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Cascade observer for a class of nonlinear systems with output delays

M.L. Fall, M. Farza, M. M'Saad, E. Pigeon, O. Gehan

Abstract— **This paper proposes a state observer with a cascade structure for a class of nonlinear systems in the presence of delayed output measurements. The first system in the cascade allows to estimate the delayed state while each of the remaining ones is a predictor. Each predictor estimates the state of the preceding one with a prediction horizon equal to a fraction of the time delay in such a way that the state of the last predictor is an estimate of the system actual state. The design of the observer is achieved by assuming a set of conditions under which the exponential convergence of the estimation error to zero is established, namely the system is uniformly observable for any any input and its nonlinearities are globally Lipschitz. Of particular interest, it is shown that the number of the systems in the cascade depends on the magnitudes of the considered delay and the Lipschitz constant. The performance of the proposed observer and its main properties are compared with those of two existing observers through a typical bioreactor model.**

Key words : delay system, delayed output, cascade systems, high gain observer.

I. INTRODUCTION

During the last two decades, an intensive research activity has been devoted to investigate the stability, control and state estimation for systems with time delays. A particular attention has been paid to the case of linear systems (See for instance [6], [5], [9], [11] and references therein) whereas only few results have been established in the nonlinear case (see for instance [10], [12]). Moreover, in most works dealing with state estimation for delay systems, the output is assumed to be free-delay. In many real-time applications, some state variables may not be available instantaneously and corresponding measurements are systematically tainted with delay. One can cite the example of bioreactors where most of the component measurements are obtained with more or less important time delays since they result from time consuming laboratory analyses. Another typical example is that of network connected systems where some output data are transmitted through low-rate communication systems. This generally introduces non negligible time-delays that have to be account for in order to ensure the viability of the control and monitoring system.

Contributions dealing with the observer design problem with delayed output measurements are very few. A promising approach was reported in [4] where the authors proposed a constant gain observer for a class of single output systems that are globally drift-observable with globally Lipschitz nonlinearities and where the output is available with a constant delay τ. The main characteristic of the proposed observer lies in its structure that consists in $m+1$ cascade subsystems where the first subsystem is an observer of the delayed state. Each one of the m remaining subsystems is a predictor: the state of the predictor at rank j is an estimation of the delayed state with a time delay equal to $\tau - \frac{j}{m}\tau$ in such a way that the state of the predictor at rank m is an estimate of the actual state. The number m of the predictors in the cascade depends on the magnitude of the system nonlinearities Lipschitz constant and the time delay. More specifically, this dependence is explained under the form of an inequality that has to be satisfied when choosing m. The idea of designing an observer with a chain structure has been reconsidered in [7] where the formulation of the nonlinear observer design is based on the resolution of first order singular partial differential equations. The resulting observer is similar to that given in [4] but it involves more design parameters that allow to relax the condition between the number of predictors in the cascade versus the nonlinearities Lipschitz constant and the time delay magnitude. In the two chain observers referenced above, each predictor in the cascade involves a correction term the expression of which becomes complex when the predictor rank in the cascade is high. This may give rise to implementation problems when a great number of predictors in the cascade is required. To overcome this problem, a recent work [1] proposed a high gain-based observer involving cascade subsystems having the same gain structure. Though the proposed observer may seem attractive since the gain structure in all the cascade subsystems is the same and is very simple, it leads to an oscillatory long transient behaviour with relatively high magnitudes as it shall be illustrated in this paper. Moreover, unlike in the classical high gain observers where the decay to zero of the observation error is generally p roportional to a constant that can be chosen arbitrarily large, the observation error of the observer proposed decreases exponentially to zero with a decay term that is proportional to a small constant ε that was set to zero by the authors in order to exhibit the condition under which the exponential convergence of the observation error is ensured. Furthermore, the involved design parameter specification is by no means useful from the convergence analysis pursued in the paper.

In the present paper, we shall propose a cascade observer

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The authors are with the GREYC Laboratory, UMR 6072 CNRS, 6 Bd Maréchal Juin, 14050 Caen Cedex, FRANCE e-mail: mondher.farza@unicaen.fr.

for a class of single output nonlinear systems that are observable for any input and where the output is available with a delay. The main property of the proposed observer lies in the simplicity of its structure and therefore the easiness of its implementation. Indeed, all the predictors in the cascade have the same gain structure thanks to the availability of a matrix design parameter that allows to assign the involved prediction dynamics.

This paper is organized as follows: in the next section the class of considered systems is introduced and some requisite preliminaries related to the observer design in the free delay output case are briefly presented. In section 3, the observer design is proposed with a full convergence analysis. In section 4, simulation results are given in order to illustrate the above theoretical results. The performance of the observer with its main convergence properties are compared throughout this section to those of two observers proposed for similar classes of systems [4] and [1]. Finally, some concluding remarks are given in section 5.

II. PROBLEM FORMULATION AND PRELIMINARIES

One shall consider the following class of single input-single output nonlinear systems

$$\left\{\begin{array}{lcl} \dot{x}(t) & = & f(x(t)) + g(x(t))u(t) \\ y_\tau(t) & = & h(x(t-\tau)) \end{array}\right. \tag{1}$$

where $\tau > 0$ is the measurement delay, $x(t) \in \mathbb{R}^n$, $u(t) \in D$ a compact subset of $\mathbb{R}$, the vector functions f, g and h are $\mathcal{C}^\infty$. The delayed available output $y_\tau(t) = y(t-\tau)$ where $y(t)$ is the undelayed output.

One assumes that system (1) is uniformly observable for any input [3]. Such a system is then diffeomorphic to a system of the form

$$\left\{\begin{array}{lcl} \dot{z}(t) & = & Az(t) + \varphi\left(u(t), z(t)\right) \\ y_\tau(t) & = & z_1(t-\tau) = Cz(t-\tau) \end{array}\right. \tag{2}$$

where

$$A = \begin{pmatrix} 0 & I_{n-1} \\ 0 & 0 \end{pmatrix}, \ C = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix} \tag{3}$$

the state $z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{R}^n$ with $z_i \in \mathbb{R}$ and I_{n-1} is the identity matrix of order $n-1$. The nonlinear vector function φ assumes a triangular structure with respect to z, i.e. $\varphi(u,z) = \begin{pmatrix} \varphi_1(u, z_1) \\ \varphi_2(u, z_1, z_2) \\ \vdots \\ \varphi_n(u, z) \end{pmatrix}$. In the free delay case ($y_\tau(t) = y(t)$), a high gain observer has been proposed for system (1). The design of the observer has been achieved under the following assumption:

(H1) The function φ is globally Lipschitz in z uniformly in u i.e. for all bounded inputs, there exists $L_\varphi > 0$ such that for all $z, \bar{z} \in \mathbb{R}^n$, one has :$\|\varphi(u,z) - \varphi(u,\bar{z})\| \leq L_\varphi \|z - \bar{z}\|$

The equations of the proposed observer can be written as follows in the z' coordinates:

$$\dot{\hat{z}}(t) = A\hat{z}(t) + \varphi(u(t), \hat{z}(t)) - \theta\Delta_\theta^{-1}K(C\hat{z}(t) - y(t)) \tag{4}$$

where K is a constant vector which is chosen such that the matrix $A - KC$ is Hurwitz, $\theta \geq 1$ is a design parameter and Δ_θ is the following diagonal matrix

$$\Delta_\theta = diag\left(1, \frac{1}{\theta}, \dots, \frac{1}{\theta^{n-1}}\right) \tag{5}$$

The equations of the observer can be written in the original coordinates by considering the inverse of the transformation jacobian as follows:

$$\begin{array}{rcl} \dot{\hat{x}}(t) & = & f(\hat{x}(t)) + g(\hat{x}(t))u(t) - \\ & & \left(\dfrac{\partial \Phi}{\partial x}(\hat{x}(t))\right)^{-1} \theta\Delta_\theta^{-1}K(h(\hat{x}) - y(t)) \end{array}$$

where $z(t) = \Phi(x(t))$ is the diffeomorphism that puts system (1) under the forme (2).
The exponential convergence to zero of the observation error for bounded inputs can be expressed as follows:

$$\|\hat{z}(t) - z(t)\| \leq \eta(\theta)e^{-\lambda(\theta)t}\|\hat{z}(0) - z(0)\| \tag{6}$$

where $\eta(\theta)$ is polynomial in θ and $\lim\limits_{\theta \to +\infty} \lambda(\theta) = +\infty$
Of course, since $z = \Phi(x)$ is a diffeomorphism, the observation error $\hat{x}(t) - x(t)$ also converges to zero, exponentially.

A detailed description of the observer design is provided in the next section. The design is brought in the z'coordinates where the observers' equations are firstly given before being derived in the original x'coordinates.

The observer design requires the adoption of some assumptions that shall be stated in due courses. At this step, one assumes that (H1) holds throughout the paper.

III. OBSERVER DESIGN

As in [4] and [7], one adopts the following notations:

$$z_j(t) \quad = \quad z\left(t - \tau + \frac{j}{m}\tau\right) \quad \text{and } u_j(t) = u\left(t - \tau + \frac{j}{m}\tau\right)$$

for $j = 0, \dots, m$ and $t \geq -\frac{j}{m}\tau$ where m is a positive integer that shall be specified more precisely later.

The observer we propose is composed by $m+1$ cascade subsystems where the first subsystem is an observer for the delayed state whereas each one of the remaining subsystems is a predictor. As in [4] and [7], the predictor of rank j in the cascade predicts the state of the preceding subsystem with a prediction horizon equal to $\dfrac{\tau}{m}$ in such a way that the state of the $m'th$ predictor is an estimate of the system actual state.

According to the adopted notation, the state of the delayed system is governed by the following dynamical model

$$\left\{\begin{array}{lcl} \dot{z}_o(t) & = & Az_o(t)+\varphi(u_0(t),z_o(t)) \\ y_{z_o}(t) & = & y_\tau(t) = Cz_o(t) \end{array}\right. \tag{7}$$

System (7) is under the form (2) where no delay is considered in the latter. As a result, a high gain observer under the form (4) can be used for the estimation of z_o. The equations of such an observer specialize as follows:

$$\dot{\hat{z}}_0 = A\hat{z}_o(t)+\varphi(u_0(t),\hat{z}_o(t)) - \theta\Delta_\theta^{-1}K(C\hat{z}_o(t)-y_\tau(t)) \tag{8}$$

where, as in (4), K is a vector such that $(A-KC)$ is Hurwitz and $\theta \geq 1$ is a design parameter.

One now shall focus on the structure of the remaining m predictors. As it has been mentioned in the introduction, a main characteristic of these predictors is that their respective gains have the same structure and this is true whatever is the rank of the predictor in the cascade. The design of these predictors is described below.

A. The predictors' structure

Recall that the variable z_j denotes the delayed state with a time delay equal to $\tau - \dfrac{j}{m}\tau$. Since observer (8) allows the estimation of the state z_o, one shall focus on the case where $j = 1,\ldots,m$: the aim is to design a predictor that estimates the state z_j. The latter is governed by the following dynamical equation

$$\dot{z}_j(t) = Az_j(t)+\varphi(u_j(t),z_j(t)) \tag{9}$$

System (9) can be rewritten as follows

$$\dot{z}_j(t) = \bar{A}z_j(t)+\varphi(u_j(t),z_j(t)) + (A-\bar{A})z_j(t) \tag{10}$$

where $\bar{A}$ is a Hurwitz matrix. As it shall be detailed later, the consideration of this matrix allows to derive a simple gain structure for the predictor.

According to (10), the state $z_j(t)$ can be explained as follows:

$$\begin{array}{rcl} z_j(t) & = & e^{\bar{A}\frac{\tau}{m}}z_{j-1}(t) + e^{\bar{A}t}\displaystyle\int_{t-\frac{\tau}{m}}^{t} e^{-\bar{A}s}\cdot \\ & & \big(\varphi(u_j(s),z_j(s)) + (A-\bar{A})z_j(s)\big)\,ds \end{array} \tag{11}$$

Let us denote by $\hat{z}_j$ the state of the predictor that shall provide an estimate of the state z_j. Motivated by implementation simplicity and miming the structure of the delayed state observer (8), one assigns the dynamics of $\hat{z}_j$ as follows:

$$\dot{\hat{z}}_j(t) = A\hat{z}_j(t)+\varphi(u_j(t),\hat{z}_j(t)) - G_j(t) \tag{12}$$

where $G_j(t)$, $j = 1,\ldots,m$ is a corrective term that has to be chosen in order to guarantee the exponential convergence to zero of the prediction error, e.g. $\tilde{z}_j(t) = \hat{z}_j(t) - z_j(t)$. Notice that the structure of the delayed state observer (8) is similar to that of predictor (12) with

$$\begin{array}{rcl} G_o(t) & = & -\theta\Delta_\theta^{-1}K(C\hat{z}_o(t)-y_\tau(t)) \\ & = & -\theta\Delta_\theta^{-1}KC(\hat{z}_o(t)-z_o(t)) \triangleq -\theta\Delta_\theta^{-1}KC\tilde{z}_o(t) \end{array}$$

where $\tilde{z}_o(t)$ is the observation error corresponding to the delayed state and it converges to zero exponentially according to the above developments.

Now, miming (10), the predictor equation (12) can be rewritten as follows:

$$\dot{\hat{z}}_j(t) = \bar{A}\hat{z}_j(t)+\varphi(u_j(t),\hat{z}_j(t)) + (A-\bar{A})\hat{z}_j(t) - G_j(t) \tag{13}$$

Again and according to (13), the prediction $\hat{z}_j$ can be explained as follows:

$$\begin{array}{rcl} \hat{z}_j(t) & = & e^{\bar{A}\frac{\tau}{m}}\hat{z}_j\left(t-\dfrac{\tau}{m}\right) + e^{\bar{A}t}\displaystyle\int_{t-\frac{\tau}{m}}^{t} e^{-\bar{A}s}\left(\varphi_j(u(s),\hat{z}_j(s))\right. \\ & & \left. +(A-\bar{A})\hat{z}_j(s) - G_j(s)\right)ds \end{array} \tag{14}$$

Now, one imposes the following relationship between the states corresponding to two successive predictors, e.g. $\hat{z}_j$ and $\hat{z}_{j-1}$, $j = 1,\ldots,m$

$$\begin{array}{rcl} \hat{z}_j(t) & = & e^{\bar{A}\frac{\tau}{m}}\left(\hat{z}_{j-1}(t) - r_j(t)\right) + e^{\bar{A}t}\displaystyle\int_{t-\frac{\tau}{m}}^{t} e^{-\bar{A}s}\cdot \\ & & \big(\varphi(u_j(s),\hat{z}_j(s)) + (A-\bar{A})\hat{z}_j(s)\big)\,ds \end{array} \tag{15}$$

where the r_j's, $j = 1,\ldots,m$, are functions that are differentiable with respect to time and shall be determined simultaneously with the G_j's in the next subsection. One notices that, the introduction of the functions r_j is motivated by the aim to derive simple expressions for the predictors gain. Indeed, one shall show later that unlike the predictors proposed in [4] and [7] where the gain expression complexity grows with the predictor rank in the cascade, the structure of the predictors gain proposed in this paper is simple and is same for all the predictors. Moreover, one shall show that the functions r_j only intervene in the analysis of the observer convergence and they do not appear in the observer equations.

B. Determination of the predictors' gain

Subtracting equation (15) from equation (14) gives

$$\begin{array}{c} e^{\bar{A}\frac{\tau}{m}}\left(\hat{z}_j\left(t-\dfrac{\tau}{m}\right) - \hat{z}_{j-1}(t) + r_j(t)\right) = \\ e^{\bar{A}t}\displaystyle\int_{t-\frac{\tau}{m}}^{t} e^{-\bar{A}s}G_j(s)ds \end{array}$$

Differentiating with respect to time each side of the above equation yields to

$$\begin{array}{l} e^{\bar{A}\frac{\tau}{m}}\left(\dot{\hat{z}}_j\left(t-\dfrac{\tau}{m}\right) - \dot{\hat{z}}_{j-1}(t) + \dot{r}_j(t)\right) = \\ \quad \bar{A}e^{\bar{A}\frac{\tau}{m}}\left(\hat{z}_j\left(t-\dfrac{\tau}{m}\right) - \hat{z}_{j-1}(t) + r_j(t)\right) + \\ \quad G_j(t) - e^{\bar{A}\frac{\tau}{m}}G_j\left(t-\dfrac{\tau}{m}\right) \end{array} \tag{16}$$

Substituting in (16) $\dot{\hat{z}}_{j-1}(t)$ and $\dot{\hat{z}}_j\left(t-\frac{\tau}{m}\right)$ by their expressions given by (12) leads to

$$\begin{aligned}
& e^{\bar{A}\frac{\tau}{m}}\left(A\left(\hat{z}_j\left(t-\frac{\tau}{m}\right)-\hat{z}_{j-1}(t)\right)\right. \\
& +\varphi(u_{j-1},\hat{z}_j\left(t-\frac{\tau}{m}\right))-\varphi(u_{j-1},\hat{z}_{j-1}(t)) \\
& \left.+G_{j-1}(t)-G_j\left(t-\frac{\tau}{m}\right)+\dot{r}_j(t)\right)= \\
& \qquad \bar{A}e^{\bar{A}\frac{\tau}{m}}\left(\hat{z}_j\left(t-\frac{\tau}{m}\right)-\hat{z}_{j-1}(t)+r_j(t)\right)+ \\
& \qquad G_j(t)-e^{\bar{A}\frac{\tau}{m}}G_j\left(t-\frac{\tau}{m}\right) \qquad (17)
\end{aligned}$$

Notice that one has used in the last equation the identity $u_j\left(t-\frac{\tau}{m}\right)=u_{j-1}(t)$. Now, using the fact that the matrices $\bar{A}$ and $e^{\bar{A}\frac{\tau}{m}}$ commute, equation (17) leads to

$$\begin{aligned}
G_j(t) \quad = \quad & e^{\bar{A}\frac{\tau}{m}}\left((A-\bar{A})\left(\hat{z}_j\left(t-\frac{\tau}{m}\right)-\hat{z}_{j-1}(t)\right)\right. \\
& \left.+\varphi(u_{j-1},\hat{z}_j\left(t-\frac{\tau}{m}\right))-\varphi(u_{j-1},\hat{z}_{j-1}(t))\right) \\
& +e^{\bar{A}\frac{\tau}{m}}\left(\dot{r}_j(t)-\bar{A}r_j(t)+G_{j-1}(t)\right) \qquad (18)
\end{aligned}$$

Now, if one chooses $r_j(t)$ such that

$$\dot{r}_j(t)=\bar{A}r_j(t)-G_{j-1}(t) \text{ with } r_j(0)=0,\ j=1,\ldots,m \qquad (19)$$

the expression of $G_j(t)$ becomes

$$\begin{aligned}
& G_j(t)=e^{\bar{A}\frac{\tau}{m}}\left((A-\bar{A})\left(\hat{z}_j\left(t-\frac{\tau}{m}\right)-\hat{z}_{j-1}(t)\right)\right. \\
& \left.+\varphi(u_{j-1},\hat{z}_j\left(t-\frac{\tau}{m}\right))-\varphi(u_{j-1},\hat{z}_{j-1}(t))\right) \qquad (20)
\end{aligned}$$

Before giving the main theorem that summarizes the results obtained through the above developments, one recalls that since the matrix $\bar{A}$ is Hurwitz, there exist positive numbers β and $\bar{a}$ such that [8], [7]

$$\forall t\geq 0:\ \|e^{\bar{A}t}\|\leq\beta e^{-\bar{a}t} \qquad (21)$$

The equations of the candidate observer for system (2) are then given by

$$\begin{aligned}
& \dot{\hat{z}}_j(t)=A\hat{z}_j(t)+\varphi(u_j(t),\hat{z}_j(t))-G_j(t)\ for\ j=0,\ldots,m \\
& G_o(t)=\theta\Delta_\theta^{-1}KC(\hat{z}_o(t)-y_\tau(t)) \qquad (22) \\
& \text{and for } j=1,\ldots,m: \\
& G_j(t)=e^{\bar{A}\frac{\tau}{m}}\left((A-\bar{A})\left(\hat{z}_j\left(t-\frac{\tau}{m}\right)-\hat{z}_{j-1}(t)\right)\right. \\
& \left.+\varphi\left(u_{j-1}(t),\hat{z}_j\left(t-\frac{\tau}{m}\right)\right)-\varphi(u_{j-1}(t),\hat{z}_{j-1}(t))\right)
\end{aligned}$$

where the matrix $\bar{A}$ is Hurwitz, the vector K is such that $A-KC$ is a Hurwitz matrix , Δ_θ is the diagonal matrix given by (5) and $\theta\geq 1$ is a design parameter.

One states the following

Theorem 1: *Consider system (2) subject to Hypothesis H1 and observer (22). If the number* m *is selected such that the following condition is satisfied*

$$\beta\left(L_\varphi+\|\bar{A}-A\|\right)\frac{\tau}{m}<1 \qquad (23)$$

where β *is the scalar appearing in the bound of* $\|e^{\bar{A}t}\|$ *given by (21) and* L_φ *is the Lipschitz constant of* φ *defined in H1, then the state trajectories of the last subsystem of the cascade exponentially converges to those of system (2).*

The proof of the theorem is given below.

C. Proof of Theorem 1:

Let $\tilde{z}_j=\hat{z}_j-z_j$ denotes the estimation error for $j=0,\ldots,m$. In order to prove the theorem, we shall show that $\tilde{z}_j$ exponentially converges to zero for $j=0,\ldots,m$. For this aim, one shall proceed by induction on j. Indeed, for $j=0$, it is clear that the first system of the cascade (22) is a high gain observer of the form (4) and it provides an estimate of the delayed state z_o. So, the associated observation error satisfies an inequality similar to that given by (6) and one has $\|\tilde{z}_o(t)\|\leq\eta(\theta)e^{-\lambda(\theta)t}\tilde{z}_o(0)$. Let us continue with the induction proof and let $j\geq 1$. Assume that $\tilde{z}_j(t)$ exponentially converges to zero for $j=1,\ldots,m-1$ and let us show that $\tilde{z}_m(t)$ also converges to zero exponentially. Indeed, from (11) and (15), one obtains

$$\begin{aligned}
\tilde{z}_m(t) \quad & = \quad e^{\bar{A}\frac{\tau}{m}}\left(\tilde{z}_{m-1}(t)-r_m(t)\right) \\
& \quad +\int_{t-\frac{\tau}{m}}^{t}e^{\bar{A}(t-s)}\left(\varphi(u_m(s),\hat{z}_m(s))-\right. \\
& \quad \left.\varphi(u_m(s),z_m(s))+(A-\bar{A})\tilde{z}_m(s)\right)ds \implies \\
\|\tilde{z}_m(t)\| \quad & \leq \quad \|e^{\bar{A}\frac{\tau}{m}}\|\left(\|\tilde{z}_{m-1}(t)\|+\|r_m(t)\|\right) \\
& \quad +\int_{t-\frac{\tau}{m}}^{t}\|e^{\bar{A}(t-s)}\|\left(\|\varphi(u_m(s),\hat{z}_m(s))-\right. \\
& \quad \left.\varphi(u_m(s),z_m(s))\|+\|(A-\bar{A})\|\|\tilde{z}_m(s)\|\right)ds \\
& \leq \quad \|e^{\bar{A}\frac{\tau}{m}}\|\left(\|\tilde{z}_{m-1}(t)\|+\|r_m(t)\|\right) \\
& \quad +\beta(L_\varphi+\|A-\bar{A}\|)\int_{t-\frac{\tau}{m}}^{t}e^{-\bar{a}(t-s)}\|\tilde{z}_m(s)\|ds \\
& \qquad (24)
\end{aligned}$$

Now, according to the induction hypothesis, $G_{m-1}(t)$ converges exponentially to zero and since $\bar{A}$ is Hurwitz, one easily concludes from (19) that r_{m-1} also exponentially converges to zero. Without loss of generality, one shall suppose that

$$\|e^{\bar{A}\frac{\tau}{m}}\|\left(\|\tilde{z}_{m-1}(t)\|+\|r_m(t)\|\right)\leq\mu_{m-1}e^{-\alpha_{m-1}t} \qquad (25)$$

where α_{m-1},μ_{m-1} are positive reals. So, inequality (24) can be written as follows:

$$\begin{aligned}
\|\tilde{z}_m(t)\| \quad & \leq \quad \mu_{m-1}e^{-\alpha_{m-1}t}+\beta(L_\varphi+\|A-\bar{A}\|) \\
& \quad \int_{t-\frac{\tau}{m}}^{t}e^{-\bar{a}(t-s)}\|\tilde{z}_m(s)\|ds \\
& \leq \quad \mu_{m-1}e^{-\alpha_{m-1}t}+\beta(L_\varphi+\|A-\bar{A}\|) \\
& \quad \int_{t-\frac{\tau}{m}}^{t}\|\tilde{z}_m(s)\| \qquad (26)
\end{aligned}$$

Now, using lemma 2 in [7] and assumptions of Theorem 1, it follows that under condition (23), $\|\tilde{x}_m(t)\|$ exponentially converges to zero. This ends the proof of the theorem.

D. Equations of the observer in the original coordinates

It is easy to deduce that observer (22) can be written in the original coordinates as follows

$$\begin{aligned}
& \text{for } j=0,\ldots,m: \\
& \dot{\hat{x}}_j(t)=f(\hat{x}_j(t))+g(\hat{x}_j(t))u(t)-\left(\frac{\partial\Phi(\hat{x}_j}{\partial\hat{x}_j}(t)\right)^{-1}G_j(t) \\
& G_o(t)=\theta\Delta_\theta^{-1}K\left(h(\hat{x}_0(t))-y_\tau(t)\right) \qquad (27) \\
& \text{and for } j=1,\ldots,m: \\
& G_j(t)=e^{\bar{A}\frac{\tau}{m}}\left((A-\bar{A})\left(\Phi(\hat{x}_j\left(t-\frac{\tau}{m}\right))-\Phi(\hat{x}_{j-1}(t))\right)\right. \\
& \left.+\varphi(u_{j-1}(t),\Phi(\hat{x}_j\left(t-\frac{\tau}{m}\right)))-\varphi(u_{j-1}(t),\Phi(\hat{x}_{j-1}(t)))\right)
\end{aligned}$$

IV. EXAMPLE

In this section, one shall illustrate the theory described in the above sections through the following example dealing with a typical bioreactor. Moreover, the performance of the observer shall be compared with those of two observers proposed in [4] and [1]. We consider a simple microbial culture which involves a single biomass x_1 growing on a single substrate x_2. The bioprocess is supposed to be continuous with a dilution rate $u(t)$ and an input substrate concentration $s_{in}(t)$. The specific growth rate is assumed to follow the Contois model [2]. The mathematical dynamical model of the process is constituted by the following two mass balance equations associated to x_1 and x_2, respectively:

$$\left\{\begin{array}{lcl} \dot{x}_1(t) & = & \dfrac{\mu^\star x_1(t)x_2(t)}{K_c x_1(t)+x_2(t)} - u(t)x_1(t) \\ \dot{x}_2(t) & = & \dfrac{-k\mu^\star x_1(t)x_2(t)}{K_c x_1(t)+x_2(t)} + u(t)(s_{in}(t)-x_2(t)) \\ y_\tau(t) & = & x_1(t-\tau) \end{array}\right. \tag{28}$$

where x_1 and x_2 respectively denote the concentration of the biomass and the substrate, $\mu^\star$ and K_c are the Contois law parameters while k is a yield coefficient. The measurements of the biomass concentration are supposed to be available with a time delay τ and the objective is to estimate the actual biomass concentration together with that of the substrate from the available delayed measurements.

System (28) has been considered in [3] where the authors exhibited a compact set $\mathcal{X} \subset \mathbb{R}^2$ which is positively invariant under the dynamics of (28). Moreover, it was shown that the following function $\Phi : \mathcal{X} \longrightarrow \Phi(\mathcal{X})$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto z = \Phi(x) = \begin{pmatrix} z_1 & = & x_1 \\ z_2 & = & \dfrac{\mu^\star x_1 x_2}{K_c x_1 + x_2} \end{pmatrix}$ is a diffeomorphism from X onto its image. System (28) can be written in the new coordinates z under form (2). As a result, the observers proposed in [4] and [1] as well as that proposed in this paper can be used for the estimation of the actual states. In the remaining of this section, results obtained with the observer we propose are compared with two other cascade observers proposed in [4] and [1], respectively. All the simulations are achieved using Matlab function for solving delay differential equations with a constant delay, namely $dde23$. We shall refer to the different observers as observer 1 for that given in [4], observer 2 for that proposed by [1] and observer 3 for that proposed in this paper. Example (28) has been considered in [7]. Here, one shall consider the same operating conditions as in [7]. The values of the model parameters are:

$$\mu^\star = 1(min^{-1}),\ K_c = 1,\ k = 1,\ u = 0.08(min^{-1})$$

$$s_{in} = 0.1(kg\ m^{-3})$$

Notice that the equilibrium point is $(x_{1,0}, x_{2,0}) = (0.092, 0.008)$. The simulation of the observers has been carried out using pseudo data measurements issued from the simulation of system (28) with the point equilibrium as an initial condition. The initial condition of the cascade observers is such that all initial conditions corresponding to the substrate concentrations are perturbed by 50% from the steady state value.

One starts the comparison of the observer performance by commenting results given in Table 1. The first row of this table contains different values of the number of subsystems in the cascade associated to an observer. This number is denoted n_c and is equal to $m+1$ for the observer we propose. The remaining rows provide the maximum of the delay value that can be recovered by each of the three considered observers. A delay value is said to be recovered if satisfactory estimates are provided by the observer. A quick examination of this table allows to claim the following:

> *For a given $n_c > 2$, the maximum value of delay recovered by observer 2 is less than the third of that corresponding to observer 1 and less than the seventh associated to observer 3.*

Moreover, one notices that the prediction error related to observer 2 oscillates and reaches some peaks with relatively important amplitudes as it shall be illustrated later. Such oscillations have also been noticed with observer 1 but the amplitude of the peaks were much more smaller. On the contrary and for all values of n_c, the peaks of oscillations before convergence do not exist for observer 3. One notices that the results for observer 2 are obtained by fixing the observer design parameter to its minimum allowed value, i.e. 1. Higher values of this parameter give rise to much more greater values for n_c with amplified oscillations.

Observer \ n_c	**2**	**3**	**5**	**10**	**15**
Observer 1	1.4	2.7	4.5	6.3	6.9
Observer 2	0.3	0.6	1.1	1.8	2.3
Observer 3	3.5	5	8.5	15	18.5

TABLE I

MAXIMUM VALUES OF THE DELAY THAT CAN BE RECOVERED BY THE OBSERVER COMPOSED BY n_c CASCADE SYSTEMS

Let us illustrate these issues by focusing on a particular value of the delay, e.g. $\tau = 1.8s$. According to table I, the minimum value required for n_c is equal to 3 for observer 1 and to 1 for observer 3 while a value not smaller than 10 is necessary for observer 2. The comparison of the prediction error behaviour corresponding to observer 3 with those issued from observers 1 and 2 are reported in Figure 1. The high values of the amplitude oscillations of observer 2 obtained with $n_c = 10$ are worth to be mentioned. In order to (partially) overcome this problem, another value for $n_c (= 20)$ has been considered for this observer and as shown in the figure the new obtained prediction error still be much higher than that corresponding to observers 3 (and 1). Notice that, for values of n_c greater than 20, the amplitude of the oscillations increase again for observer 2 (results are not shown here). For observer 3, the results shown in Figure 1 are obtained with $K^T = [2\ 1], \theta = 20$ and $\bar{A} = A - I_2$. The speed under which the estimates converge to the actual state depends on the magnitude of the time delay and it

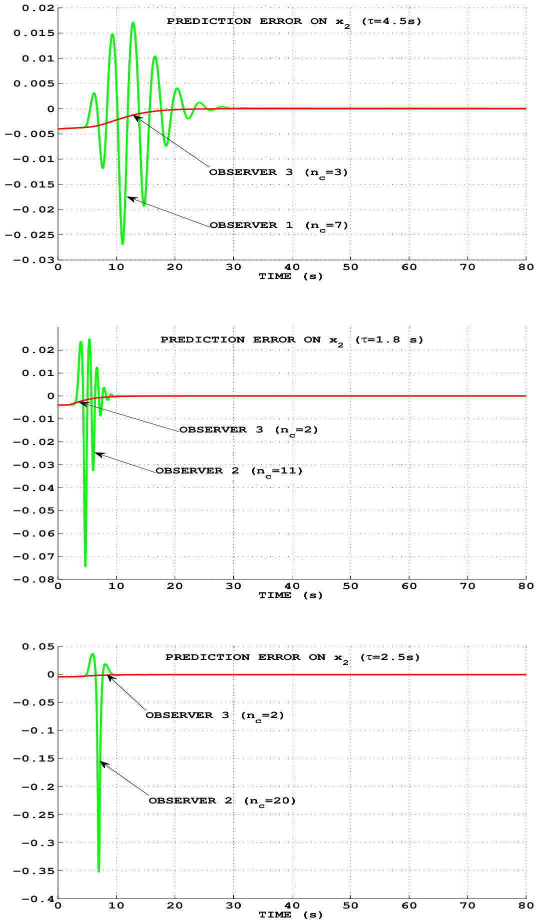

Fig. 1. Comparison of observer 3 with observers 1 and 2

is also a function of the eigenvalues of the matrix $\bar{A}$. We have reproduced in figure 2 different estimates provided by the observer with three different values for the matrix $\bar{A}$ while keeping the other observer design parameters constant ($n_c = 4$, $K^T = [2\ 1], \theta = 20$). The obtained results clearly show that as the eigenvalues become small in magnitude, the speed of convergence to the actual state increases. Notice that, relatively high values for the eigenvalues of $\bar{A}$ are to be avoided since such values give raise to high values for β and $\|A - \bar{A}\|$ and as result condition (23) may be violated.

V. CONCLUSION

The design of a cascade observer to estimate the state of a class of nonlinear systems involving delay measurements is presented. The performance of the observer with its main properties are highlighted and compared with two existing observers through an academic example. The obtained results clearly put forward the following features:

- For a fixed value of the delay, the number of subsystems to be cascaded is less than the seventh of the number required by the observer proposed in [1]. Moreover, it has been shown that the latter leads to a relatively poor performances with respect the the proposed observer up to a similar implementation simplicity.
- Unlike in [1], the conditions ensuring the exponential convergence of the observation error are similar to those corresponding to the observer proposed in [4] and they allow to provide intuitive insights about the design parameter specifications.
- The main motivation of the observer design in [1] consists in its structure simplicity with respect to the available ones, namely the observer proposed in [4]. Such an argument could be valid if the achieved performances are comparable and this is not the case.
- There are two design features that should be pointed out with respect to the design proposed in [4]. The first one consists in the gain structure simplicity while the second one concerns the design parameters specification. The latter allows to achieve a good compromise between convergence and performance.

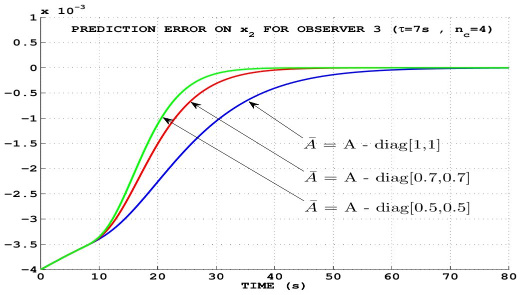

Fig. 2. Estimation with different values for $\bar{A}$, $n_c = m+1 = 4$, $\tau = 4.5s$

REFERENCES

[1] T. Ahmed-Ali, E. Cherrier, and F. Lamnabhi-Lagarrigue. Cascade high gain predictors for a class of nonlinear systems. *IEEE Trans. Automatic Control*, 57(1):224–229, 2012.
[2] J.E. Bailey and D.F. Ollis. *Biochemical Engineering Fundamentals*. McGraw-Hill, New York, 1986.
[3] J.P. Gauthier, H. Hammouri, and S. Othman. A simple observer for nonlinear systems - application to bioreactors. *IEEE Trans. Auto. Control*, 37(6):875–880, 1992.
[4] A. Germani, C. Manes, and P. Pepe. A new approach to state observation of nonlinear systems with delayed output. *IEEE Trans. Automatic Control*, 47(1):96–101, 2002.
[5] K. Gu, V.L. Kharitonov, and J. Chen. *Stability of time-delay systems*. Birkhäuser, 2003.
[6] M. Hou, P. Zitek, and R.J. Patton. An observer design for linear time-delay systems. *IEEE Trans. Automatic Control*, 47(1):121–125, 2002.
[7] N. Kazantzis and R. A. Wright. Nonlinear observer design in the presence of delayed output measurements. *Sys. & Control Letters*, 54(9):877–886, 2005.
[8] H. K. Khalil. *Nonlinear systems*. Printice Hall, third edition, New Jersey, 2003.
[9] V.L. Kharitonov and D. Hinrichsen. Exponential estimates for time delay systems. *Sys. & Control Letters*, 53(5):395–405, 2004.
[10] F. Mazenc and S.I. Niculescu. Lyapunov stability analysis for nonlinear delay systems. *Sys. & Control Letters*, 42(4):245–251, 2001.
[11] S. Mondie and V.L. Kharitonov. Exponential estimates for retarded time-delay systems: an LMI approach. *IEEE Trans. Automatic Control*, 50(2):268–273, 2005.
[12] H. Trinh, M. Aldeen, and M. Nahavandi. An observer design for a class of nonlinear time-delay systems. *Computers & Electrical Engineering*, 1(30):61–72, 2004.