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A Contribution to Parameter Identification in Infinite-dimensional Systems

Torsten Knüppel and Frank Woittennek[1]

***Abstract*— Parameter identification in infinite-dimensional systems with lumped measurements is considered. The proposed method is based on a convolutional input-output relation of the underlying system of partial differential equations. The derivation of this input-output relation is presented in a framework involving (ultra-)distributions. The unknown parameters are determined by minimizing an associated error functional.**

I. Introduction

The application of model-based control strategies requires, as the notion already suggests, an appropriate mathematical model of the considered system. This has to be determined such that the actual behaviour of the physical system is accurately reproduced. Apart from an appropriately chosen structure the quality of such a model depends on the accurate knowledge of the involved parameters. One of the most popular methods for the identification of these parameters in finite-dimensional systems is based on the error defined by means of the differential or difference input-output relation (cf. [1], [2]). The main problem in generalizing this approach to infinite-dimensional systems lies in the difficulty of deriving an analogous input-output-relation.

This problem is discussed in the present contribution for certain infinite-dimensional models. To this end, a convolution system over an appropriate ring of (ultra-)distributions is associated with the boundary value problem under consideration. This system involves lumped variables only, e.g. boundary values. This alternative system representation forms the basis for the identification procedure.

Note that, contrary to previously published results (see e.g. [3]), the proposed approach involves lumped measurements only.

II. General setting

A. Mathematical models considered

In this section the derivation of an input-output-relation is detailed for systems governed by the partial differential equation[1] (p.d.e.)

$$\frac{\partial \boldsymbol{x}}{\partial z}(z,t) = \sum_{\nu=0}^{\gamma} A_\nu(z;\boldsymbol{\vartheta}) \frac{\partial^\nu \boldsymbol{x}}{\partial t^\nu}(z,t), \tag{1a}$$

that involves derivatives of the system variable $\boldsymbol{x}$ w.r.t. the time t and the spatial coordinate $z \in \Omega = [0,\ell]$. The coefficient matrices $A_\nu(z;\boldsymbol{\vartheta})$ are time invariant, but depend on both, the spatial coordinate z and the unknown parameters in $\boldsymbol{\vartheta}$. This model is completed by the boundary conditions[2]

$$\sum_{\nu=0}^{\delta} L_\nu(\boldsymbol{\vartheta}) \frac{\partial^\nu \boldsymbol{x}}{\partial t^\nu}(\cdot,t) = D(\boldsymbol{\vartheta})\boldsymbol{u}(t), \tag{1b}$$

that are written in terms of the control input $\boldsymbol{u}$ and the linear operators $L_\nu : C^0(\Omega,\mathbb{R}^n) \to \mathbb{R}^n$ and $D \in \mathbb{R}^{n\times m}$. As only boundary measurements are considered in this contribution the output $\boldsymbol{y}$ can be written as

$$\boldsymbol{y}(t) = \sum_{\nu=0}^{\mu} \left(C_{\nu,0}(\boldsymbol{\vartheta}) \frac{\partial^\nu \boldsymbol{x}}{\partial t^\nu}(0,t) + C_{\nu,\ell}(\boldsymbol{\vartheta}) \frac{\partial^\nu \boldsymbol{x}}{\partial t^\nu}(\ell,t) \right),$$

where $C_{\nu,0}, C_{\nu,\ell} \in \mathbb{R}^{p\times n}$.

B. Deriving an input-output-relation

In order to avoid unwanted technicalities in the following a distributional setting is used: All the time signals are considered as distributions or ultra-distributions. In this setting equation (1a) can be rewritten as an abstract differential equation w.r.t. the spatial variable z:

$$\frac{\partial \boldsymbol{x}}{\partial z}(z) = A(z;\boldsymbol{\vartheta})\boldsymbol{x}(z),\ \boldsymbol{x} \in \mathcal{F} := C^1(\Omega,\mathcal{D}'(\mathbb{R})^n), \tag{2}$$

where $\mathcal{D}'$ is an appropriate space of (ultra-)distributions[3] and $A(z;\boldsymbol{\vartheta}) : \mathcal{D}'(\mathbb{R})^n \to \mathcal{D}'(\mathbb{R})^n$ is a differential operator w.r.t. the time t, that is defined by the right-hand side of (1a). Similarly one can rewrite the associated boundary conditions (1b) by introducing an operator $L(\boldsymbol{\vartheta}) : \mathcal{F} \to \mathcal{D}'(\mathbb{R})^n$

$$L(\boldsymbol{\vartheta})\boldsymbol{x} = D(\boldsymbol{\vartheta})\boldsymbol{u}. \tag{3}$$

The measurements $\boldsymbol{y}$ are given in a similar form using an operator $C(\boldsymbol{\vartheta}) : \mathcal{F} \to \mathcal{D}'(\mathbb{R})^p$:

$$\boldsymbol{y} = C(\boldsymbol{\vartheta})\boldsymbol{x}. \tag{4}$$

The main assumption is constituted by the fact that the solution of the Cauchy problem associated with the differential equation (2) can be expressed as convolution product of the boundary values $\boldsymbol{x}(0)$ and a matrix-valued function

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[1]Both authors are with the Faculty of Electrical and Computer Engineering, Chair of Control Theory, Technische Universität Dresden, 01062 Dresden, Germany. Their email addresses are t.knueppel@tu-dresden.de and frank.woittennek@tu-dresden.de respectively.

[1]Explicit higher order equations w.r.t. the spatial coordinate z can be easily written in the form (1a) by introducing additional variables (cf. the examples).

[2]Note that this setting is more general than the usual one. Boundary conditions in the literal sense can be obtained by choosing the entries of L_ν as Dirac distributions centered in 0 or ℓ.

[3]Whether spaces of distributions or ultradistributions are used depends on the problem considered. In the case of hyperbolic equations distributions serve as an appropriate space of generalized functions while ultradistributions of appropriate Gevrey order come into play in the parabolic case.

$\Phi \in C^1\left(\Omega, \mathcal{E}'(\mathbb{R})^{(n\times n)}\right)$, the entries of which are (ultra-) distributions with compact support:

$$\boldsymbol{x}(z) = \Phi(z;\boldsymbol{\vartheta}) \star \boldsymbol{x}(0). \tag{5}$$

This representation exists always for hyperbolic and parabolic p.d.e. for which the curve $s \mapsto (0, s)$ in the (z, t)-plane is not characteristic. In order to obtain equations comprising only lumped quantities, i.e., the measurements and control inputs, the general solution (5) is plugged into the boundary conditions (3) and the output equation (4) yielding

$$L(\boldsymbol{\vartheta})\Phi(\cdot;\boldsymbol{\vartheta}) \star \boldsymbol{x}(0) = D(\boldsymbol{\vartheta})\boldsymbol{u}$$

and

$$\boldsymbol{y} = C(\boldsymbol{\vartheta})\Phi(z;\boldsymbol{\vartheta}) \star \boldsymbol{x}(0).$$

These two equations can be used to eliminate the boundary trajectory $\boldsymbol{x}(0)$ by means of purely algebraic manipulations leading to

$$R_u(\boldsymbol{\vartheta}) \star \boldsymbol{u} = R_y(\boldsymbol{\vartheta}) \star \boldsymbol{y}. \tag{6}$$

C. Calculating parameters

Since the distributions $R_u \in \mathcal{E}'(\mathbb{R})^{(p\times m)}$ and $R_y \in \mathcal{E}'(\mathbb{R})^{(p\times p)}$ may contain derivatives of the Dirac distribution of arbitrary order, this equation may involve derivatives of $\boldsymbol{u}$ and $\boldsymbol{y}$ of arbitrary order. Therefore, it is necessary to regularize $\boldsymbol{u}$ and $\boldsymbol{y}$ by convolving with an appropriate test function[4] $\psi \in \mathcal{D}(\mathbb{R})$:

$$(R_u \star \psi) \star \boldsymbol{u} = (R_y \star \psi) \star \boldsymbol{y}. \tag{7}$$

Note that even in the case where the distributions R_u and R_y do not contain derivatives of the Dirac distribution it might be advantageous to regularize the involved signals, especially with regard to the subsequent application of numerical methods. The physical parameters can be obtained from equation (7) by further algebraic manipulations (cf. [4]). However, in this contribution this will not be carried out as the parameters are obtained by numerically minimizing the error given by

$$e_\psi(t,\boldsymbol{\vartheta}) = \left((R_u(\boldsymbol{\vartheta}) \star \psi) \star \boldsymbol{u} - (R_y(\boldsymbol{\vartheta}) \star \psi) \star \boldsymbol{y}\right)(t)$$

w.r.t. a certain norm. In the following examples the 2-norm will be used, i.e. the parameter estimates $\hat{\boldsymbol{\vartheta}}$ are obtained from solving

$$\hat{\boldsymbol{\vartheta}} = \arg\min J(\boldsymbol{\vartheta}), \quad \text{with} \quad J(\boldsymbol{\vartheta}) = \int_{T_{\min}}^{T_{\max}} \left(e_\psi(\tau,\boldsymbol{\vartheta})\right)^2 \, d\tau, \tag{8}$$

The interval $[T_{\min}, T_{\max}]$ follows from the requirement that in evaluating the error e_ψ only known values of $\boldsymbol{u}$ and $\boldsymbol{y}$ are used, i.e. values in the measuring interval $[0, T]$. From the Titchmarsh theorem on convolution (cf. [5]) results

$$T_{\min} = \max_{\nu=\{u,y\}} \left\{\sup \operatorname{supp}(R_\nu(\boldsymbol{\vartheta}) \star \psi)\right\}$$

and

$$T_{\max} = T + \min_{\nu=\{u,y\}} \left\{\inf \operatorname{supp}(R_\nu(\boldsymbol{\vartheta}) \star \psi)\right\}.$$

It becomes obvious from these equations that in general $T_{\min}$ and $T_{\max}$ may depend on the unknown parameters $\boldsymbol{\vartheta}$.

In order to solve the above minimization problem (8) it is necessary to find the zeros of the gradient

$$\frac{\partial J}{\partial \boldsymbol{\vartheta}}(\boldsymbol{\vartheta}) = \frac{\partial T_{\max}}{\partial \boldsymbol{\vartheta}} \left(e_\psi(T_{\max},\boldsymbol{\vartheta})\right)^2 - \frac{\partial T_{\min}}{\partial \boldsymbol{\vartheta}} \left(e_\psi(T_{\min},\boldsymbol{\vartheta})\right)^2 + 2\int_{T_{\min}}^{T_{\max}} e_\psi(\tau,\boldsymbol{\vartheta}) \frac{\partial e_\psi}{\partial \boldsymbol{\vartheta}}(\tau,\boldsymbol{\vartheta})\, d\tau. \tag{9}$$

One might simplify this expression by priorly defining relevant intervals for the unknown parameters and fix $T_{\min}$ and $T_{\max}$ such that the error can be evaluated for all relevant parameter combinations. Then the first two terms on the right-hand side of (9) would vanish. However, this is not done in the following examples.

The zeros of the gradient (9) are determined numerically. For the one-dimensional case, i.e. only one unknown parameter, an algorithm proposed by Brent (cf. [6]) that is contained in the SciPy library (cf. [7]) under the name `brentq` is used. In the general case the zeros are determined by means of the Levenberg-Marquardt algorithm[5] (cf. [8]). Note that with these gradient-based algorithms only local minima can be found. Therefore, the initial guesses have to be carefully selected.

Note that identifiability of the parameters from the equations (6) is always assumed. The question of identifiability (not parameter identifiability) for general convolutional systems over rings of distributions is discussed in [9].

III. The Wave Equation Example

In order to illustrate the algorithm described in the previous section a vibrating string of length ℓ is considered. Its displacement w is modelled by the wave equation

$$\frac{\partial^2 w}{\partial t^2}(z,t) - c^2 \frac{\partial^2 w}{\partial z^2}(z,t) = 0, \;\; z \in [0,\ell], \;\; t \in \mathbb{R}. \tag{10}$$

Given that the string is clamped at $z = 0$ and it can be excited by a force u at $z = \ell$ the boundary conditions read

$$w(0,t) = 0 \qquad \text{and} \qquad \frac{\partial w}{\partial z}(\ell,t) = u. \tag{11}$$

The aim of the identification process is to obtain an estimate of the speed of propagation c appearing in (10). As this parameter appears only quadratically, it will only be possible to identify its absolute value $|c|$. In the following two scenarios are considered that differ in the chosen output. In the first one the output comprises the gradient at the left boundary

[4]The regularizing procedure can be slightly modified in the hyperbolic case: Since the entries of the coefficient matrices are distributions with compact support, they are necessarily of finite order γ. As a consequence the regularity assumptions on the ψ can be weakened, i.e., ψ is required to be a $C^\gamma(\mathbb{R})$ function with compact support, only.

[5]The first and second derivatives of the error e_ψ w.r.t. the unknown parameters $\boldsymbol{\vartheta}$, that are required for evaluating (9) and the application of the Levenberg-Marquardt algorithm can be obtained by solving the differentiated boundary value problem.

$z = 0$, while in the second scenario the displacement at the right boundary $z = \ell$ is measured.

In order to apply the proposed algorithm the system equations are formulated as first order abstract differential equation w.r.t. the spatial coordinate z. This is achieved by considering w as distribution-valued function of the spatial coordinate z and introducing the vector $\boldsymbol{x} = (x_1, x_2)^T = \left(w, \frac{\partial w}{\partial z}\right)^T$. This way the operators introduced in section II-B, namely equations (2) and (3), read

$$A(c)\boldsymbol{x} = \begin{pmatrix} x_2 \\ \frac{1}{c^2}\frac{\partial^2 x_1}{\partial t^2} \end{pmatrix}, \; L\boldsymbol{x} = \begin{pmatrix} x_1(0) \\ x_2(\ell) \end{pmatrix} \text{ and } Du = \begin{pmatrix} 0 \\ u \end{pmatrix}.$$

Depending on the actual identification scenario the output operator either reads

$$C_1\boldsymbol{x} = x_2(0) \qquad \text{or} \qquad C_2\boldsymbol{x} = x_1(\ell),$$

where the subscript is used to indicate the respective case.

As pointed out in section II it is necessary to solve the Cauchy problem associated with the abstract differential equation. The explicit solution can be obtained for example by operational calculus or the method of characteristics and reads

$$\begin{aligned} (x_1(z))(t) &= \frac{c}{2}\int_{t-\frac{z}{c}}^{t+\frac{z}{c}} (x_2(0))(\tau)\, d\tau \\ (x_2(z))(t) &= \frac{1}{2}\left((x_2(0))\left(t+\frac{z}{c}\right) + (x_2(0))\left(t-\frac{z}{c}\right)\right). \end{aligned} \tag{12}$$

Note that in these equations the boundary value $x_1(0)$ has already been eliminated using the boundary condition $x_1(0) = 0$ in order to shorten the notation. The solutions (12) can be formulated as convolution product

$$x_i(z) = \Phi_i(z) \star x_2(0), \;\; i = 1, 2,$$

where the involved distribution-valued functions $\Phi_i : \Omega \to \mathcal{E}'(\mathbb{R})$ are defined using the Heaviside function H and the Dirac distribution δ:

$$(\Phi_1(z))(t) = \frac{c}{2}\left(H\left(t+\frac{z}{c}\right) - H\left(t-\frac{z}{c}\right)\right)$$

and

$$\Phi_2(z) = \frac{1}{2}\left(\delta_{-\frac{z}{c}} + \delta_{\frac{z}{c}}\right).$$

In the next step of the algorithm this convolutional formulation of the solution is substituted into the boundary conditions and the output equation yielding

$$u = \Phi_2(\ell) \star x_2(0) \tag{13}$$

and either $y_1 = \Phi_2(0) \star x_2(0)$ or $y_2 = \Phi_1(\ell) \star x_2(0)$. Here the subscript denotes again the different scenarios.

It remains to eliminate the unknown boundary value $x_2(0)$. In the first scenario this unknown value is actually measured, because $\Phi_2(0) = \delta$ and, thus, $y_1 = x_2(0)$. Substituting this relation into the boundary condition (13) yields

$$u = \Phi_2(\ell) \star y_1$$

or

$$u(t) = \frac{1}{2}\left(y_1\left(t+\frac{\ell}{c}\right) + y_1\left(t-\frac{\ell}{c}\right)\right), \tag{14}$$

respectively. For the second scenario one may exploit the commutativity of the distributions $\Phi_1(\ell)$ and $\Phi_2(\ell)$ in the following way: By convolving (13) with $\Phi_1(\ell)$ and the output equation for y_2 with $\Phi_2(\ell)$ one obtains

$$\Phi_1(\ell) \star u = \Phi_1(\ell) \star (\Phi_2(\ell) \star x_2(0))$$

and

$$\Phi_2(\ell) \star y_2 = \Phi_2(\ell) \star (\Phi_1(\ell) \star x_2(0)).$$

From the associativity of the convolution and the priorly mentioned commutativity of the involved distributions follows the equality of these two equations:

$$\Phi_1(\ell) \star u = \Phi_2(\ell) \star y_2$$

or

$$c\int_{t-\frac{\ell}{c}}^{t+\frac{\ell}{c}} u(\tau)\, d\tau = y_2\left(t+\frac{\ell}{c}\right) + y_2\left(t-\frac{\ell}{c}\right), \tag{15}$$

respectively. The two equations (14) and (15) are used to define the generalized errors for each identification scenario:

$$e_1(t) = u(t) - \frac{1}{2}\left(y_1\left(t+\frac{\ell}{c}\right) + y_1\left(t-\frac{\ell}{c}\right)\right) \tag{16a}$$

and

$$e_2(t) = c\int_{t-\frac{\ell}{c}}^{t+\frac{\ell}{c}} u(\tau)\, d\tau - \left(y_2\left(t+\frac{\ell}{c}\right) + y_2\left(t-\frac{\ell}{c}\right)\right). \tag{16b}$$

The limits of the integral in the definition of the error functional (8) are $T_{\min} = \frac{\ell}{c}$ and $T_{\max} = T - \frac{\ell}{c}$, because both distributions $\Phi_1(\ell)$ and $\Phi_2(\ell)$ have support in

$$\operatorname{supp}(\Phi_1(\ell)) = \operatorname{supp}(\Phi_2(\ell)) = \left[-\frac{\ell}{c}, \frac{\ell}{c}\right].$$

A. Simulation Results

The signals depicted in figure 1 were obtained by solving the boundary value problem by means of a generalized Fourier expansion. The physical parameters were chosen such that the speed of propagation is $c = 1$ and the length ℓ of the string equals 2. The system was excited by a sinusoidal trajectory for the control input $\frac{\partial w}{\partial z}(\ell)$ and its initial conditions were generated randomly.

By inspecting the delays and predictions appearing in the error equations (16), it becomes obvious that one has to measure at least twice the time $T_c = \frac{\ell}{c}$ in order to obtain the correct value for c. This minimal measuring time is indicated by a vertical dashed line in figure 2 depicting the relative errors of the identified parameter. It becomes apparent from this figure that the proposed algorithm produces accurate estimates of the unknown parameter c. The remaining error is due to discretization of the integrals involved.

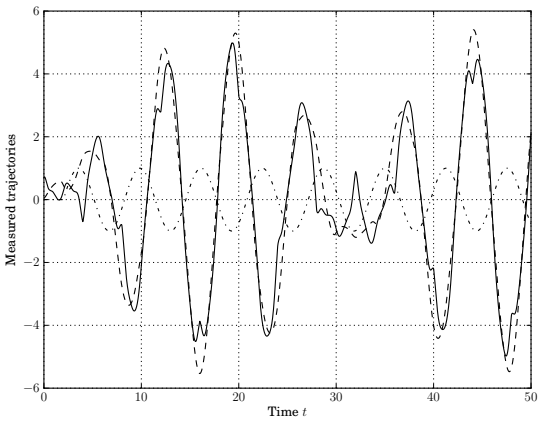

Fig. 1. Simulated signals – control input $\frac{\partial w}{\partial z}(\ell)$ (dot-dash line), gradient $\frac{\partial w}{\partial z}(0)$ (solid line) and displacement $w(\ell)$ (dashed line)

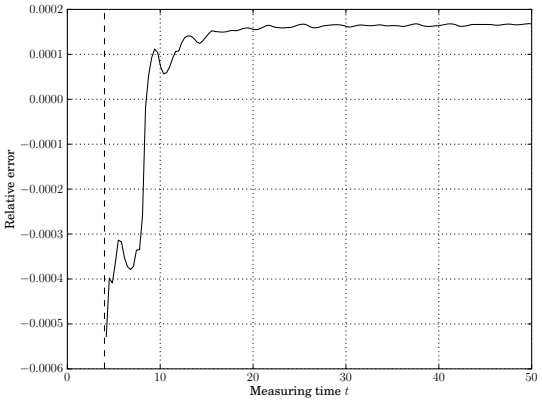

Fig. 2. Scenario 1: Relative error of identified parameter

IV. Heat equation example

Another example that the proposed algorithm can be applied for is the heat equation. The one-dimensional form of this equation can be used to model the evolution of the temperature w in a rod, whose length ℓ is considerably greater than its diameter. It is given by the partial differential equation

$$\frac{\partial w}{\partial t}(z,t) = \alpha \frac{\partial^2 w}{\partial z^2}(z,t),\ z \in [0,\ell],\ t \in \mathbb{R}. \qquad (17)$$

The only involved parameter that needs to be identified is the thermal diffusivity α. The boundary conditions

$$\frac{\partial w}{\partial z}(0,t) = 0 \quad \text{and} \quad \frac{\partial w}{\partial z}(\ell,t) = u(t)$$

follow from the assumptions that the conductor is isolated at $z = 0$ and a heat flow u can be prescribed at $z = \ell$. As in the priorly considered wave equation example two identification scenarios are discussed, that differ in the position of the temperature measurement: While in the first one the output is composed of the temperature at the left boundary, i.e. $y_1(t) = w(0,t)$, it comprises the temperature at the right boundary, i.e. $y_2(t) = w(\ell,t)$, in the second scenario.

A representation of the heat equation as first-order abstract differential equation w.r.t. the spatial coordinate z is obtained by introducing the function $\boldsymbol{x} = (w, \frac{\partial w}{\partial x})^T \in C_1([0,\ell],(D_2'(\mathbb{R}))^2)$, where $D_2'(\mathbb{R})$ is the space of Beurling ultra-distributions of Gevrey order 2 (cf. [10]). This way the operators corresponding to the notation in section II-B read

$$A\boldsymbol{x} = \begin{pmatrix} x_2 \\ \frac{1}{\alpha}\frac{\partial x_1}{\partial t} \end{pmatrix},\ L\boldsymbol{x} = \begin{pmatrix} x_2(0) \\ x_2(\ell) \end{pmatrix} \text{ and } Du = \begin{pmatrix} 0 \\ u \end{pmatrix}.$$

The output equation is given by either $C_1\boldsymbol{x} = x_1(0)$ or $C_2\boldsymbol{x} = x_1(\ell)$ depending on the actual identification scenario, that is denoted by the subscript. Taking into account the boundary condition at $z = 0$, the solution of the associated Cauchy problem can be written as

$$x_1(z) = \sum_{\nu=0}^{\infty} \frac{z^{2\nu}}{\alpha^\nu (2\nu)!} \frac{d^\nu x_1}{dt^\nu}(0)$$
$$x_2(z) = \sum_{\nu=1}^{\infty} \frac{z^{2\nu-1}}{\alpha^\nu (2\nu-1)!} \frac{d^\nu x_1}{dt^\nu}(0).$$

This solution can be formulated as system of convolutional equations

$$x_i(z) = \Phi_i(z) \star x_1(0),\ i = 1,2, \qquad (18)$$

with the ultradistributions

$$\Phi_1(z) = \sum_{\nu=0}^{\infty} \frac{z^{2\nu}}{\alpha^\nu (2\nu)!} \delta^{(\nu)}$$
$$\Phi_2(z) = \sum_{\nu=1}^{\infty} \frac{z^{2\nu-1}}{\alpha^\nu (2\nu-1)!} \delta^{(\nu)}.$$

Herein, $\delta^{(\nu)}$ denotes the ν-th derivative of the Dirac distribution. Both ultra-distributions Φ_1 and Φ_2 are Beurling ultra-distributions of Gevrey order 2 and support in 0.

In order to eliminate the boundary values $x_1(0)$ from (18), this expression is plugged into the boundary condition at $z = \ell$ and the output equations yielding

$$u = \Phi_2(\ell) \star x_1(0) \qquad (19)$$

and, depending on the identification scenario, either

$$y_1 = \Phi_1(0) \star x_1(0) \qquad \text{or} \qquad y_2 = \Phi_1(\ell) \star x_1(0).$$

The elimination of the boundary values $x_1(0)$ is obvious in the case of the first identification scenario, since this boundary value is directly measured. Thus, one obtains

$$u = \Phi_2(\ell) \star y_1. \qquad (20)$$

In the second scenario $x_1(0)$ can be eliminated by exploiting the commutativity of the involved distributions and the properties of the convolution product as it has been shown for the wave equation example in the preceding section. Thus, one obtains the equality

$$\Phi_1(\ell) \star u = \Phi_2(\ell) \star y_2. \qquad (21)$$

The equations (20) and (21) are the basis for the definition of the generalized error. However, as the involved distributions comprise derivatives of arbitrary order, these equations hold in the distributional sense only and have to be regularized

for numerical evaluations. To this end, both equations are convolved with an appropriate test function $\psi \in \mathcal{D}_2(\mathbb{R})$ of Gevrey order 2. Under consideration of the properties of the convolution product this yields

$$\psi \star u = (\Phi_2(\ell) \star \psi) \star y_1$$

and

$$(\Phi_1(\ell) \star \psi) \star u = (\Phi_2(\ell) \star \psi) \star y_2.$$

These equations can be evaluated and lead to the following generalized errors:

$$e_1 = \psi \star u - (\Phi_2(\ell) \star \psi) \star y_1$$

and

$$e_2 = (\Phi_1(\ell) \star \psi) \star u - (\Phi_2(\ell) \star \psi) \star y_2.$$

More explicitely these two errors read

$$e_1 = \psi \star u - \sum_{\nu=1}^{\infty} \frac{\ell^{2\nu-1}}{\alpha^\nu (2\nu-1)!} \frac{d^\nu \psi}{dt^\nu} \star y_1. \tag{22a}$$

and

$$e_2 = \sum_{\nu=0}^{\infty} \frac{\ell^{2\nu}}{\alpha^\nu (2\nu)!} \frac{d^\nu \psi}{dt^\nu} \star u - \sum_{\nu=1}^{\infty} \frac{\ell^{2\nu-1}}{\alpha^\nu (2\nu-1)!} \frac{d^\nu \psi}{dt^\nu} \star y_2. \tag{22b}$$

A. Simulation Results

In order to numerically evaluate the errors (22) only a finite number N of the summands of the involved series are taken into account. As proposed (and justified) in [11] this allows to use a C^N-function only, instead of a smooth test function of Gevrey order 2 for regularization. Here, a piecewise polynomial function with support in $[-T_r, T_r]$ is used:

$$\psi(t) = \begin{cases} 0 & , |t| \geq T_r \\ \kappa (T_r - t)^N (T_r + t)^N & , |t| < T_r. \end{cases}$$

The factor

$$\kappa = \frac{(2N+1)!}{(N!)^2} (2T_r)^{-(2N+1)}$$

was chosen such that $\int_{-\infty}^{\infty} \psi(t)\, dt = 1$.

The trajectories for the identification were obtained from the solution of the boundary value problem with parameters $l = 2$ and $\alpha = 1.25$ by means of a generalized Fourier series. To this end, the system was excited by a sinusoidal heat flow at the right boundary. The trajectories obtained by this procedure are depicted in figure 3. The relative error of the identified parameter for the first scenario is shown in figure 4. The dashed lines indicate the part of the measured values that has to be cut off due to the regularization, because in this step the interval of available data is restricted from $[0, T]$ to $[T_r, T - T_r]$. The remaining error is due to discretization of the integrals involved and the truncation error of the involved series.

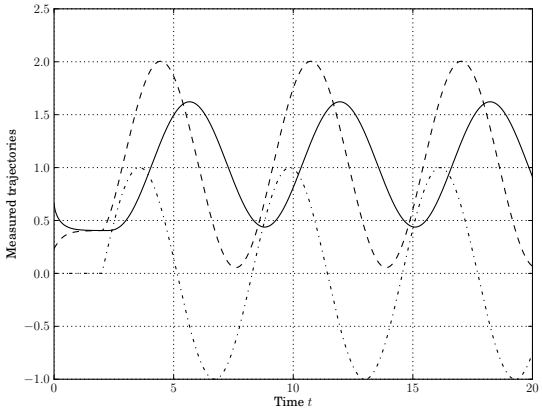

Fig. 3. Simulated signals – control input $\frac{\partial w}{\partial z}(\ell)$ (dot-dash line), temperature $w(0)$ (solid line) and temperature $w(\ell)$ (dashed line)

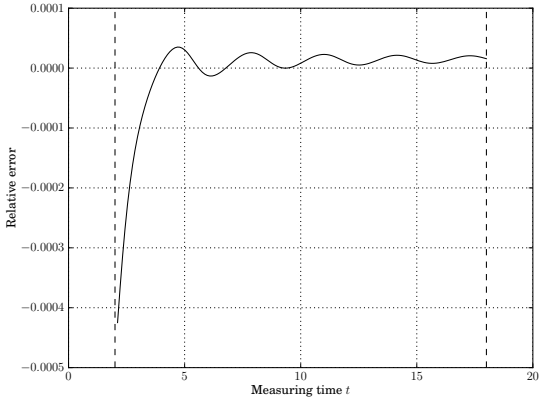

Fig. 4. Scenario 1: Relative error of identified parameter

V. Experiment: The Telegrapher's Equations

To demonstrate the usefulness of the proposed algorithm for real applications, the propagation of current and voltage in a coaxial cable is experimentally investigated. This phenomenon can be modelled by the telegrapher's equations

$$\begin{aligned} \frac{\partial u}{\partial z}(z,t) &= -L \frac{\partial i}{\partial t}(z,t) - R i(z,t) \\ \frac{\partial i}{\partial z}(z,t) &= -C \frac{\partial u}{\partial t}(z,t) - G u(z,t), \end{aligned} \tag{23}$$

where the parameters denote the distributions, i.e. the physical quantity per unit length, of the resistance R, the inductance L, the capacitance C and the conductance G. The experimental setup is depicted in figure 5. It consists of a coaxial cable of length $\ell = 50\,\mathrm{m}$, that is connected at both ends to linear networks. The voltage of the signal generator u_{Sig} is available as input and known. In order to derive the boundary condition at $z = 0$ one applies Kirchhoff's laws to the corresponding network and obtains

$$R_0 i(0,t) + u(0,t) = u_{\mathrm{Sig}}(t). \tag{24}$$

It will become obvious that the boundary conditions at $z = \ell$ are of no interest and therefore not made explicit here. The output of the system comprises the voltages at both ends of

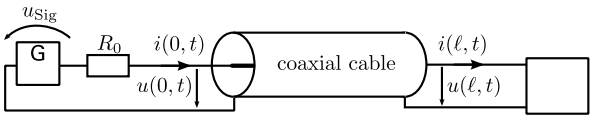

Fig. 5. Sketch of the experimental setup

the cable, i.e.

$$y_1 = u(0) \quad \text{and} \quad y_2 = u(\ell). \tag{25}$$

As the system is already given as first-order abstract differential equation (23), the state vector $\boldsymbol{x} = (u, i)^T$ comprises both physical quantities. The explicit solution of the associated Cauchy problem is cumbersome to obtain, if at all possible. However, a numerical solution can be easily obtained by means of the method of characteristics. Another consequence from the applicability of the method of characteristics is the existence of a formulation of the solution as convolution equation of the form $u(z) = \Phi_u(z) \star u(0) + \Phi_i(z) \star i(0)$.

Note that the solution for the current i can be expressed in a similar way, but is not required in the following and, therefore, neglected. In order to eliminate the boundary values $u(0)$ and $i(0)$, observe that both are directly available from the boundary condition at $z = 0$, namely equation (24), and the output equations (25). Thus, they can be considered known. It remains to plug the solution into the boundary condition at $z = \ell$ to obtain $u(\ell) = \Phi_u(\ell) \star u(0) + \Phi_i(\ell) \star i(0)$. The application of the distributions Φ_u and Φ_i does not involve any derivatives, therefore this relation can be evaluated pointwise without further regularization. However, the experience with the experimentally obtained data showed that it is beneficial for the numerical minimization of the error

$$e = u(\ell) - \Phi_u(\ell) \star u(0) + \Phi_i(\ell) \star i(0).$$

to convolve with a smooth test function ψ. This step is not a mathematical necessity, but rather motivated by the occurrence of inevitable noise in the measurements. As result one obtains a new error function

$$e_\psi = \psi \star u(\ell) - \Phi_u(\ell) \star (\psi \star u(0)) + \Phi_i(\ell) \star (\psi \star i(0)),$$

that is the basis for the numerical treatment.

A. Identification results

It turned out that the parallel estimation of all four parameters L, C, R and G required the initial values of the minimization algorithm to be very close to the final values. Therefore, only the inductance L and the capacitance C were determined from the experimentally obtained data to illustrate the performance of the proposed method. The remaining parameters were held constant at priorly determined optimal values. Figure 7 depicts the equation error e_ψ before and after the application of the minimization algorithm.

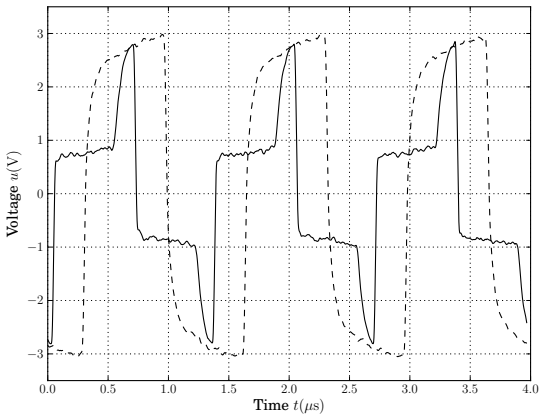

Fig. 6. Regularized signals – Voltages $u(0)$ (solid) and $u(\ell)$ (dashed)

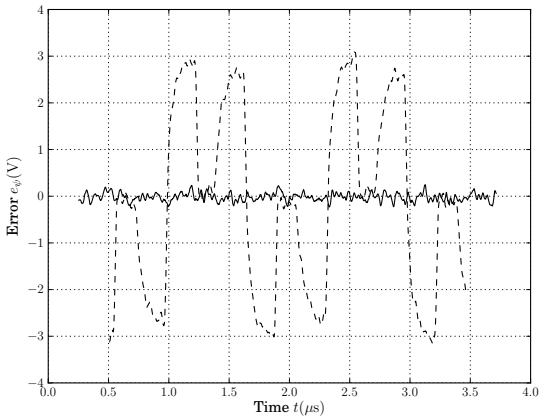

Fig. 7. Equation error e_ψ before (dashed line) and after (solid line) minimization

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