

Control of Polynomial Dynamical Systems on Rectangles

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Abstract—In this paper we focus on a particular class of nonlinear dynamical systems given by polynomial vector fields in rectangular domains (boxes). This is a generalization of the work of Belta and Habetts dealing with multi-affine dynamical systems on rectangles. The main idea is to use the blossoming principle which allows us to relate our polynomial dynamical system to a multi-affine one. This technique allows us to establish sufficient conditions for invariance of a rectangle or exit of a rectangle through a given facet. We extend these results to handle control synthesis. Finally, we show how our approach can be used to solve motion planning problem.

I. INTRODUCTION

Motion planning and control of dynamical systems is a fundamental problem that has received a lot of attention thanks to its various applications. In the past decade, abstraction methods have shown interesting results especially in the case of dynamics with low nonlinearities. In [2], discrete abstractions for robot motion planning and control in polygonal environments are considered. It consists on the use of triangulations of the environment to provide correct control laws for planar robots when affine systems are considered to model the dynamics of the robot. In [5], [7] the reachability and the control synthesis problems are addressed in the case of piecewise affine hybrid systems on simplices. An approach given by Belta and Habetts [1] for a particular class of nonlinear dynamical systems called multi-affine systems solves the problem for rectangular domains based on a convexity property of multi-affine functions on such domains. For polynomial systems, an abstraction technique based on SOS techniques has recently been proposed in [6].

In this paper, we present an approach allowing us to solve the motion planning problem for a class of continuous nonlinear systems defined by polynomial vector fields using only simple linear programs. This approach can be seen as a generalization of the approach given in [1]. It is also related to [3], [4] where invariance of polyhedral domains is also treated using the blossoming principle. In this paper, we present an abstraction method based on this principle allowing us to recast our polynomial dynamical system as a multi-affine one. Then, we find conditions ensuring the invariance and the exit by a facet problem for trajectories of autonomous polynomial dynamical system in rectangular domains and then we generalize these results for the purpose of control synthesis. As an application, we solve the motion

planning problem for polynomial dynamical systems in a given sequence of adjacent rectangles.

The rest of the paper is organized as follows. In section 2, we state some preliminary results that will be useful for the rest of the paper. In section 3, we present our abstraction method. In section 4, we solve some verification problems for polynomial dynamical systems in rectangles. In section 5, we extend previous results to the controlled case by showing how a polynomial controller having the same degree as the polynomial vector field can be built. Finally, in section 6, we solve a motion planning problem for a polynomial dynamical system.

II. PRELIMINARIES

In this section, we introduce notations and preliminary results that will be useful for subsequent discussions. All the results in this section are quite standard and are therefore stated without proofs. Let $R = \prod_{k=1}^{k=n} [a_k, b_k]$ be a rectangle of \mathbb{R}^n and let $V = \prod_{k=1}^{k=n} \{a_k, b_k\}$ be its set of vertices.

A. Multi-variate polynomials

A multivariate polynomial of degree Δ is any function $p : \mathbb{R}^n \rightarrow \mathbb{R}$ of the form:

$$p(x) = p(x_1, \dots, x_n) = \sum_{(k_1, \dots, k_n) \in \Delta} p_{k_1, \dots, k_n} x_1^{k_1} \dots x_n^{k_n}.$$

with $\Delta = \{0, \dots, \delta_1\} \times \dots \times \{0, \dots, \delta_n\}$ where $\delta_1, \dots, \delta_n$ are the degrees of p in the respective variables x_1, \dots, x_n and $\{p_{k_1, \dots, k_n} \in \mathbb{R}, (k_1, \dots, k_n) \in \Delta\}$ denotes its coefficient set. Another writing can be given using multi-indices. Let $I = (i_1, \dots, i_n) \in \mathbb{N}^n$ and $\delta = (\delta_1, \dots, \delta_n) \in \mathbb{N}^n$, the polynomial p of degree δ can be written as follows:

$$p(x) = \sum_{I \leq \delta} p_I x^I \text{ with } p_I \in \mathbb{R} \forall I \leq \delta,$$

where $I \leq \delta$ is the order relation which is equivalent to say that $i_j \leq \delta_j$ for all $j \in \{1, \dots, n\}$.

B. Multi-affine functions

A multivariate polynomial is multi-affine if it is affine in each of its variables when the other variables are regarded as constant:

Definition 1: A multi-affine function $p : \mathbb{R}^n \rightarrow \mathbb{R}$ is a multivariate polynomial in the variables x_1, \dots, x_n where the degree of p in each of the variable is at most 1. For $x = (x_1, \dots, x_n)$,

$$p(x) = \sum_{(l_1, \dots, l_n) \in \{0,1\}^n} p_{l_1, \dots, l_n} x_1^{l_1} \dots x_n^{l_n}$$

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where $p_{l_1, \dots, l_n} \in \mathbb{R}$ for all $(l_1, \dots, l_n) \in \{0, 1\}^n$.

It is shown in [1] that a multi-affine function is uniquely determined by its values at the vertices of a rectangle. In particular, the following result holds:

Lemma 1: For all $x \in R$, $p(x)$ is a convex combination of the values at the set of vertices V .

We easily deduce that: $\min_{x \in R} p(x) = \min_{v \in V} p(v)$.

C. Blossoming principle

Let $p : \mathbb{R}^n \rightarrow \mathbb{R}$ be an arbitrary multivariate polynomial of degree δ . The blossoming principle maps polynomials to symmetric multi-affine functions (see [8] and references therein).

Definition 2: The blossom or polar form of the polynomial $p : \mathbb{R}^n \rightarrow \mathbb{R}$ is the function $q : \mathbb{R}^{\delta_1 + \dots + \delta_n} \rightarrow \mathbb{R}$ given for $z = (z_{1,1}, \dots, z_{1,\delta_1}, \dots, z_{n,1}, \dots, z_{n,\delta_n})$ by

$$q(z) = \sum_{(l_1, \dots, l_n) \in \Delta} p_{l_1, \dots, l_n} \prod_{i=1}^n B_{l_i, \delta_i}(z_{i,1}, \dots, z_{i,\delta_i})$$

with

$$B_{l,r}(z_1, \dots, z_r) = \frac{1}{\binom{r}{l}} \sum_{\sigma \in C(l,r)} z_{\sigma_1} \dots z_{\sigma_l}$$

where $C(l,r)$ denotes the set of combinations of l elements in $\{1, \dots, r\}$.

An example may help to understand the definition; the blossom of the polynomial $p(x) = 3x_1 + 2x_2^3 + x_1^2 x_2^2$ is

$$q(z) = \frac{3}{2}(z_{1,1} + z_{1,2}) + 2z_{2,1}z_{2,2}z_{2,3} + \frac{1}{3}z_{1,1}z_{1,2}(z_{2,1}z_{2,2} + z_{2,1}z_{2,3} + z_{2,2}z_{2,3}).$$

We define a relation on $\mathbb{R}^{\delta_1 + \dots + \delta_n}$: for $z, z' \in \mathbb{R}^{\delta_1 + \dots + \delta_n}$, with $z = (z_{1,1}, \dots, z_{1,\delta_1}, \dots, z_{n,1}, \dots, z_{n,\delta_n})$ and $z' = (z'_{1,1}, \dots, z'_{1,\delta_1}, \dots, z'_{n,1}, \dots, z'_{n,\delta_n})$, we denote $z \cong z'$ if, for all $k = 1, \dots, n$, there exists a permutation π_k such that $(z_{k,1}, \dots, z_{k,\delta_k}) = \pi_k(z'_{k,1}, \dots, z'_{k,\delta_k})$. It is easy to see that \cong is an equivalence relation. A characterization of blossoms that is equivalent to Definition 2 is given by the following proposition:

Proposition 1: $q : \mathbb{R}^{\delta_1 + \dots + \delta_n} \rightarrow \mathbb{R}$ is the blossom of the polynomial $p : \mathbb{R}^n \rightarrow \mathbb{R}$ if and only if:

- 1) q is a multi-affine function;
- 2) q is a symmetric function of its arguments:

$$\forall z \cong z', q(z) = q(z');$$

- 3) q satisfies the diagonal property:

$$q(z_1, \dots, z_1, \dots, z_n, \dots, z_n) = p(z_1, \dots, z_n).$$

We define R' the associated rectangle of $\mathbb{R}^{\delta_1 + \dots + \delta_n}$ and its set of vertices V' given by: $R' = \prod_{k=1}^n [a_k, b_k]^{\delta_k}$ and $V' = \prod_{k=1}^n \{a_k, b_k\}^{\delta_k}$. For $v = (v_{1,1}, \dots, v_{1,\delta_1}, \dots, v_{n,1}, \dots, v_{n,\delta_n}) \in V'$ and $k \in \{1, \dots, n\}$, $l_k(v)$ denotes the number of elements $v_{k,1}, \dots, v_{k,\delta_k}$ that are equal to b_k . It is easy to verify that for $v, v' \in V'$, $v \cong v'$ if and only if $l_k(v) = l_k(v')$ for all $k \in \{1, \dots, n\}$. We denote by $\overline{V'} = (V' / \cong)$ the set of equivalence classes of the relation \cong on the set V' ; $\overline{V'}$ has

$(\delta_1 + 1) \times \dots \times (\delta_n + 1)$ elements. Then, for $\bar{v} \in \overline{V'}$ and $k \in \{1, \dots, n\}$ it makes sense to write $l_k(\bar{v})$. Also from the second property in Proposition 1, it makes sense to write $q(\bar{v})$.

There exists an interesting relation between the blossom values $q(\bar{v})$ and the Bernstein polynomials (see e.g. [8]). Indeed, the values $q(\bar{v})$ for $\bar{v} \in \overline{V'}$ are the coordinates of the polynomial p in the Bernstein basis:

$$p(x) = \sum_{\bar{v} \in \overline{V'_n}} q(\bar{v}) \prod_{k=1}^n B_{l_k(\bar{v}), \delta_k} \left(\frac{x_k - a_k}{b_k - a_k} \right) \quad (1)$$

where

$$B_{i,\delta}(x) = \binom{\delta}{i} x^i (1-x)^{\delta-i}.$$

are the Bernstein polynomials. The previous equation can be useful when one needs to compute the values $q(\bar{v})$. The explicit computation of the blossom $q(z)$ (which can count up to $2^{\delta_1 + \dots + \delta_n}$ terms) is computationally expensive. However, the previous equation shows that it is sufficient to compute the $(\delta_1 + 1) \times \dots \times (\delta_n + 1)$ coordinates of the polynomial $p(x)$ in the Bernstein basis.

III. ABSTRACTION OF POLYNOMIAL SYSTEMS USING THE BLOSSOMING PRINCIPLE

In this section, we show how a polynomial dynamical system can be abstracted by a multi-affine one. The main tool of this abstraction is the blossoming principle. We consider the following dynamical system S :

$$\dot{x}(t) = f(x(t)), \quad (2)$$

where $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a polynomial vector field. Let us denote $\delta_{1,j}, \dots, \delta_{n,j}$ the respective degrees of the variables x_1, \dots, x_n of the polynomial f_j for all $j \in \{1, \dots, n\}$ and let $\delta_i = \max_{j \in \{1, \dots, n\}} \delta_{i,j}$ for all $i \in \{1, \dots, n\}$. All the polynomials f_i can be considered as polynomials of higher degrees $\delta_1, \dots, \delta_n$ in the variables x_1, \dots, x_n by adding zeros coefficients if necessary. For all $i = 1, \dots, n$, let $f_{i,\delta}$ be the polar form of f_i regarded as a polynomial of degree δ and let $f_\delta = (f_{1,\delta}, \dots, f_{n,\delta})$. For all $\alpha \geq 0$, our abstract dynamical system S'_α will be of the form:

$$\dot{z}(t) = g_\alpha(z(t)), \quad (3)$$

where the vector field $g_\alpha : \mathbb{R}^{\delta_1 + \dots + \delta_n} \rightarrow \mathbb{R}^{\delta_1 + \dots + \delta_n}$ is given by:

$$g_{\alpha,i,j}(z) = f_{i,\delta}(z) + \alpha \left(\sum_{k \in \{1, \dots, \delta_i\} \setminus \{j\}} z_{i,k} - (\delta_i - 1) z_{i,j} \right).$$

for all $i = 1, \dots, n$ and all $j = 1, \dots, \delta_i$ where $z = (z_{1,1}, \dots, z_{1,\delta_1}, \dots, z_{n,1}, \dots, z_{n,\delta_n})$. We also define the vector space H given by :

$$H = \{z \in \mathbb{R}^{\delta_1 + \dots + \delta_n} \mid \forall i \in \{1, \dots, n\}, z_{i,1} = \dots = z_{i,\delta_i}\}.$$

The advantage of the abstraction method is that the abstract system is multi-affine and we have the following result:

Theorem 1: For all $\alpha \geq 0$, the dynamical systems S and S_{α}' are equivalent on the vector space H : if x, z are trajectories of S and S_{α}' such that $x_i(0) = z_{i,j}(0)$ for all $i = 1, \dots, n, j = 1, \dots, \delta_i$ ($z(0) \in H$) then $x_i(t) = z_{i,j}(t)$ for all $t \geq 0$. In addition, for all $\alpha > 0$, H is an attractor for S_{α}' .

Proof: Using the diagonal property of the polar form we show that the systems (2) et (3) are equivalent if we restrict ourselves to the set H . Now, let $i \in \{1, \dots, n\}$ such that $\delta_i > 1$ and let j_1 and j_2 two distinct elements of $\{1, \dots, \delta_i\}$. We have:

$$\dot{z}_{i,j_1}(t) - \dot{z}_{i,j_2}(t) = -\delta_i \alpha (z_{i,j_1}(t) - z_{i,j_2}(t))$$

Implying that for all $\alpha > 0$:

$$z_{i,j_1}(t) - z_{i,j_2}(t) = C.e^{-\delta_i \alpha t} \rightarrow_{t \rightarrow +\infty} 0$$

Then, H is an attractor for S_{α}' for all $\alpha > 0$. ■

Example 1: For illustration, we will consider the Van Der Pol oscillator given by the following polynomial dynamical system S :

$$(S) \quad \begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_2(1 - x_1^2) - x_1. \end{cases}$$

In the Figure 1 we plot some trajectories of S which show the existence of a limit cycle.

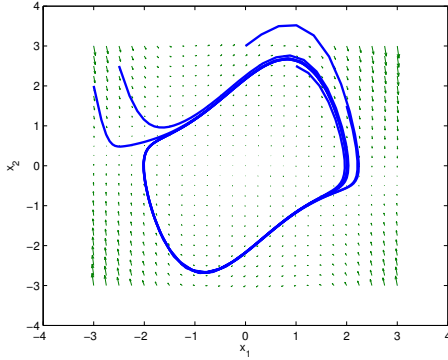


Fig. 1. Some trajectories of S showing the existence of a limit cycle.

Now, for all $\alpha \geq 0$, the abstract dynamical system S_{α}' is given by:

$$(S_{\alpha}') \quad \begin{cases} \dot{z}_{1,1} = \frac{1}{2}(z_{1,1} + z_{1,2}) + \alpha(z_{1,2} - z_{1,1}), \\ \dot{z}_{1,2} = \frac{1}{2}(z_{1,1} + z_{1,2}) + \alpha(z_{1,1} - z_{1,2}), \\ \dot{z}_{2,1} = z_{2,1}(1 - z_{1,1}z_{1,2}) - \frac{1}{2}(z_{1,1} + z_{1,2}). \end{cases}$$

For $\alpha = 50$, we plot (see Figure 2) some trajectories of the system S'_{50} including those having the same initial conditions that the trajectories of Figure 1 after the abstraction. For example, the trajectory corresponding to the initial condition $x_0 = (-3, 2)$ will be replaced by the trajectory having initial condition $z_0 = (-3, -3, 2)$. We can remark the equivalence between the original system S and the system S'_{50} on the attractor $H = \{z \in \mathbb{R}^3 \mid z_{1,1} = z_{1,2}\}$.

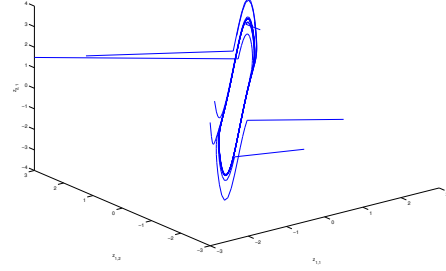


Fig. 2. Some trajectories of S'_{50} showing the equivalence with S on H .

IV. ANALYSIS OF POLYNOMIAL SYSTEMS ON RECTANGLES

In this section, we apply our abstraction method to find conditions ensuring the resolution of the following problems.

A. Problem formulation

Let $R = \prod_{k=1}^n [a_k, b_k]$ be a given rectangle of \mathbb{R}^n . We consider three different problems:

Problem 1 (Exit the rectangle) :

It consists in finding sufficient conditions ensuring that all the trajectories of the system (2) leave R .

Problem 2 (Invariant rectangle):

It consists in finding sufficient conditions ensuring that all the trajectories of the system (2) starting on R stay inside R , i.e if x is trajectory of (2) such that $x(t_0) \in R$, then $x(t) \in R$ for all $t \geq t_0$.

Problem 3 (Exit through a given facet F_0):

It consists in finding sufficient conditions ensuring that all the trajectories of the system (2) leave R through the facet F_0 .

B. Case of multi-affine systems

In this section, we recall solutions in the case of a multi-affine vector field based on the values of the vertices of R . All these results are known from [1]. We present them without proofs.

The first result concerning the **Problem 1** is the following:

Theorem 2: All the trajectories of the system (2) starting inside R at time $t = t_0$ leave R if it exists a direction \vec{n} such that: $\vec{n} \cdot f(v) > 0$ for all $v \in V$.

This result can be seen as an admissibility result of a linear program with $|V| = 2^n$ constraints.

For **Problem 2** we will need the following notations [1]:

- $\xi_k : \{a_k, b_k\} \mapsto \{0, 1\}$ when for all $k \in \{1, \dots, n\}$, $\xi_k(a_k) = 0$ and $\xi_k(b_k) = 1$.
- $F_{j, \xi_j(w_j)} = \{x \in R \mid x_j = w_j\}$: the set of facets of R where for all $j \in \{1, \dots, n\}$, $w_j \in \{a_j, b_j\}$.
- $n_{j, \xi_j(w_j)} = (-1)^{(\xi_j(w_j)+1)} e_j$: the outer normal of the facet $F_{j, \xi_j(w_j)}$ where the vectors e_j form the canonical basis of \mathbb{R}^n .

We have the following result:

Theorem 3: All the trajectories of the system (2) starting inside R at time $t = t_0$ remain in R for all $t \geq t_0$ if and only if $n_{j,\xi_j(v_j)} \cdot f(v) \leq 0$ for all $v = (v_1, \dots, v_n) \in V$ and all integer $j \in \{1, \dots, n\}$.

For **Problem 3**, we have the following theorem:

Theorem 4: All the trajectories of the system (2) starting inside R leave it through the facet $F_{j,\xi_j(w_j)}$ if for all vertex $v = (v_1, \dots, v_n) \in V$ we have:

$$n_{j,\xi_j(w_j)} \cdot f(v) > 0 \text{ and } \forall i \in \{1, \dots, n\} \setminus \{j\}, n_{i,\xi_j(v_i)} \cdot f(v) \leq 0.$$

C. Case of polynomial systems

We are going to generalize the previous results in the case of polynomial systems using our abstraction method. More precisely, we are going to use the following lemma:

Lemma 2: For all $i = 1, 2$, **Problem i** is resolved for system (2) and rectangle R if it is resolved for the abstract system (3) and the abstract rectangle R' , for some $\alpha \geq 0$.

Proof: The proof is an immediate consequence of Theorem 1 and the equivalence between systems S and S_α' . ■

For **Problem 1** we have the following result:

Theorem 5: All the trajectories of (2) starting in R at time $t = t_0$ will leave R if it exists a direction $\vec{w} \in \mathbb{R}^n$ such that $\vec{w} \cdot f_\delta(\bar{v}) > 0$ for all vertices $\bar{v} \in \bar{V}'$.

Proof: Using our previous lemma with $\alpha = 0$ and Theorem 2, we can deduce that all the trajectories of the system (2) starting in R leave it if it exists a direction $\vec{w} \in \mathbb{R}^n$ such that $\vec{w} \cdot f_\delta(v) > 0$ for all $v \in V'$. Then, thanks to the symmetric property of f_δ , we have just to find a direction $\vec{w} \in \mathbb{R}^n$ such that $\vec{w} \cdot f_\delta(\bar{v}) > 0$ for all $\bar{v} \in \bar{V}'$. ■

For **Problem 2**, we need to generalize the notation of a facet and its outer normal for the abstract rectangle R' . More precisely, for all $i \in \{1, \dots, n\}$, $j \in \{1, \dots, \delta_i\}$, we denote:

- $F_{i,\xi_i(w_i)}^j = \{z \in R' \mid z_{i,j} = w_i\}$, $\forall w_i \in \{a_i, b_i\}$.
- $n_{i,\xi_i(w_i)}^j = (-1)^{(\xi_i(w_i)+1)} e_{i,j}$: where the vectors $e_{i,j}$ form the canonical basis of $\mathbb{R}^{\delta_1 + \dots + \delta_n}$.

A first result for the invariance of R is the following:

Proposition 2: R is invariant for the system (2) if it exists a real $\alpha \geq 0$ such that for all $\bar{v} \in \bar{V}'$:

$$n_{i,\xi_i(\bar{v}_{i,j})}^j g_\alpha(\bar{v}) \leq 0 \quad \forall i \in \{1, \dots, n\}, \forall j \in \{1, \dots, \delta_i\}. \quad (4)$$

Proof: Using Lemma 2 we show that R is invariant for the system (2) if it exists a real $\alpha \geq 0$ such that the abstract rectangle R' is invariant for the system (3). Then, thanks to Theorem 3 and to the symmetric property of g_α , R' is invariant for the system (3) if and only if:

$$n_{i,\xi_i(\bar{v}_{i,j})}^j g_\alpha(\bar{v}) \leq 0 \quad \forall i \in \{1, \dots, n\}, \forall j \in \{1, \dots, \delta_i\}. \quad \blacksquare$$

After some complexity reductions due to the blossom properties, we find the following result:

Theorem 6: R is invariant for the system (2) if for all $\bar{v} \in \bar{V}'$ and for all $i = 1, \dots, n$, the following constraints are verified:

- If $l_i(\bar{v}) = 0$, we should have $f_{i,\delta}(\bar{v}) \geq 0$.

- If $l_i(\bar{v}) = \delta_i$, we should have $f_{i,\delta}(\bar{v}) \leq 0$.

Proof: We fix $\alpha \geq 0$, $i \in \{1, \dots, n\}$, $j \in \{1, \dots, \delta_i\}$ and a vertex $\bar{v} \in \bar{V}'$, we have: $n_{i,\xi_i(\bar{v}_{i,j})}^j g_\alpha(\bar{v}) =$

$$\chi_{i,j}(\bar{v}) \left(g_{i,j}(\bar{v}) + \alpha \left(\sum_{k \in \{1, \dots, \delta_i\} \setminus \{j\}} \bar{v}_{i,k} - (\delta_i - 1) \bar{v}_{i,j} \right) \right).$$

where $\chi_{i,j}(\bar{v}) = (-1)^{(\xi_i(\bar{v}_{i,j})+1)}$.

$$\text{Let } C = \left(\sum_{k \in \{1, \dots, \delta_i\} \setminus \{j\}} \bar{v}_{i,k} - (\delta_i - 1) \bar{v}_{i,j} \right).$$

If $0 < l_i(\bar{v}) < \delta_i$:

If $\bar{v}_{i,j} = a_i$ we will have: $C > 0$ and $(-1)^{(\xi_i(a_i)+1)} = -1$ then the condition (4) of Proposition 2 can be written in the form $-g_{i,j}(\bar{v}) - \alpha C \leq 0$. As the choice of $\alpha \geq 0$ is arbitrary, it suffices to take a real α sufficiently large for that this constraint is checked.

If $\bar{v}_{i,j} = b_i$ we will have $g_{i,j}(\bar{v}) + \alpha C \leq 0$ with now a constant $C < 0$. So this constraint can also be satisfied by taking a real α sufficiently large.

If $l_i(\bar{v}) = 0$:

We will have $C = 0$ and $(-1)^{(\xi_i(\bar{v}_{i,j})+1)} = -1$, then the condition (4) of Proposition 2 can be written in the form: $g_{i,j}(\bar{v}) = f_{i,\delta}(\bar{v}) \geq 0$.

If $l_i(\bar{v}) = \delta_i$:

We will have $C = 0$ and $(-1)^{(\xi_i(\bar{v}_{i,j})+1)} = 1$, then the condition (4) of Proposition 2 can be written in the form: $g_{i,j}(\bar{v}) = f_{i,\delta}(\bar{v}) \leq 0$. ■

For $j = 1, \dots, n$ and $\bar{v} \in \bar{V}'$, we will use the following notations:

$$\bar{V}'_j = \{v \in \bar{V}' \text{ such that } l_j(\bar{v}) = 0 \text{ or } \delta_j\}$$

$$\sigma_j(\bar{v}) = \begin{cases} 1 & \text{if } l_j(\bar{v}) = 0 \\ -1 & \text{if } l_j(\bar{v}) = \delta_j \\ 0 & \text{else} \end{cases}$$

Remark 1: The result of Theorem 6 can be formulated using the previous notations as follows. Let:

$$d^* = \min_{j=1, \dots, n, \bar{v} \in \bar{V}'_j} \sigma_j(\bar{v}) f_{j,\delta}(\bar{v})$$

If d^* is positive then the rectangle R is invariant for (2).

Now for **Problem 3**, let us fix a facet $F_{j,\xi_j(w_j)}$ of R , we obtain the following result:

Theorem 7: All the trajectories of the system (2) starting inside R leave it through the facet $F_{j,\xi_j(w_j)}$ if:

- 1) $\sigma_i(\bar{v}) f_{i,\delta}(\bar{v}) \geq 0$, $\forall i \in \{1, \dots, n\} \setminus \{j\}$, $\forall \bar{v} \in \bar{V}'_i$.
- 2) $(-1)^{(\xi_j(w_j)+1)} f_{j,\delta}(\bar{v}) > 0$, $\forall \bar{v} \in \bar{V}'$.

Proof: The proof is just an adaptation to that given in [1] for Theorem 4. In fact we can show using our abstraction and similar reasoning than Theorem 3 that a facet $F_{j,\xi_j(w_j)}$ is blocked¹ if it exists a real $\alpha \geq 0$ such that for all $\bar{v} \in \bar{V}'$ with $\bar{v}_{i,j} = w_j$, the condition (4) of Proposition 2 is satisfied. Then using the complexity reduction results of Theorem 6, we can show that all the facets of

¹A facet F is said to be blocked if and only if $\forall x \in F, n_F \cdot f(x) \leq 0$ where n_F is its outer normal

R different from $F_{j,\xi_j(w_j)}$ are blocked if $\sigma_i(\bar{v})f_{i,\delta}(\bar{v}) \geq 0$ for all $i \in \{1, \dots, n\} \setminus \{j\}$ and all $\bar{v} \in \bar{V}'_i$. Now, as $(-1)^{(\xi_j(w_j)+1)}f_{j,\delta}(\bar{v}) > 0$ for all $\bar{v} \in \bar{V}'$, then the normal $\bar{n} = n_{j,\xi_j(w_j)}$ gives us our exit direction. Therefore, using Theorem 2, we deduce that we will exit R . Therefore, the only possibility will be to leave it through the facet $F_{j,\xi_j(w_j)}$. ■

V. CONTROLLER SYNTHESIS

In this section, we will generalize the results obtained previously for the following controlled dynamic system

$$\dot{x}(t) = f(x(t)) + Bu(t) \quad x \in R, \quad (5)$$

where $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a polynomial vector field, R is a rectangle of \mathbb{R}^n , $B \in \mathbb{R}^{n \times p}$ is a constant matrix of control directions and $u \in U$ is a convex set.

We will describe how one can build a polynomial controller having the same degrees as the vector field ensuring the invariance of the rectangle R or ensuring the exit through a given facet of R .

A. Problem 2 (Invariance of the rectangle)

To ensure the invariance of a rectangle R , we will have to build a controller whose values at the vertices of its polar form can block all the facets of R . This can be done using the following proposition:

Proposition 3: Let $D = |\bar{V}'| = (\delta_1 + 1) \times \dots \times (\delta_n + 1)$ and B_i denotes the i -th line vector of the matrix B .

If it exists $\{\lambda = (\lambda_{\bar{v}}) \in \mathbb{R}^{p \times D} \mid \bar{v} \in \bar{V}'\}$ such that $\sigma_i(\bar{v})(f_{i,\delta}(\bar{v}) + B_i\lambda_{\bar{v}}) \geq 0$ for all $i = 1, \dots, n$ and all $\bar{v} \in \bar{V}'_i$, then we can build using the vector λ a polynomial controller u having the same degree as f ensuring the invariance of the rectangle R for the dynamical system (5).

Proof: First, we will describe the construction of the controller u using the vector $\lambda = (\lambda_1, \dots, \lambda_D) \in \mathbb{R}^{p \times D}$:

Let us denote u_δ the polar form with respect to δ of the controller u that we want to build and let $u_\delta(\bar{v}^r) = \lambda_r$ for all $r = 1, \dots, D$ where \bar{v}^r is r -th element of \bar{V}' . Then, using Bernstein polynomials and equation (1) of the Preliminary section, we will be able to construct a polynomial controller $u(x)$ for all $x \in R$.

Now, let $F(x) = f(x) + Bu(x)$ and apply Theorem 6 on its polar form $F_\delta = f_\delta + Bu_\delta$ with respect to δ to prove that R is invariant for the new vector field F . ■

B. Problem 3 (Exit through a given facet)

Now, we want to build a controller ensuring the exit of all the trajectories of the system (5) through a given facet F of R . Indeed, to solve this problem we must block all the facets of R different from F and impose the exit through the direction given by the outer normal of this facet. Specifically, if we fix an arbitrary facet $F_{i_0,\xi_{i_0}(w_{i_0})}$ of R where $i_0 \in \{1, \dots, n\}$ and $w_{i_0} \in \{a_{i_0}, b_{i_0}\}$, a solution to this problem is given by the following proposition:

Proposition 4: If it exists $\{\lambda = (\lambda_{\bar{v}}) \in \mathbb{R}^{p \times D} \mid \bar{v} \in \bar{V}'\}$ such that:

- 1) $\sigma_i(\bar{v})(f_{i,\delta}(\bar{v}) + B_i\lambda_{\bar{v}}) \geq 0$ for all $i \in \{1, \dots, n\} \setminus \{i_0\}$ and all $\bar{v} \in \bar{V}'_i$.
- 2) $(-1)^{(\xi_{i_0}(w_{i_0})+1)}(f_{i_0,\delta}(\bar{v}) + B_{i_0}\lambda_{\bar{v}}) > 0$ for all $\bar{v} \in \bar{V}'$.

Then we can build using the vector λ a polynomial controller u having the same degree as f ensuring the exit through the facet $F_{i_0,\xi_{i_0}(w_{i_0})}$ of R .

Proof: The construction of the polynomial controller u is the same that in Proposition 3. Then, it suffices to apply Theorem 7 with $F(x) = f(x) + Bu(x)$ to show that all the trajectories corresponding to the vector field F leave R through $F_{i_0,\xi_{i_0}(w_{i_0})}$. ■

Example 2: We will consider the polynomial dynamical system introduced in [6]:

$$\begin{cases} \dot{x}_1 &= -x_2 - 1.5x_1 - 0.5x_1^3 - x_2 + u_1(x_1, x_2), \\ \dot{x}_2 &= x_1 + u_2(x_1, x_2), \end{cases} \quad (6)$$

Let $R = [-2, 2] \times [-1.5, 3]$. We propose to find a polynomial controller $u(x) \in U = [-10, 10]$ of degree 3 in x_1 and 1 en x_2 for the system (6) ensuring :

- 1) The resolution of **Problem 2**.
- 2) The resolution of **Problem 3** where the exit facet is $F = F_{2,1} = \{(x, y) \in R \mid y = 3\}$.

Using Proposition 3, we can resolve **Problem 2** by solving the following linear program:

$$\begin{aligned} &\text{maximise } t \\ &\text{s.c} \quad t \in \mathbb{R}, \lambda \in [-10, 10]^{p \times D} \\ &\quad t \leq \sigma_1(\bar{v})(f_{1,\delta}(\bar{v}) + B_1\lambda_{\bar{v}}) + \hat{\sigma}_1(\bar{v}), \quad \bar{v} \in \bar{V}'_1, \\ &\quad t \leq \sigma_2(\bar{v})(f_{2,\delta}(\bar{v}) + B_2\lambda_{\bar{v}}) + \hat{\sigma}_2(\bar{v}), \quad \bar{v} \in \bar{V}'_2. \end{aligned} \quad (7)$$

where for $i = 1, 2$, $\hat{\sigma}_i(\bar{v}) = \begin{cases} 1 & \text{if } \bar{v} \in \bar{V}'_i \\ \infty & \text{else} \end{cases}$.

Let (t^*, λ^*) be the solution of the linear program. We obtain an optimal value $t^* = 8 > 0$ then we can build using the vector λ^* a polynomial controller u making invariant the rectangle $R = [-2.5, 2.5] \times [-1.5, 3.5]$ for the system (6) and solves then the **Problem 2** as shown in Figure 3.

For the facet $F_{2,1} = \{(x, y) \in R \mid y = 3.5\}$, we use the

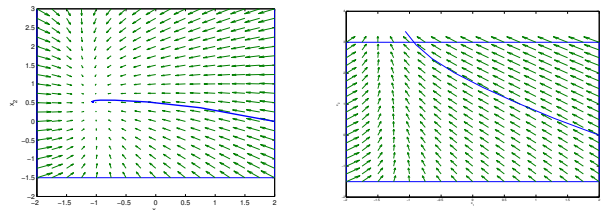


Fig. 3. The vector fields of the system (6) associated to the polynomial controller u and a trajectory illustrating the invariance of R (left). The vector fields of the system (6) associated to the polynomial controller u and a trajectory illustrating the exit through the facet F of R (right).

Proposition 4 and then resolve **Problem 3** by resolving the

following linear program:

$$\begin{aligned}
 & \text{maximise } t \\
 & \text{s.t } t \in \mathbb{R}, \lambda \in [-10, 10]^{p \times D} \\
 & t \leq \sigma_1(\bar{v}) (f_{1,\delta}(\bar{v}) + B_1 \lambda_{\bar{v}}) + \hat{\sigma}_1(\bar{v}), \quad \bar{v} \in \bar{V}', \\
 & t \leq f_{2,\delta}(\bar{v}) + B_2 \lambda_{\bar{v}}, \quad \bar{v} \in \bar{V}'.
 \end{aligned} \tag{8}$$

Let (t^*, λ^*) be the solution of the linear program. We obtain an optimal value $t^* = 8 > 0$. Therefore, the polynomial controller u built using λ^* solves the **Problem 3** for the exit facet $F_{2,1}$ as shown in Figure 3.

VI. APPLICATION: MOTION PLANNING

In this section, we will use the results of the previous section to plan the path of the trajectories of the polynomial dynamical system (5). More precisely, let Γ a set of indices corresponding to a set of rectangles $R_i, i \in \Gamma$, two by two adjacent. In our context, the motion planning of trajectories consists on fixing an initial rectangle $R_{i_1} = R_{initial}$ and a final rectangle $R_{i_2} = R_{final}$ where i_1 and i_2 are two distinct indices of Γ and requires that the trajectories of the system (5) starting in R_{i_1} end in R_{i_2} while passing through all the rectangles between this two rectangles.

Let m be the number of rectangles that the trajectories should visit and let us reorder the indices of such rectangles by denoting $R_1 = R_{i_1}, R_2$ the rectangle adjacent to R_1 and so on until the final one $R_m = R_{i_2}$. Therefore, for all $j = 1, \dots, m-1$, it suffices to find a controller ensuring the passage from R_j to R_{j+1} : this can be achieved by constructing a controller that blocks all the facets of R_j except the adjacent facet to R_{j+1} and ensures the exit through this facet. Then by the mean of Proposition 4, a polynomial controller ensuring the exit through the common facet of R_j and R_{j+1} can be obtained. Now, for $j = m$, if a trajectory enters to the rectangle R_m , it should still there forever: this can be done by looking for a controller which blocks all the facets of R_m to make it invariant. Then using Proposition 3 we will be able to synthesize a polynomial controller ensuring the invariance of R_m .

An illustration is given using the following example:

Example 3: Again we consider Example 2. Let $R_1 = R_{initial} = [-2, -1] \times [-1.5, -0.5]$ and $R_8 = R_{final} = [1, 2] \times [2.5, 3]$. We propose to find polynomial controllers (one for each sub rectangle) belonging to $U = U_i = [-10, 10]$ for all $i = 1, \dots, 8$, ensuring that the trajectories of the system (6) starting in R_1 ends in R_8 while passing respectively by the following rectangles: $R_2 = [-1, 0] \times [-1.5, -0.5]$, $R_3 = [-1, 0] \times [-0.5, 0.5]$, $R_4 = [0, 1] \times [-0.5, 0.5]$, $R_5 = [0, 1] \times [0.5, 1.5]$, $R_6 = [1, 2] \times [0.5, 1.5]$, and $R_7 = [1, 2] \times [1.5, 2.5]$.

Using the results of the previous section we can find in each rectangle $R_i, i = 1, \dots, 7$, a polynomial controller $u_i \in U_i$ which resolves the **Problem 3** for the common facet between R_i and R_{i+1} , and a polynomial controller $u_8 \in U_8$ resolving the **Problem 2** for the rectangle R_8 (see Figure 4).

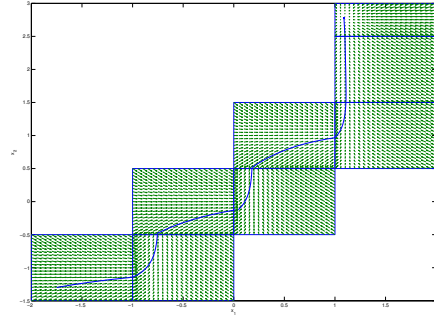


Fig. 4. The vector fields of the system (6) on the rectangles R_1, \dots, R_8 respectively associated to the controllers u_1, \dots, u_8 and a trajectory illustrating the motion planning.

VII. CONCLUSIONS

In this paper, we generalize the work of Belta and Habets [1] in the case of a polynomial dynamical system using an abstraction method. The main tool is the blossoming principle which allow us to map the polynomial dynamic in rectangles to a multi-affine one with higher dimension on abstracted rectangles. Using, the properties of the polar form we show that efficient results for solving some verification analysis problem can be established and can be easily formulated using Bernstein coefficients. We extend all these results to the case of controlled polynomial dynamical systems and show how one can synthesis a polynomial controller. As an application we show that we can solve the motion planning problem of polynomial dynamical system in a sequence of adjacent rectangles. The approach gives just some sufficient conditions and then may fail to resolve the motion planning problem. For example one can fail to find a controller ensuring the exit through a facet for a rectangle R but succeed if we cut it into two sub rectangles. A future work will be to use some decomposition techniques like the branch-and-bound algorithm to improve our results.

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