2013 European Control Conference (ECC)
July 17-19, 2013, Zürich, Switzerland.

Multi-objective Predictive Control for Non Steady-State Operation

J.P. Maree*, L. Imsland*

***Abstract*— The concept of dynamic-mean Pareto optimality is introduced for multi-objective Model Predictive Control. Dynamic-mean Pareto optimal solutions are obtained by solving a *free* initial state and final time optimal control problem. Subsequently, we propose a receding horizon tracking formulation with dynamic-mean Utopia set-points. A Dynamic-mean Utopia point is defined as the intersection of average minima, of underlying performance indices, over a dynamic horizon. The latter is compared with recently proposed steady-state Utopia tracking and Pareto optimally weighted Economic MPC. Incorporating dynamic-mean Utopia set-points in a tracking formulation, one attains economic performance at least equal to that of steady-state Utopia tracking, and, performance close to that of Pareto optimal, weighted Economic MPC. The latter is illustrated for a CSTR numerical case example.**

I. Introduction

Multi-objective optimization in control involves optimizing a collection of performance criteria, often conflicting, systematically and simultaneously, taking into account different control specifications [1].

Since the performance criteria are usually incommensurate and in conflict, it often implies that no unique solution to the multi-objective optimization problem is attainable. Instead, the underlying problem of interest is not in finding a single optimal solution, but rather efficient, non-inferior and dominated solutions that attain the priorities of the multiple performance criteria as well as possible. Such Pareto optimal solutions consequently contribute towards the construction of a Pareto front. Based on expert knowledge, and predefined decision criteria, a single Pareto optimal solution is chosen on this Pareto front as a compromise solution for which the process needs to be operated at [2].

The key technical challenge for multi-objective optimization in control problems is constructing the Pareto front. Finding the Pareto optimal solutions along the Pareto front can be computationally exhaustive for large dimensional systems. Furthermore, once such a Pareto front is constructed, expert knowledge is still required, in addition, to formulate decision criteria for choosing the preferred compromise solution [3].

A steady-state Utopia tracking MPC formulation has been proposed [3] as an alternative to solving Pareto optimal optimization problems for the construction of a Pareto front. In the proposed method, the tracking objective is that of minimizing the distance norm between a vector of performance indices, and the intersection of steady-state minima of the respective performance indices. This intersection was subsequently termed the steady-state Utopia point. A fundamental property of the aforementioned approach, for multi-objective optimization formulations, is that the tracking controller can exploit system dynamics by leaving the steady-state Pareto front, getting closer to the steady-state Utopia point [3].

This approach retain characteristics of unreachable set-point tracking, earlier discussed by [4], in the sense that beneficial fast/slow asymmetric set-point tracking is observed. Fast convergence towards, and slow divergence away, from the set-point, is achieved.

A. Contribution

For selected cases, the benefits of increased process performance during periodic operation in contrast to steady-state operation is now well established [5]. Angeli et al. [6] shows that optimal periodic operation may outperform, on a time average, the best steady-state operation. Hence, the steady-state Utopia set-points used in the tracking formulation of [3] may be sub-optimal set-points for non steady operation. If periodic operation is indeed favourable, and admissible for a certain control philosophy, the later Utopia tracking strategy may impose limitations on possible time average economic performance. Hence, we present the concept of *dynamic-mean Pareto optimal* and construct a dynamic-mean Pareto front which entails the optimal limiting bound on average process performance. It is shown that such a front defines limiting process performance which is at least as good as that of the steady-state Pareto front. Next, we extend the notion of steady-state Utopia set-point tracking [3] to dynamic-mean Utopia set-point tracking. The latter results in a favourable Utopia point shift, in the performance criterion space, for optimal set-point tracking. Lastly, a result on convergence for the proposed dynamic-mean Utopia set-point tracking strategy is presented.

II. Preliminaries

A. System definition

We consider the discrete, time-invariant model

$$x^{+} = f\left(x, u\right) \tag{1}$$

with state $x \in \mathbb{X} \subseteq \mathbb{R}^{n_x}$ and control input $u \in \mathbb{U} \subseteq \mathbb{R}^{n_u}$. We define the mixed constraint set $(x(k), u(k)) \in \mathbb{Z}$, $\forall k \in \mathbb{I}_{\geqslant 0}$ for a compact, mixed constraint set $\mathbb{Z} = \mathbb{X} \times \mathbb{U}$. We denote $\eta(k; x, \mathbf{u})$ the solution to (1) at sample time $k \in \mathbb{I}_{0:N}$ with initial condition x and control sequence $\mathbf{u} \in \mathbb{U}^N$.

B. Multi-objective Optimization

The problem of multi-objective optimization is the problem of simultaneously minimizing $n_\Phi \geqslant 2$ performance

*Department of Engineering Cybernetics, Norwegian University of Science and Technology, 7034 Trondheim, Norway. {phillip.maree;lars.imsland}@itk.ntnu.no

indices $\phi_i(x,u),\ i \in \mathbb{I}_{1:n_\Phi}$. If we denote $\Phi(x,u) \in \mathbb{R}^{n_\Phi}$ a vector of performance indices

$$\Phi\left(x,u\right) := \left[\phi_1\left(x,u\right), \cdots, \phi_{n_\Phi}\left(x,u\right)\right]^T \tag{2}$$

then the solution to the weighted multi-objective optimization problem

$$\left(x^0, u^0\right) = \arg \min_{(x,u)\in\mathbb{Z}} \mathbf{w}^T \Phi\left(x,u\right) \tag{3}$$

is assumed unique, in which $\mathbf{w} = \left[w_1, \cdots, w_{n_\Phi}\right]^T \in \mathbb{R}^{n\Phi}_{\geqslant 0}$ denotes a particular positive weighting vector for the n_Φ respective performance indices. It holds that $\sum_{i=1}^{n_\Phi} w_i = 1$.

In multi-objective optimization, optimality is defined in the sense of Pareto optimality [7], since no single global optimal solution exists due to the often conflicting nature of the underlying performance indices. We denote the set $\Omega \subseteq \mathbb{R}^{n_\Phi}$ to be the admissible criterion space, given system constraints, in which the performance vector $\Phi(x,u)$ can be evaluated. Figure 1, adapted from [7], gives an graphical representation of the criterion space $\Omega \subseteq \mathbb{R}^2$ for two performance indices $\phi_1(x,u)$ and $\phi_2(x,u)$.

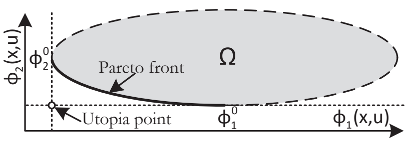

Fig. 1. Evaluation mapping of multi-objective problen.

Definition 1 (Pareto optimal [7]): A state control pair $\left(x_p^0, u_p^0\right)$, being the optimal solution to optimization problem (3), given a particular weight vector $\mathbf{w}$, is Pareto optimal iff there does not exist another state control pair $(x,u) \in \mathbb{Z}$ such that $\mathbf{w}^T\Phi\left(x,u\right) \leqslant \mathbf{w}^T\Phi\left(x_p^0, u_p^0\right)$ and $\exists\left(x,u\right) \in \mathbb{Z}$: $w_i\phi_i\left(x,u\right) < w_i\phi_i\left(x_p^0, u_p^0\right)$ for at least one index $i \in \mathbb{I}_{1:n_\Phi}$.

Definition 2 (Pareto set/front [7]): Define the Pareto set $\varphi := \left\{\Phi\left(x_p^0, u_p^0\right) \in \Omega \,\middle|\, \left(x_p^0, u_p^0\right) \in \mathbb{Z}\right\}$ being non-empty, which encapsulate all evaluated Pareto optimal points. The Pareto front is defined as the boundary of the Pareto set φ.

Definition 3 (Utopia point [7]): We define the Utopia point $\Phi^0 := \left[\phi_1^0, \phi_2^0, \cdots, \phi_{n_\Phi-1}^0, \phi_{n_\Phi}^0\right]^T$ in which $\phi_i^0 = \min_{(x,u)\in\mathbb{Z}} \phi_i(x,u),\ \forall i \in \mathbb{I}_{1:n_\Phi}$

Definition 4 (Compromise point [7]): The compromise point is Pareto optimal, evaluated on the Pareto front as close as possible to the Utopia point, given some decision criteria norm measure.

Norms used in [3] for evaluating the compromise point with respect to the Utopia point are either 1-, 2- or ∞-norm measures. We will restrict the discussion in this work to the Euclidean distance 2-norm, denoted $\left\|\cdot\right\|_2$.

III. Pareto Optimality for Dynamic Processes

Attempting to regulate a constrained, dynamic process (1) at a compromise point, being Pareto optimal in light of Definition 1, may contribute towards infeasible process operation. The latter may be due to controllability that is lost when one is limited by constraints on the control inputs, and process operation outside the admissible region of operation. Hence, authors [3] defines Pareto optimality, and a Pareto front for a dynamic process (1) in the sense of steady-state operation. It follows that choosing a compromise point along a steady-state Pareto front implies that the process can admissibly be regulated at steady-state for this compromise point, $k \in \mathbb{I}_{\geqslant 0}$.

A. On Pareto for steady-state operation

Given a constrained dynamic process (1), and multi-objective performance vector (2), define the steady-state optimization problem

$$\left(x^0, u^0\right) = \arg \min_{(x,u)\in\mathbb{Z}} \mathbf{w}^T \Phi\left(x,u\right) \tag{4a}$$

$$\text{s.t. } x = f\left(x,u\right) \tag{4b}$$

Definition 5 (Steady-state Pareto optimal [3]): Define the set $\mathbb{Z}_s := \left\{(x,u) \in \mathbb{Z} \middle| x = f\left(x,u\right)\right\}$. The state control pair $\left(x_p^0, u_p^0\right)$, being the optimal solution to the steady-state optimization problem (4), given a particular weight vector $\mathbf{w}$, is steady-state Pareto optimal iff there does not exist another state control pair $(x,u) \in \mathbb{Z}_s$ such that $\mathbf{w}^T\Phi\left(x,u\right) \leqslant \mathbf{w}^T\Phi\left(x_p^0, u_p^0\right)$ and $\exists\left(x,u\right) \in \mathbb{Z}_s$ such that $w_i\phi_i\left(x,u\right) < w_i\phi_i\left(x_p^0, u_p^0\right)$ for at least one index $i \in \mathbb{I}_{1:n_\Phi}$.

Definition 6 (Steady-state Utopia Point [3]): We define $\Phi_s^0 := \left[\phi_{s,1}^0, \phi_{s,2}^0, \cdots, \phi_{s,n_\Phi-1}^0, \phi_{s,n_\Phi}^0\right]^T$ the steady-state Utopia point, in which $\forall i \in \mathbb{I}_{1:n_\Phi}$ we have the i^{th} optimal performance index being the solution the steady-state optimization problem (4) in which the performance index weight is chosen as $w_j = \{1, j = i; 0, j \neq i\}$.

With Definitions 2 and 5 in mind, we denote φ_s to be the steady-state Pareto front.

B. On Pareto optimality for non steady-state operation

Consider the optimal, steady-state control pair (x^0, u^0) being the solution to (4), given a particular weight vector $\mathbf{w}$. For performance indices $\phi_i(x,u),\ \forall i \in \mathbb{I}_{1:n_\Phi}$, it is conceivable, as previously noted by [6], [8]; during dynamic process operation we may have $\exists\left(x,u\right) \in \mathbb{Z} : \mathbf{w}\Phi\left(x,u\right) < \mathbf{w}\Phi\left(x^0,u^0\right)$. More fundamentally, there may exist a state control sequence $(x, \mathbf{u}) \in \mathbb{X} \times \mathbb{U}^N$ such that the average dynamic performance of $\mathbf{w}\Phi(x,u)$, over horizon $\mathrm{N} \in \mathbb{I}_{\geqslant 1}$, evaluates $\frac{1}{N}\sum_{k=0}^{N-1} \mathbf{w}\Phi\left(\eta\left(k;x,\mathbf{u}\right), u\left(k\right)\right) \leqslant \mathbf{w}\Phi\left(x^0,u^0\right)$.

Remark 3.1: The latter reasoning bears close resemblance to Pareto optimality in light of Definition 1. Section III-A diverged away from the latter framework in attempt of circumventing implementing Pareto optimal set-points which may inevitably lead to infeasible process operation.

We are interested in formulating an admissible, dynamic optimization problem, which will optimize a given objective, with respect to on time average performance, over one cyclical period, where the optimal period length is dynamically determined by varying the horizon length $N \in \mathbb{I}_{\geqslant 1}$. The Dynamic-mean Pareto multi-objective Optimization (DPO) problem is defined as:

$$\begin{aligned}&\min_{\{N,(x,\mathbf{u})\in\mathbb{X}\times\mathbb{U}^N\}} \frac{1}{N}\sum_{k=0}^{N-1}\mathbf{w}^T\Phi\left(x\left(k\right),u\left(k\right)\right)\\ &\text{s.t.}\quad x\left(k+1\right)=f\left(x\left(k\right),u\left(k\right)\right),\ k\in\mathbb{I}_{0:N-1}\\ &\qquad x\left(0\right)=x\left(N\right),\quad x\left(0\right)=\textbf{free}\end{aligned} \tag{DPO}$$

The optimal solution to (DPO), denoted $(x^0, \mathbf{u}^0, N^0)$, is assumed unique.

Remark 3.2: It is important to note that the horizon length N in (DPO), is *free*. Also, the equality constraint $x(0) = x(N)$ enforces cyclical operation with *free* initial state x. Enforcing periodicity does not restrict the generality of the approach much (i.e., steady-state operation is a special case).

Remark 3.3: For N being very large, one may have numerical difficulties in solving (DPO). For computational considerations one can restrict horizon N to some admissible horizon length. The latter, however, may lead to a sub-optimal, average dynamic performance.

Definition 7 (Dynamic-mean Pareto optimal): The horizon N^0 and state-control sequence pair $\left(x_p^0, \mathbf{u}_p^0\right) \in \mathbb{X}\times\mathbb{U}^{N^0}$ is dynamic-mean Pareto optimal iff there does not exists another horizon N and pair $(x, \mathbf{u}) \in \mathbb{X}\times\mathbb{U}^N$ such that

$$\begin{aligned}&\frac{1}{N}\sum_{k=0}^{N-1}\mathbf{w}^T\Phi\left(\eta\left(k;x,\mathbf{u}\right),u\left(k\right)\right)\\ &\quad\leqslant\frac{1}{N^0}\sum_{k=0}^{N^0-1}\mathbf{w}^T\Phi\left(\eta\left(k;x_p^0,\mathbf{u}_p^0\right),u_p^0\left(k\right)\right)\end{aligned} \tag{5}$$

and $\exists\,(x, \mathbf{u}) \in \mathbb{X}\times\mathbb{U}^N$ such that

$$\begin{aligned}&\frac{1}{N}\sum_{k=0}^{N-1}w_i\phi_i\left(\eta\left(k;x,\mathbf{u}\right),u\left(k\right)\right)\\ &\quad<\frac{1}{N^0}\sum_{k=0}^{N^0-1}w_i\phi_i\left(\eta\left(k;x_p^0,\mathbf{u}_p^0\right),u_p^0\left(k\right)\right)\end{aligned} \tag{6}$$

for at least one performance index $i \in \mathbb{I}_{1:n_\Phi}$.

Definition 8 (Dynamic-mean Utopia Point): We denote $\Phi_d^0 := [\phi_{d,1}^0, \phi_{d,2}^0, \cdots, \phi_{d,n_\Phi-1}^0, \phi_{d,n_\Phi}^0]^T$ the dynamic-mean Utopia point, in which $\forall i \in \mathbb{I}_{1:n_\Phi}$, the optimal i^{th} performance index $\phi_{d,i}^0$ is the average performance of objective $\phi_i(x, u)$, evaluated over an optimal period N^0, for (DPO), i.e., $\phi_{d,i}^0 := \frac{1}{N^0}\sum_{i=0}^{N^0-1} w_j\phi_i\left(\eta\left(k; x_p^0, \mathbf{u}_p^0\right), u_p^0(k)\right)$ in which $w_j = \{1, j = i; 0, j \neq i\}$.

With Definitions 2 and 7 in mind, we denote φ_d to be the dynamic-mean Pareto front. Next, equivalence and performance results for the respective dynamic-mean Utopia point and steady-state Utopia point is presented.

Proposition 1 (Utopia point equivalence): Assume horizon length N is restricted to 1 for optimal control problem (DPO). Then, the dynamic-mean Utopia point Φ_d^0, as per Definition 8, is equivalent to the steady-state Utopia point, Φ_s^0, as per Definition 6. Consequently, steady-state and dynamic-mean Pareto optimal points, evaluated in the criterion space $\Omega \subseteq \mathbb{R}^{n_\Phi}$, give equivalent Pareto fronts, i.e., $\varphi_d \equiv \varphi_s$.

Proof: For horizon length $N = 1$, periodicity constraint in (DPO) is equivalent to the steady-state constraint (4b). Furthermore, objectives for optimization problems (DPO) and (4) are both evaluated at a single time instant. It follows that optimization problems (4) and (DPO) are equivalent steady-state optimization problems. Equivalence of dynamic-mean and steady-state Utopia points follows. It is clear that the latter extends to equivalence of steady-state and dynamic-mean Pareto fronts. ■

Proposition 2 (Utopia point performance): Suppose horizon length $N > 1$, for optimal control problem (DPO). Then, the realized dynamic-mean Utopia point Φ_d^0 is at least as good as the steady-state Utopia point, Φ_s^0, *element wise* i.e., $\Phi_{d,i}^0 \leqslant \Phi_{s,i}^0,\ \forall i \in \mathbb{I}_{i:n_\Phi}$.

Proof: Suppose the optimal solution to (4) is the steady-state control pair (x_s^0, u_s^0) given a weight vector $\mathbf{w}_i = [w_1, w_j, \cdots, w_{n_\Phi}]$, in which $w_j = \{1, j = i; 0, j \neq i\}$. The i^{th} element of the steady-state Utopia point, i.e., $\Phi_{s,i}^0$, is subsequently evaluated. For horizon $N = 1$, it follows from Proposition 1 that $\Phi_{d,i}^0 = \Phi_{s,i}^0,\ \forall i \in \mathbb{I}_{1:n_\Phi}$. Next, for horizon $N > 1$, suppose for weight $\mathbf{w}_i$, choosing a control sequence, $\tilde{\mathbf{u}} = [u(0), \cdots, u(N-1)] = \left[u_s^0, \cdots, u_s^0\right]$ system (1) evolves, given this control sequence, as

$$\tilde{\mathbf{x}} = \left[x_s^0, \eta\left(1; x_s^0, u_s^0\right), \cdots, \eta\left(N-1; x_s^0, u_s^0\right), x_s^0\right]$$

We have that the state control-sequence $(\tilde{\mathbf{x}}(0), \tilde{\mathbf{u}})$ is an admissible solution to the optimal control problem (DPO). Furthermore, given the particular choice of control sequence we have that $\mathbf{w}_i^T\Phi_{d,i}(\tilde{\mathbf{x}}(0), \tilde{\mathbf{u}}) = \mathbf{w}_i^T\Phi_{s,i}^0$ since steady-state performance on average is equivalent to steady-state performance at given time step k. From optimality of (DPO) it follows that $\mathbf{w}_i^T\Phi_{d,i}^0 \leqslant \mathbf{w}_i^T\Phi_{d,i}(\tilde{\mathbf{x}}(0), \tilde{\mathbf{u}}) = \mathbf{w}_i^T\Phi_{s,i}^0$. Hence, $\Phi_{d,i}^0 \leqslant \Phi_{s,i}^0$ since all weight elements of $\mathbf{w}_i$ are zero except the i^{th} element being one. Generalising the latter reasoning for all $i \in \mathbb{I}_{1:n_\Phi}$, concludes the proof. ■

IV. Multi-objective Predictive Control

Section III extended the concept of steady-state Pareto optimality to that of dynamic-mean Pareto optimality. Definitions on steady-state and dynamic-mean Utopia points followed. This section entails the incorporation of these Utopia points in respective receding horizon control tracking strategies. For steady-state Utopia set-point tracking, we briefly present the Utopia tracking approach proposed earlier by [3]. Motivation for applying dynamic-mean Utopia set-points in an averaging receding horizon tracking formulation, in contrast to steady-state Utopia point tracking used by [3], becomes apparent when one consider case studies where non-steady process operation, on time average, deems more beneficial than steady-state operation. Cases where the latter is apparent include processes with consecutive-competitive reactions [5].

A. Steady-state Utopia Tracking MPC

Given the optimal steady-state Utopia point, Φ_s^0, as per Definition 6, define the steady-state Utopia tracking MPC (ss-UT MPC) value function [3]

$$V_{N,1}(x,\mathbf{u}) = \sum_{k=0}^{N-1} \left\| \Phi(x(k), u(k)) - \Phi_s^0 \right\|_2 \quad (7)$$

which measures the tracking distance from the optimal steady-state Utopia point over a $N-$step receding horizon. The MPC optimal control problem is formulated as

$$\min_{x,\mathbf{u}} V_{N,1}(x,\mathbf{u}) \quad (8a)$$

$$\text{s.t.} \quad x^+ = f(x,u), \ \forall k \in \mathbb{I}_{0:N-1} \quad (8b)$$

$$(x(k), u(k)) \in \mathbb{Z}, \ \forall k \in \mathbb{I}_{0:N-1} \quad (8c)$$

$$\eta(k; x, \mathbf{u}) \in \mathbb{X}, \ \forall k \in \mathbb{I}_{0:N} \quad (8d)$$

B. Dynamic-mean Utopia Tracking MPC

Given the dynamic-mean Utopia point, Φ_d^0, as per Definition 8, we define dynamic-mean Utopia tracking MPC (dm-UT MPC) value function

$$V_{N,2}(x,\mathbf{u}) = \left\| \frac{1}{N} \sum_{k=0}^{N-1} \Phi(x(k), u(k)) - \Phi_d^0 \right\|_2 \quad (9)$$

Remark 4.1: The particular formulation of the dm-UT MPC value function, penalizes (minimize) the 2-norm *average* tracking distance from the dynamic-mean Utopia point Φ_d^0. This formulation is in contrast to the more often used tracking formulation, where one penalize (minimize) the sum of individual 2-norm tracking measures from the desirable set-point at each time instant k. Motivation is that we would like to exploit beneficial transients past the dynamic-mean Utopia set-point, increasing on time average performance.

The MPC optimal control problem is formulated as

$$\min_{x,\mathbf{u}} V_{N,2}(x,\mathbf{u}) \quad (10a)$$

$$\text{s.t.} \quad (8b) - (8d) \quad (10b)$$

$$x(0) = x(N) \quad (10c)$$

in which (10c) is required for stabilizing formulations. Denote the optimal solution, and MPC value function by $\mathbf{u}^0$ and $V_{N,2}^0(x)$. The optimal first move then defines the MPC control law, $\kappa_{N,2}(x) = \mathbf{u}^0(0;x)$. The closed-loop system under MPC is then $x^+ = f(x, \kappa_{N,2}(x))$.

C. Convergence

A brief discussion on convergence for (10) is provided. We denote the state-control sequence $(x_c^0, \mathbf{u}_c^0)$, and horizon N^0, a compromise solution of the dm-UT MPC strategy, iff $(x_c^0, \mathbf{u}_c^0)$ is a dynamic-mean Pareto optimal solution to (DPO), and choosing horizon length N^0 for (10), is a unique solution that minimize (10). For the purpose of analysis, assume horizon length $N = N^0$ for (10). Next, define the set of admissible initial states, $\mathcal{X}_N$,

$$\mathcal{X}_N := \{x \in \mathbb{X} | \exists \mathbf{u} \in \mathbb{U}^N \text{ s.t. } x = \eta(N; x, \mathbf{u}), \ \eta(k; x, \mathbf{u}) \in \mathbb{X}, \forall k \in \mathbb{I}_{0:N-1}, \}$$

and corresponding control constraint set, $\mathcal{U}_N$,

$$\mathcal{U}_N(x) := \left\{ \mathbf{u} \in \mathbb{U}^N \,|\eta(k; x, \mathbf{u}) \in \mathcal{X}_N, \forall k \in \mathbb{I}_{0:N} \right\}$$

Note that $\mathcal{X}_N$ is control invariant by definition. Next, define $\Sigma := \left\{ x \in \mathcal{X}_N \left| \exists k \in \mathbb{I}_{0:N-1} \text{ s.t. } x = f\left(\eta\left(k; x_c^0, \mathbf{u}_c^0\right), \mathbf{u}_c^0(k)\right) \right.\right\}$ the compromise solution set which contains periodic solutions of form $\eta\left(k + iN; x_c^0, \mathbf{u}_c^0\right), k \in \mathbb{I}_{0:N}, i \in \mathbb{I}_{\geqslant 0}$. It is easy to show that Σ is a bounded, control invariant set i.e., $\forall x \in \Sigma \Rightarrow \exists k \in \mathbb{I}_{0:N-1}$ s.t. $f\left(x, \mathbf{u}_c^0(k)\right) \in \Sigma$. Constraint (10c) assists in the proof for convergence of $x^+ = f(x, \kappa_{N,2}(x))$ towards the set Σ.

Lemma 4.1 (Decrease of $V_{N,2}^0(x)$): For all $x \in \mathcal{X}_N$, $\exists \mathbf{u} \in \mathcal{U}_N$ such that $V_{N,2}^0(f(x, \mathbf{u}(0))) \leqslant V_{N,2}^0(x)$.
Proof: (Sketch) Enforced periodic constraint (10c) implies there exists $\tilde{\mathbf{u}} \in \mathbb{U}_N$ such that $V_{N,2}(f(x, \tilde{\mathbf{u}}(0))) = V_{N,2}^0(x)$ for all $\forall x \in \mathcal{X}_N$. From optimality it follows that $V_{N,2}^0\left(f\left(x, \mathbf{u}^0(0;x)\right)\right) \leqslant V_{N,2}^0(x)$. ■

Assumption 1 (Identical solutions): Suppose no solution $(x, \mathbf{u}) \in \mathcal{X}_N \times \mathcal{U}_N$ can stay identical in the set $\mathcal{S} := \left\{ x \in \mathcal{X}_N \left| V_{N,2}^0(f(x, \mathbf{u}(0)))) = V_{N,2}^0(x) \right.\right\}$ other than the trivial solution $(x, \mathbf{u}) = \left(x_c^0, \mathbf{u}_c^0\right)$.

Remark 4.2: What Assumption 1 implicitly implies is that we assume, given horizon length $N > 1$, that $\exists k \in \mathbb{I}_{0:N-1}$ such that $V_{N,2}^0(\eta(k+1; x, \kappa_{N,2}(x))) < V_{N,2}^0(\eta(k; x, \kappa_{N,2}(x)))$ exists for all $x \in \mathcal{X}_N - \Sigma$.

Theorem 4.1 (Convergence): Let Assumption 1 hold. The system $x^+ = f(x, \kappa_{N,2}(x))$ converge to the compromise solution set Σ.
Proof: (Sketch) By Assumption 1 and Lemma 4.1, convergence of $x^+ = f(x, \kappa_{N,2}(x))$ to the compromise solution set Σ is a direct consequence when one apply invariance set principles stipulated in Lasalle's invariance Theorem [9] for discrete-time systems. ■

V. NUMERICAL CASE STUDY

A. Isothermal CSTR with heat flux

We consider a continuous flow stirred-tank reactor (CSTR) with heat flux control [5] as numerical example. The process exhibits parallel reactions, $R \to P_1$, $R \to P_2$, in which R being the reaction concentration, P_1 the desired product and P_2 the waste product. The general form of the dimensionless conservation equations of the CSTR can be written as

$$\begin{aligned} \dot{x}_1 &= 1 - a_1 e^{-1/x_3} x_1^\alpha - a_2 e^{-\delta/x_3} x_1 - x_1 \\ \dot{x}_2 &= a_1 e^{-1/x_3} x_1^\alpha - x_2 \\ \dot{x}_3 &= u - x_3 \end{aligned} \quad (11)$$

We denote the dimensionless control input u as the heat-flux. States x_1 ,x_2 and x_3 are the dimensionless concentration R, desired product P_1 and waste product P_2 respectively. We assume the following parameter values $\alpha = 2$, $\delta = 0.55$, $a_1 = 10^4$, $a_2 = 400$. In this example we only consider two objectives of interest. The first objective aims at maximizing product profit P_1, i.e., minimize the objective $\phi_1(x,u) = -x_2$. The second objective aim is to minimize the waste product cost P_2, i.e., minimize the objective $\phi_2(x,u) = 0.3x_3$.

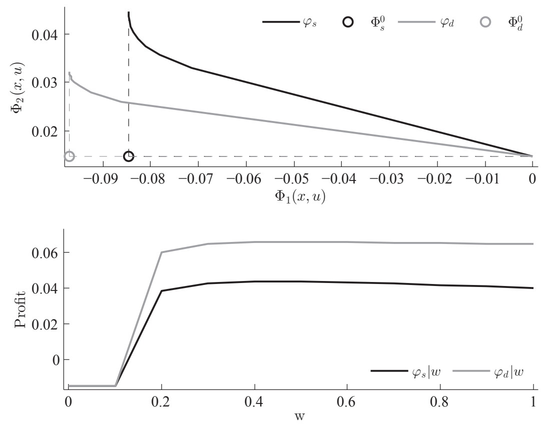

Fig. 2. Steady-state and dynamic-mean Pareto fronts, Utopia points and respective compromise solutions.

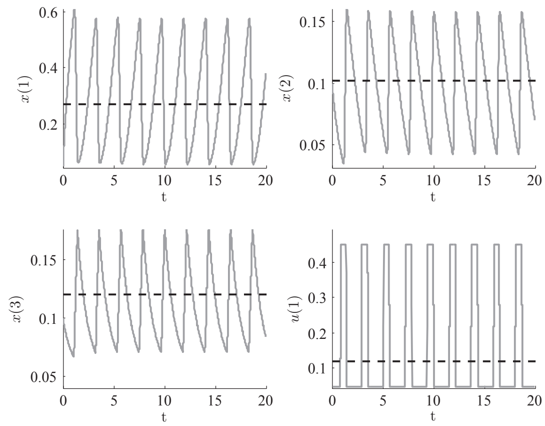

Fig. 3. Trajectories for the Economic MPC formulation with Pareto optimal weighting (dashed line denote averages).

B. Computational Considerations

The respective dynamic optimization problems are solved in a Modelica-based open source platform called JModelica.org [10]. The CasADi [11] package was incorporated for evaluating first and second order derivatives and for integration. A collocation-based optimization algorithm was employed, where a Legendre-Gauss-Radau collocation scheme was used to discretize the problem. An interior-point optimization package, IPOPT [12], was used as NLP solver for the discretized problem.

C. On Construction of Pareto Fronts

This section reports on how the dynamic-mean and steady-state Pareto fronts, evaluated in the cost criterion space, perform, and also the subsequent effect on achievable Utopia points (note that these fronts need not be constructed in the case of the ss-UT- and dm-UT MPC strategies). For scalar weight $w \in [0,1]$, sweep over $n_w = 11$ individual, equally space points, such that the i^{th} index of weight vector $\mathbf{w}$ is defined as $\mathbf{w}_i = [w_i, 1-w_i]$, $\forall i \in \mathbb{I}_{1:n_w}$. We define $\Phi(x,u) = [\phi_1(x,u), \phi_2(x,u)]^T \in \mathbb{R}^{n_\Phi}$ the performance vector for $n_\Phi = 2$. The steady-state and dynamic-mean Pareto fronts can be constructed respectively by solving the n_w steady-state optimization problems (4), and n_w dynamic optimization problems (DPO). Figure 2 illustrates the obtained steady-state Pareto front φ_s and dynamic-mean Pareto front φ_d, and attainable profit evaluated for different objective weights. For optimal non-steady operation, the dynamic-mean Pareto front improves the limiting performance bound, on time average. As consequence the optimal dynamic-mean Utopia point Φ_d^0 undergoes a left-wise shift in the economic criterion space, providing a more favourable set-point for average process operation.

D. On Performance for Utopia-tracking Formulations

This section entails the results obtained for the respective ss-UT MPC (8) and dm-UT MPC (10) formulations. We will also compare performance of the latter two formulations with that of Economic MPC (12), when the single underlying objective of interest for Economic MPC, is the Pareto optimal weighted combination of objectives $\phi_1(x,u)$ and $\phi_2(x,u)$. From Figure 2, we can conclude that the best decision criteria for optimal profitable operation of the process along both steady-state, and dynamic Pareto fronts, is for optimal weight $w^0 \approx 0.6$. The optimal performance weight vector is defined as $\mathbf{w}^0 = [w^0, 1-w^0]$ and is needed only for the multi-objective Economic MPC formulation. We define the Economic MPC value function

$$V_{N,3}(x,\mathbf{u}) = \sum_{k=0}^{N-1} \mathbf{w}^{0^T} \Phi(x(k), u(k))$$

and Economic MPC optimal control problem

$$\min_{x,\mathbf{u}} V_{N,3}(x,\mathbf{u}) \text{ s.t. } (8b)-(8d) \tag{12}$$

A MPC horizon length of $N = 50$, sampling period $\Delta t = 0.04$, simulation time $t_f = 20s$, and initial condition $x(0) = [0.05, 0.1, 0.1]^T$ was chosen. Figure 3 illustrates how Economic MPC performs under the Pareto optimal weighting. For the consecutive-competitive reactions in the CSTR case, optimal operation is during non-steady state or periodical operation. One observes how economic gains are obtained by switching the heat flux control input between extremes.

Figure 4 illustrates trajectories for the ss-UT MPC strategy. No stabilizing constraints, as proposed by [3], was enforced. It is observed that since periodical operation is optimal, the ss-UT MPC formulation operates at non-steady state. Since, steady-state Utopia set-points are unreachable, and on time average, periodical operation outperforms steady-state, one observes how on time average the product being produced outperforms the steady-state tracking set-point. However, due to the conservative steady-state set-points being implemented, any further gains past the steady-state set-point is penalized. The latter motivates the implementation of dynamic-mean Utopia set-points.

Figure 5 illustrates the process trajectories for the dm-UT MPC strategy. As discussed in Section IV-B, the unique MPC value function (9) penalizes the average tracking distance with respect to a dynamic-mean Utopia point. Any process

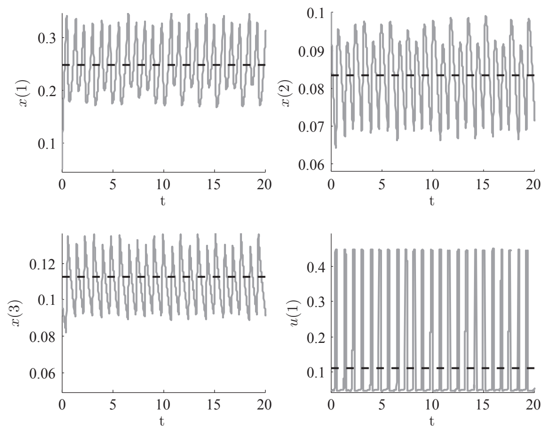

Fig. 4. Trajectories for the steady-state Utopia Tracking MPC formulation with steady-state Utopia set-point $\bar{\Phi}_s^0$ (dashed line denote averages).

transients past the Utopia point, at a specific point in time, will not be penalized, as in contrast to the steady-state MPC value function (7). The latter allows the full exploitation of non-steady process behaviour. Allowing process transients past the Utopia points results in less frequent switching, i.e., less control effort is utilized for greater average gains.

Table I tabulates the achieved performance for the respective ss-UT and dm-UT MPC strategies. The latter is compared with Economic MPC. As expected Economic MPC with optimal Pareto weighting outperforms dm-UT MPC, however at the off-line computational cost of finding the Pareto optimal weights along a constructed Pareto front. Also for an 1.5% profit increase, the latter is realized at the cost of increasing the control effort and waste product by 26.8% and 9.1% respectively. By implementing sub-optimal steady-state Utopia points, a ss-UT control strategy results in a 14.6% product decrease, 1.7% waste increase, and 29.8% netto profit drop with respect to the dm-UT MPC strategy for the numerical case study. The switching frequency of heat flux input also increases by over 13.2%, to minimize the soft constraint violation with respect to the steady-state Utopia point of the product P_1, (x_2), which is undesirable for control performance.

VI. Conclusion

In this work we have generalized the ss-UT MPC approach, proposed by [3], for cases which also include optimal process operation during non steady-state. The latter led to the concept of dynamic-mean Pareto optimality, which is equivalent to steady-state Pareto optimality, when steady-state process operation is optimal. Numerical simulations have verified that dm-UT MPC outperforms ss-UT MPC for cyclical process operation, and, retains almost as good performance as Economic MPC. The dm-UT MPC approach, in contrast to Economic MPC, avoids the off-line computational burden of constructing a Pareto front, and, the subsequent need to apply expert process operational insight for the selection of the best compromise point of operation.

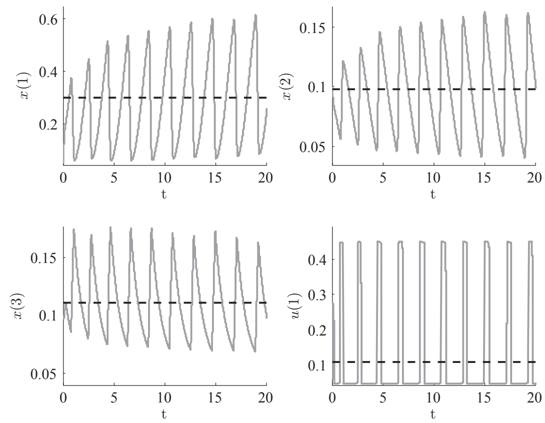

Fig. 5. Trajectories for the dynamic-mean Utopia Tracking MPC formulation with dynamic-mean Utopia set-point $\bar{\Phi}_d^0$ (dashed line denote averages).

TABLE I

Average Performance

	ss-UT MPC	dm-UT MPC	Economic MPC
Av$\{\Phi_1(x,u)\}$	0.083	0.098	0.102
Av$\{\Phi_2(x,u)\}$	(0.034)	(0.033)	(0.036)
Av$\{u\}$	0.142	0.125	0.159
Av{Profit}	0.050	0.065	0.066

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