t \in \mathcal{T}, \tag{5}$$

where $\overline{x}_i(t)$ denotes the inflexible consumption and $\tilde{x}_i(t)$ denotes flexible consumption. It is reasonable to consider $\overline{x}_i(t)$ as the known, traditional consumption, *i.e.*, a baseline consumption. The losses presented in (2) are typically in the order of 5 % of this baseline [12].

D. Consumer behavior

As also outlined in Section I, the flexible consumption of each individual consumer, may have different characteristics, which may also effect the way energy is consumed. In the following we outline two different types of consumers. The purpose of this is to illustrate, that when relying on indirect control, there is no way of deciding, or perhaps even knowing how any individual consumer will behave and react to a price signal. In other words, the local objective of any consumer remains unknown. Below we present the behavior of two types or classes of consumers which are fundamentally different, but both likely to be present in the grid. In Section IV, we present a stability analysis of the indirect control, in the extreme cases where a community consists solely of consumers of either type. We refer to the two consumer types as *greedy* and *comfort* consumers.

Greedy consumer

The greedy consumer-type represents consumers who are only concerned with the total price of energy across the horizon. This could be consumers who only provide flexibility to the grid through, for example, an EV. The consumer is thereby only concerned that the EV is charged to some required level, at the end of the horizon, *i.e.*, that the flexible consumption integrates to some fixed value. If this is achieved, the specific charge pattern is of no concern, or discomfort.

Given the relation between flexible consumption and total consumption (5), the local optimization problem of the greedy consumer is then

$$\begin{array}{ll} \underset{\mathbf{x}_i}{\text{minimize}} & c_i(\mathbf{x}_1, \ldots, \mathbf{x}_n) \\ \text{subject to} & \mathbf{1}^T \tilde{\mathbf{x}}_i = \alpha_i, \end{array} \tag{6}$$

for $i = 1, \ldots, n$, where we have extended the bold notation introduced previously, *i.e.*, $\tilde{\mathbf{x}}_i = (\tilde{x}_i(1), \cdots, \tilde{x}_i(m)) \in \mathbf{R}^m$, for all i. Above, $\alpha_i > 0$, in units of Wh, is the required accumulated consumption of flexible energy during the horizon. The constraint ensures that the EV is charged to the required level.

As we have argued previously, the cost of any single consumer is affected by the consumption of all consumers, through the cost of losses. We have indicated this in the notation of (6), as the cost function depends on $\mathbf{x}_1, \ldots, \mathbf{x}_n$, but the consumer can only control his local consumption $\mathbf{x}_i$.

Comfort consumer

The second type of consumer, is the comfort consumer. As opposed to the greedy consumer, the comfort consumer is not only accounting for the cost of energy and losses when optimizing a consumption profile, in that the comfort consumer also includes a cost of discomfort. In this work, we shall model this as a consumer with an installed EHP, where the discomfort is measured as deviation from a desired set-point temperature in units of °C:

$$\mathbf{T}_{\text{sp},i} = (T_{\text{sp},i}(1), \ldots, T_{\text{sp},i}(m)) \in \mathbf{R}^m,$$

for $i = 1, \ldots, n$. Let the indoor temperature of household i be denoted $T_i \in \mathbf{R}, i = 1, \ldots, n$, also in units of °C. Then, given some initial value $T_{i,0}$, the household temperature is modeled as

$$T_i(t+1) = a_i T_i(t) + b_i \tilde{x}_i(t) + e_i T_a(t), \quad t \in \mathcal{T}, \tag{7}$$

for all i, where $T_a(t) \in \mathbf{R}$ is the ambient temperature, and

$$0 < a_i < 1, \quad b_i, e_i > 0,$$

are known, household specific parameters, accounting for heat dissipation, EHP efficiency and outside conditions. The model could be expanded to also include elements such as direct and indirect solar radiation, but we shall save this for future work. For more thorough discussion of the thermal modeling, consult [2].

With the same bold-font notation as previously, we let

$$\mathbf{T}_i(\tilde{\mathbf{x}}_i) = (T_i(1), \ldots, T_i(m)) \in \mathbf{R}^m,$$

where we have written $\mathbf{T}_i(\tilde{\mathbf{x}}_i)$ since the temperature $T_i(\tau)$ for any $\tau \in \mathcal{T}$, depends on $\tilde{x}_i(t)$ for $t = 1, \ldots, \tau - 1$. Given the temperature model and the desired set-point, the discomfort of the consumer is modeled by $d_i : \mathbf{R}^n \to \mathbf{R}$ as

$$d_i(\tilde{\mathbf{x}}_i) = (\mathbf{T}_{\text{sp},i} - \mathbf{T}_i(\tilde{\mathbf{x}}_i))^T (\mathbf{T}_{\text{sp},i} - \mathbf{T}_i(\tilde{\mathbf{x}}_i)), \tag{8}$$

that is; the discomfort is quadratic in temperature deviation. Including both the total cost of energy from (4), and the cost of discomfort in (8), results in the following optimization problem for the comfort consumer

$$\underset{\tilde{\mathbf{x}}_i}{\text{minimize}} \quad c_i(\mathbf{x}_1, \ldots, \mathbf{x}_n) + \lambda_i d_i(\tilde{\mathbf{x}}_i) \tag{9}$$

for $i = 1, \ldots, n$, where $\lambda_i > 0$ is a unitless local trade-off parameter.

We could include a number of constraints, such as discomfort limits or actuator limits as bounds on $\mathbf{x}_i$. For the time being, we shall omit to do so, and the reader is referred to, *e.g.*, [2].

III. Indirect control

In this section we outline the indirect control framework. As evident, the local cost function of both Problem (6) and Problem (9), is affected by the consumption pattern across the entire community, which is a consequence of the cost of losses in (3). To simplify notation, we let

$$\mathbf{q}_i = \sum_{j \neq i} \mathbf{x}_j.$$

With this notation, we have

$$\begin{aligned} c_i(\mathbf{x}_i, \mathbf{q}_i) &= \mathbf{x}_i^T \mathbf{W} \mathbf{1} + \beta \mathbf{x}_i^T \mathbf{W}(\mathbf{x}_i + \mathbf{q}_i) \\ &= (\mathbf{1} + \beta \mathbf{q}_i)^T \mathbf{W} \mathbf{x}_i + \beta \mathbf{x}_i^T \mathbf{W} \mathbf{x}_i \end{aligned} \tag{10}$$

and we further write

$$\tilde{\mathbf{x}}_i^\star(\mathbf{q}_i) = \underset{\{\tilde{\mathbf{x}}_i | \mathbf{1}^T \tilde{\mathbf{x}}_i = \alpha_i\}}{\arg\inf} (c_i(\mathbf{x}_i, \mathbf{q}_i)),$$

and

$$\tilde{\mathbf{x}}_i^\star(\mathbf{q}_i) = \underset{\tilde{\mathbf{x}}_i}{\arg\inf} (c_i(\mathbf{x}_i, \mathbf{q}_i) + \lambda_i d_i(\tilde{\mathbf{x}}_i)),$$

to denote the solution of the optimization for greedy and comfort consumers respectively, given $\mathbf{q}_i$.

For each individual household, the consumption of the remaining community, $\mathbf{q}_i$, is unknown. Our approach to indirect control, is therefore to assume that an approximation of $\mathbf{q}_i$ is available to each household. Such approximations could be made available, for instance, by the distribution system operator (DSO). Based on the approximation of $\mathbf{q}_i$ for $i \in \{1, \ldots, n\}$, each consumer can optimize consumption locally, and we can subsequently improve the approximation of all $\mathbf{q}_i$, through iterative updates. This is summarized in Algorithm 1.

Algorithm 1 Indirect control of consumers

Initialize estimates $\mathbf{q}_i^{(0)}, i = 1, \ldots, n$
for k=0,1, ... **do**
  Obtain local solutions:
    $\mathbf{x}_i^\star(\mathbf{q}_i^{(k)})$, $i = 1, \ldots, n$
  Update estimates:
    $\mathbf{q}_i^{(k+1)} = \sum_{j \neq i} \mathbf{x}_j^\star(\mathbf{q}_j^{(k)})$, $i = 1, \ldots, n$
end for

In Algorithm 1, we have used $\mathbf{q}_i^{(k)}$, to denote the estimated $\mathbf{q}_i$, at iteration k. Convergence of Algorithm 1, in the sense that

$$\lim_{k \to \infty} (\mathbf{x}_i^{(k+1)} - \mathbf{x}_i^{(k)}) = 0, \quad \forall i, \tag{11}$$

would entail that there is no update of the consumption patterns for any consumer. The process has then reached a Nash equilibrium, in the sense that no consumer desires to alter their current consumption pattern, provided that the remaining community refrains from changing theirs as well.

The approach outlined above, allows each individual consumer to privately plan and optimize the optimization pattern, accounting for private comfort concerns, as well as the cost of energy and cost of grid-losses. However, it requires some exchange service, or shared data center for collecting

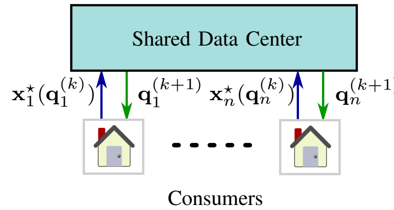

Fig. 2. Accounting for grid losses, requires data exchange with a shared data center.

$\mathbf{x}_i(\mathbf{q}_i^{(k)})$ and distributing $\mathbf{q}_i^{(k+1)}$ for each iteration of the algorithm, for all i. This is illustrated in Fig. 2.

Granting consumers access to information in this fashion, is what allows them to account for grid losses, by considering the action of the remaining members of the community. This poses a benefit for both the consumer, and for society as a whole, since the consumer desires to consider losses as an expense, and society desires to reduce the waste of energy.

Notice, that no consumer receives direct information about any other consumer. Instead, only information concerning the community as a whole is distributed, so no privacy concerns are violated.

With this outline of the indirect control framework, the following section examines the stability for the two types of consumers described in Section II-D, in the sense of convergence of Algorithm 1, as given by (11).

IV. Stability Analysis

The following analysis is performed for the two extremal cases, where the community consists solely of greedy or comfort consumers. We shall leave mixed populations for future work. Following the stability analysis, Section V gives numerical examples.

A. Stability of Greedy Consumers

For greedy consumers, with $c_i(\mathbf{x}_i, \mathbf{q}_i)$ replaced by the expression (10), the Lagrangian of Problem (6) becomes
$L_i(\mathbf{x}_i, \mu_i) =$
$(\mathbf{1}+\beta\mathbf{q}_i)^T \mathbf{W}(\overline{\mathbf{x}}_i+\tilde{\mathbf{x}}_i)+\beta(\overline{\mathbf{x}}_i+\tilde{\mathbf{x}}_i)^T \mathbf{W}(\overline{\mathbf{x}}_i+\tilde{\mathbf{x}}_i)+\mu_i(\mathbf{1}^T\tilde{\mathbf{x}}_i-\alpha_i)$,
where we remind the reader that $\mathbf{x}_i = \overline{\mathbf{x}}_i + \tilde{\mathbf{x}}_i$, and the optimization variable is only $\tilde{\mathbf{x}}_i$. The baseline consumption $\overline{\mathbf{x}}_i$ is assumed fixed and known. Similarly, at each iteration of the previously described algorithm, $\mathbf{q}_i$ represents a fixed and known parameter. Above, $\mu_i \in \mathbf{R}$ is the Lagrange multiplier for the equality constraint. The Karush-Kuhn-Tucker (KKT) conditions are then

1) $\nabla_{\tilde{\mathbf{x}}_i} L(\mathbf{x}_i, \mu) = \mathbf{W}(\mathbf{1} + \beta(\mathbf{q}_i + 2\overline{\mathbf{x}}_i)) + 2\beta\mathbf{W}\tilde{\mathbf{x}}_i + \mathbf{1}\mu_i = 0.$
2) $\mathbf{1}^T \tilde{\mathbf{x}}_i = \alpha_i$

For fixed $\overline{\mathbf{x}}_i$ and $\mathbf{q}_i$, Problem (6) is convex, and the KKT conditions are both necessary and sufficient [13]. We let $\mathbf{I}$ denote the identity, and introduce the matrix

$$\tilde{\mathbf{I}} = \begin{bmatrix} \mathbf{I} \\ 0 \ldots 0 \end{bmatrix} \in \mathbf{R}^{n+1 \times n}.$$

The conditions above may then be formulated as

$$\underbrace{\begin{bmatrix} 2\beta\mathbf{W} & \mathbf{1} \\ \mathbf{1}^T & 0 \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \tilde{\mathbf{x}}_i \\ \mu \end{bmatrix}}_{\mathbf{y}_i} = \underbrace{\begin{bmatrix} -\mathbf{W}(\mathbf{1} + 2\beta\overline{\mathbf{x}}_i) \\ \alpha_i \end{bmatrix}}_{\mathbf{h}_i} - \beta\tilde{\mathbf{I}}\mathbf{W}\mathbf{q}_i.$$

From this we have

$$\tilde{\mathbf{x}}_i^\star(\mathbf{q}_i) = \tilde{\mathbf{I}}^T\mathbf{y}_i^\star(\mathbf{q}_i) = \tilde{\mathbf{I}}^T\mathbf{M}^{-1}\mathbf{h}_i - \beta\tilde{\mathbf{I}}^T\mathbf{M}^{-1}\tilde{\mathbf{I}}\mathbf{W}\mathbf{q}_i. \quad (12)$$

Introducing (12) in the iterative process of Algorithm 1 entails that

$$\mathbf{x}_i^\star(\mathbf{q}_i^{(k)}) = \tilde{\mathbf{x}}_i^\star(\mathbf{q}_i^{(k)}) + \overline{\mathbf{x}}_i = \mathbf{p}_i + \Phi\mathbf{q}_i^{(k)},$$

where

$$\mathbf{p}_i = \tilde{\mathbf{I}}^T\mathbf{M}^{-1}\mathbf{h}_i + \overline{\mathbf{x}}_i, \quad \Phi = -\beta\tilde{\mathbf{I}}^T\mathbf{M}^{-1}\tilde{\mathbf{I}}\mathbf{W}.$$

Let $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$, $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_n)$, $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$, and further

$$\mathbf{H} = \begin{bmatrix} \Phi & & \\ & \ddots & \\ & & \Phi \end{bmatrix}, \quad \mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \cdots & \mathbf{I} \\ \mathbf{I} & \mathbf{0} & \cdots & \mathbf{I} \\ \vdots & & \ddots & \vdots \\ \mathbf{I} & \mathbf{I} & \cdots & \mathbf{0} \end{bmatrix},$$

then

$$\mathbf{x}^\star(\mathbf{q}^{(k)}) = \mathbf{p} + \mathbf{H}\mathbf{q}^{(k)}, \quad (13)$$

and

$$\mathbf{q}^{(k+1)} = \mathbf{J}\mathbf{x}^\star(\mathbf{q}^{(k)}) = \mathbf{J}\mathbf{p} + \mathbf{J}\mathbf{H}\mathbf{q}^{(k)}. \quad (14)$$

Let $\nu_i(\mathbf{JH}), i = 1, \ldots, n$ denote the eigenvalues of $\mathbf{JH}$, and let $\mathcal{V}(\mathbf{JH}) = \{\nu_i(\mathbf{JH}), i = 1, 2, \ldots\}$ denote the spectrum. Then the iterative update in (14), converges only in the sense of (11), provided that

$$\max_i |\nu_i(\mathbf{JH})| < 1, \quad (15)$$

as this is the requirement for (14) to be a contraction [14].

The block-diagonal structure of $\mathbf{H}$, entails that ([15])

$$\nu \in \mathcal{V}(\Phi) \Rightarrow \nu \in \mathcal{V}(\mathbf{H}).$$

Further, it is easily shown that for $n \geq 2$

$$\nu \in \mathcal{V}(\mathbf{H}) \Rightarrow (n-1)\nu \in \mathcal{V}(\mathbf{JH}).$$

Finally, as elaborated in the Appendix, $-1/2 \in \mathcal{V}(\Phi_i), \forall i$, independent of β. With the arguments above, this entails that for $n > 1$

$$\frac{1-n}{2} \in \mathcal{V}(\mathbf{JH}),$$

whereby the requirement in (15) is only obeyed for $n \leq 2$. Therefore, if a community consists solely of greedy consumers, and contains more than 2 households, convergence and thereby stability of Algorithm 1, is not achieved. This shows that there is a risk pertaining to the indirect control, since a certain behavior of consumers could lead to instability. To ensure stability against greedy consumers would require a revision of the framework. For instance, by including some filtering scheme for the estimates of $\mathbf{q}_i$. We shall illustrate the behavior of greedy consumers with examples in Section V, and comment further on this in Section VI.

B. Stability of Comfort Consumers

We perform a similar analysis of the stability of comfort consumers. From a logical perspective, it would make sense that indirect control in this case, would be stable, at least when supplying a sufficiently large trade-off coefficient λ_i. This is so, since it would not make sense for the consumer to enforce a very large positive or negative consumption, since this would incur a significant cost of discomfort. We shall show that contrary to the greedy consumers, stability of comfort consumers can be guaranteed for arbitrary large community, n, provided that some lower limit on the trade-off parameter λ_i is guaranteed, *i.e*, comfort is sufficiently important for all consumers, compared to the cost of energy. To this end, we introduce the notation

$$\mathbf{A}_i = \begin{bmatrix} a_i \\ a_i^2 \\ \vdots \\ a_i^m \end{bmatrix}, \quad \mathbf{G}_i = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ a_i & 1 & 0 & \cdots & 0 & 0 \\ a_i^2 & a_i & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_i^{m-1} & a_i^{m-2} & a_i^{m-3} & \cdots & a_i & 1 \end{bmatrix},$$

$$\mathbf{D}_i = b_i^2\mathbf{G}_i^T\mathbf{G}_i, \quad \boldsymbol{\zeta}_i = b_i\mathbf{G}_i^T(\mathbf{T}_{\text{i,sp}} - \mathbf{A}_iT_{i,0} - e_i\mathbf{G}_i\mathbf{T}_a)$$

where $\mathbf{D}_i$ is symmetric and positive definite. When employing the cost of energy (10), the KKT conditions of Problem (9) reduces to the objective function having zero-gradient, *i.e.*

$$\nabla_{\tilde{\mathbf{x}}_i}c_i(\mathbf{x}_i, \mathbf{q}_i) + \lambda\nabla_{\tilde{\mathbf{x}}_i}d_i(\tilde{\mathbf{x}}_i) = 0.$$

Given the dynamics in (7) for the thermal process, and the notation introduced above, the zero-gradient requirement is expressed by

$$\mathbf{W}(\mathbf{1}+\beta(\mathbf{q}_i+2\overline{\mathbf{x}}_i))+2\beta\mathbf{W}\tilde{\mathbf{x}}_i+2\lambda_i\mathbf{D}_i\tilde{\mathbf{x}}_i-2\lambda_i\boldsymbol{\zeta}_i = 0, \quad (16)$$

from which it is clear that

$$\tilde{\mathbf{x}}_i^\star(\mathbf{q}_i) = \Phi_i^{-1}\underbrace{\left(\lambda\boldsymbol{\zeta} - \frac{1}{2}\mathbf{W}\mathbf{1} - \beta\mathbf{W}\overline{\mathbf{x}}_i)\right)}_{\mathbf{h}_i} - \frac{1}{2}\beta\Phi_i^{-1}\mathbf{W}\mathbf{q}_i,$$

where we introduce $\Phi_i = \beta\mathbf{W} + \lambda_i\mathbf{D}_i$, with the matrix Φ_i being full rank if $a_i, b_i, \lambda_i \neq 0$, and therefore, it is nonsingular. The reader will notice that we have reintroduced and redefined some of the notation from Section IV-A. We do this partly to limit the extend of our notation, and partly in order to emphasize the similarities between the two consumer types, even though the basic behavior between them, is different.

Similar to the greedy case, we can collect the expression for each household into matrix form;

$$\mathbf{x}^\star(\mathbf{q}^{(k)}) = \mathbf{p} + \mathbf{H}\mathbf{q}^{(k)}, \quad (17)$$

and

$$\mathbf{q}^{(k+1)} = \mathbf{J}\mathbf{x}^\star(\mathbf{q}^{(k)}) = \mathbf{J}\mathbf{p} + \mathbf{J}\mathbf{H}\mathbf{q}^{(k)}. \quad (18)$$

with $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$, $\mathbf{p}_i = \Phi_i^{-1}\mathbf{h}_i + \overline{\mathbf{x}}_i$ and

$$\mathbf{H} = \frac{-\beta}{2}\begin{bmatrix} \Phi_1^{-1}\mathbf{W} & & \\ & \ddots & \\ & & \Phi_n^{-1}\mathbf{W} \end{bmatrix},$$

and $\mathbf{J}$ is similar to previous. As before, convergence for comfort consumers requires

$$\max_i |\nu_i(\mathbf{JH})| < 1, \tag{19}$$

In the following we show that for any $n > 0$, there exists a sufficiently large λ_i, $i = 1, \ldots, n$, such that (19), is obeyed. Notice that

$$\Phi_i^{-1}\mathbf{W} = (\beta\mathbf{W} + \lambda_i\mathbf{D}_i)^{-1}\mathbf{W} = \frac{1}{\lambda_i}\left(\frac{\beta}{\lambda_i}\mathbf{W} + \mathbf{D}_i\right)^{-1}\mathbf{W}.$$

Hence, increasing λ_i entails that the row sum of any row of $\Phi_i^{-1}\mathbf{W}$ can get arbitrarily close to zero. So, increasing λ_i for all i, entails that any row sum of $\mathbf{JH}$ can come arbitrarily close zero, and so by the Geršgorin disc theorem [15], the eigenvalues of $\mathbf{JH}$ can come arbitrarily close to zero. Therefore, if all consumers increase λ_i sufficiently, the convergence criteria (19) is guaranteed.

This concludes our stability analysis for the two consumer types. The following section illustrates our results with a few numerical examples. As mentioned initially, convergence of the indirect control, in the case of arbitrary many consumers, can only be guaranteed for comfort consumers, and this only if a sufficiently large trade-off parameter is employed by the local optimization.

V. Examples

In the following, we present numerical examples to illustrate the main results from the previous sections. In all examples we consider a time-horizon for the local optimization of $m = 24$, corresponding to for instance 24 hours. The energy price and ambient temperature during this period is depicted in Fig. 3(Top) and (Middle). The curve in Fig. 3(Bottom) is representative for the baseline consumption; however, the baseline consumption for each individual consumer will contain some stochastic perturbation of this generic curve. In the examples, these perturbations are known in advance.

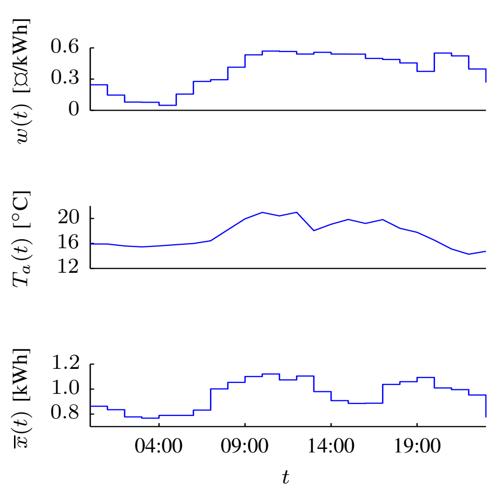

Fig. 3. Top: Price of energy through the period of optimization. Middle: Ambient temperature. Bottom: Baseline consumption.

The following examples illustrate our results for the following scenarios:

1. Community of greedy consumers with convergence
2. Community of greedy consumers where convergence is not obtained
3. Community of comfort consumers where convergence is obtained

1) Greedy consumers, convergence obtained: A community of greedy consumers in a setting where convergence is obtained, implies as we have shown, that $n \leq 2$. Below we illustrate the case for $n = 2$. In the example, we assume that the greedy consumers represent owners of EVs, that needs to be charged during the simulation horizon. Since no constraints towards maximum EV battery charge capacity where included in the convergence analysis, the converged solution cannot be expected to obey any considerations towards such constraints. Hence, to make the example more realistic, we include a maximum battery capacity constraint in the optimization for each of the two consumers. This does not affect the convergence.

After a number of iterations of Algorithm 1, Fig. 4 shows the converged flexible consumption pattern for each of the two consumers. By analyzing the figure, it is seen that energy is bought whenever there is a local minimum in the price, and subsequently sold, *i.e.* the battery is discharged, when prices are subsequently high.

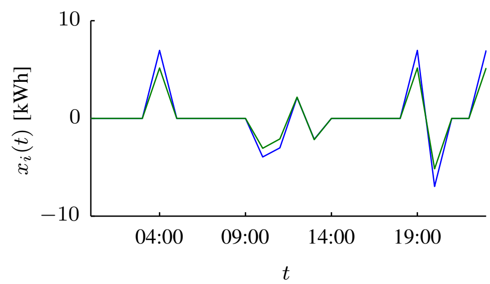

Fig. 4. Optimized, flexible consumption pattern, for the two households.

The accumulated flexible consumption is shown in Fig. 5, along with the consumption constraint and charge limit. Even though the battery is both charged and discharged several times throughout the horizon, the final charge level meets the constraint.

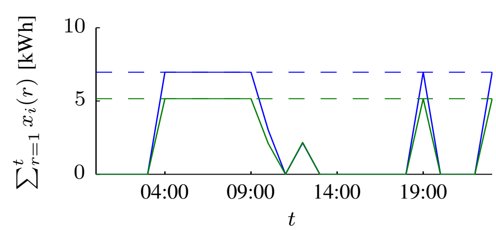

Fig. 5. Accumulated flexible consumption (Solid), and consumption constraint (Dashed), for both households.

We remark that the reader should focus on the characteristics of the consumption behavior, rather than the actual numbers presented in the example. That is, since greedy consumers acquires no discomfort by their flexible consumption, there can be introduced fluctuating consumption from buying and selling power, whenever the price signal indicates this to be beneficial.

2) Greedy consumers, no convergence: For $n = 20$, the indirect control framework diverges for greedy consumers. This is illustrated by Fig. 6, where the maximum entry of $\mathbf{q}_i^{(k)}$ is plotted for $k = 1, 2, 3$, for each of the 20 households. As evident, the algorithm does not converge, and the community estimates $\mathbf{q}^{(k)}$ diverges.

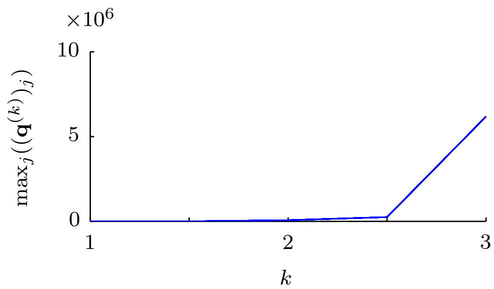

Fig. 6. Divergence of the community consumption estimates $\mathbf{q}_i^{(k)}$, for the unstable case of $n = 20$.

Adding constraints on battery charge limits as in the previous example, will not remedy this situation. Introducing such constraints would simply entail that rather than diverging towards infinity, as in Fig. 6, the local optimization of each household will simply alternate between two solutions, between each iteration of Algorithm 1, but it will not converge.

3) Comfort consumers, convergence obtained: Below we present an example including $n = 100$ comfort consumers, characterized by the parameters

$$a_i \in [0.936;\ 0.961],\ b_i \in [0.9;\ 0.95],\ e_i \in [0.039;\ 0.064]$$

$$T_{i,0} = 21\ ^\circ\text{C}, \quad T_{\text{i,sp}}(t) = 21\ ^\circ\text{C},\ \forall t,$$

for all i. The parameters above entail that the thermal dynamics of each household has a time-constant of 15-25 hours. The baseline consumption is again represented by Fig. 3(Bottom), and the grid losses corresponds to about 5 % of this consumption. The trade-off parameter has been set equal for all households, such that $\lambda_i = 0.5, i \in \{1, \ldots, 100\}$, which is large enough to ensure stability.

After convergence of Algorithm 1, the flexible consumption pattern of each household appears as in Fig. 7(Top).

Initially, an energy storage is built when the price is low. The storage is then depleted later when prices are higher in the period 09.00-19.00. This is visible in Fig. 7(Bottom) where the temperature is increased beyond the set-point when prices are low. The temperature then drops gradually when prices are higher.

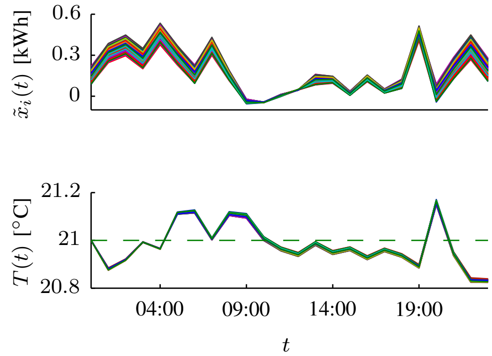

Fig. 7. Top: Consumption pattern in the implementation of 100 comfort consumers. Bottom: The temperature setpoint (Dashed), and the household temperatures, resulting from the consumption patterns above (Solid).

This shows that even for larger communities, the coordination among consumers as described through Algorithm 1, can be stable, provided that the trade-off parameter λ_i between cost and discomfort, is sufficiently large, for each consumer. In other words; each consumer needs to put a sufficiently large weight on comfort as a trade-off with cost of energy.

We have omitted to present an example illustrating an unstable implementation of a community of comfort consumers, since this would simply result in similar results as illustrated in Fig. 6.

In the examples presented here, we have adjusted the loss parameter β according to community size, such that the losses are roughly 5 % of the baseline consumption. Since the baseline increases when more consumers are added to the community, this entails that β is reduced accordingly. This would correspond to a distribution network being designed with wider cables, ensuring lower resistance for larger communities.

VI. Conclusion

In this work we have outlined a simple and intuitive framework for the concept of indirect control. We have described two types of consumers that are likely to exist in a smart grid, and their appertaining optimization problem for planning consumption across a time horizon, when faced with a price signal. We have included a cost of energy losses throughout the grid, which ties together all consumers. In order to account for cost of losses, we have introduced an iterative approach for sharing information between consumers. Despite the intuitive formulation, we have shown that the iterative approach for local optimization of consumption, in general fails to converge for communities consisting of greedy consumer. However, convergence can be guaranteed for arbitrary large communities of comfort consumers, provided the comfort trade-off parameter is sufficiently large.

With this we have illustrated that since local consumer behavior cannot be controlled, and perhaps is unknown in the framework of indirect control, several stability issues could potentially complicate the balance of the grid, and the risk of instability is related to the specific behavior and strategy of consumers. Therefore, deeper analysis of main consumer types and behavior, and the consequences for stability should be considered.

The results presented here also illustrates that a more sophisticated incentive scheme should be formulated, or a more elaborate data exchange must be devised, compared to the intuitive and naive approach outlined here. A first attempt could be to include some filtering process before distributing the estimates $\mathbf{q}_i$ to the households, in order to prevent divergence. It can be shown that even simple filtering schemes present stabilizing capabilities for both types of consumers outlined here. In that case however, it becomes necessary to analyze the risk of cheating, *i.e.* the risk of some consumers deviating from the agreed data exchange, if this presents potential benefits. However, such game theoretic considerations have been deemed outside the scope of this study. Further, when filtering processes, or similar attempts to affect consumers, are included, the framework shifts to some extend from indirect control, to direct control, where consumers are controlled directly, rather than through incentives. In our framework, we have not affected consumers in any other way, than to present them with incentives through price signals, and information about the total community consumption.

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APPENDIX

In the section relating to greedy consumers, we have introduced the matrix

$$\Phi = -\beta\tilde{\mathbf{I}}^T\mathbf{M}^{-1}\tilde{\mathbf{I}}\mathbf{W}, \quad \text{with} \quad \mathbf{M} = \begin{bmatrix} 2\beta\mathbf{W} & \mathbf{1} \\ \mathbf{1}^T & 0 \end{bmatrix}$$

and $\mathbf{W} = \operatorname{diag}(w(1), \ldots, w(m) \in \mathbf{R}_+^{m\times m}$, where $\mathbf{M} \in \mathbf{R}^{m+1\times m+1}$, and $\Phi \in \mathbf{R}^{m\times m}$. Let $w_i \equiv w(i)$, and $\rho = \sum_{i=1}^m w_i^{-1}$, then by straight calculation, we see that

$$\Phi = \frac{-1}{2\rho}\begin{bmatrix} \sum_{j\neq 1} w_j^{-1} & -1/w_1 & \cdots & -1/w_1 \\ -1/w_2 & \sum_{j\neq 2} w_j^{-1} & \cdots & -1/w_2 \\ \vdots & & \ddots & \vdots \\ -1/w_m & -1/w_m & \cdots & \sum_{j\neq m} w_j^{-1} \end{bmatrix}$$

Let $\mathbf{u} = (u_1, \cdots, u_m) \in \mathbf{R}^{m+1}$ be any vector that satisfies

$$\sum_{i=1}^m u_i = 0 \Rightarrow \sum_{j\neq i} u_j = -u_i, \quad \forall i, \tag{20}$$

and observe that

$$\begin{aligned} \Phi\mathbf{u} &= \frac{-1}{2\rho}\left((\sum_{j\neq i} w_j^{-1})u_i - w_i^{-1}\sum_{j\neq i} u_j\right) \\ &= \frac{-1}{2\rho}\left((\sum_{j\neq i} w_j^{-1})u_i + w_i^{-1}u_i\right) \\ &= \frac{-1}{2\rho}\rho u_i = \frac{-1}{2}u_i, \end{aligned}$$

for all $i \in \{1, \ldots, n\}$. This entails that $\mathbf{u}$ is an eigenvector for Φ, with eigenvalue $\nu = -1/2$.

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