

# Reactive Navigation of a Mobile Robot to the Moving Extremum of a Dynamic Unknown Environmental Field without Derivative Estimation

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**Abstract**— We consider a single kinematically controlled mobile robot traveling in a planar region supporting an unknown and unsteady field distribution. A single sensor provides the distribution value at the current robot location. We present a reactive navigation strategy that drives the robot to the time-varying location where the field distribution attains its spatial maximum and then keeps the robot in the pre-specified vicinity of the maximizer. The proposed control algorithm employs estimation of neither the entire field gradient nor derivative-dependent quantities, like the rate at which the available measurement evolves over time, and is non-demanding with respect to both computation and motion. Its mathematically rigorous analysis and justification are provided. Simulation results confirm the applicability and performance of the proposed guidance approach.

## I. INTRODUCTION

The paper addresses the problem of driving a single robot to the maximizer of an unknown scalar environmental field. This may be thermal, electric, or optical field, concentration of a chemical, physical, or biological agent, intensity of a spatial (electromagnetic, acoustic, etc.) signal, minus the distance to an unknown moving target, etc. Apart from source seeking/localizing [6], [22], [23], this problem is often referred to as gradient climbing/ascent [2], [4], [5], [21], which highlights the employed kinematic control paradigm: try to align the velocity vector of the robot with the field gradient.

For recent excellent surveys on extremum seeking control methods and algorithms, we refer the reader to [8], [14]. A good deal of related research is concerned with gradient climbing based on on-line estimation of the gradient. This approach is especially beneficial for mobile sensor networks thanks to exchange of field measurements in many locations [2], [4], [11], [20]–[22]. However even in this scenario, data exchange degradation due to e.g., communication constraints may require each robot in the team to operate autonomously during a considerable time and space. Similar algorithms can be basically used for a single robot equipped with several sensors that are distant enough from each other. In any case, multiple vehicle/sensor scenario involves bulky and costly hardware.

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The lack of multiple sensor data can be compensated via exploring multiple nearby locations by "dithering" the position of the single sensor during special maneuvers, which may be excited by probing high-frequency sinusoidal [5]–[7], [23] or stochastic [15] inputs. A similar in spirit approach is extremum seeking by means of many robots performing biased random walks [19] or by two robots with access to relative positions of each other and rotational actuation [9]. These methods rely, either implicitly or explicitly, on systematic side exploration maneuvers to collect rich enough data. Another approach limits the field gradient information to only the time-derivative of the measured field value obtained via e.g., numerical differentiation, [2], [3], [16], [18] and partly employs switching controllers [16], [18]. These give rise to concerns about amplification of the measurement noises and chattering, respectively. Adaptive extremum seeking approach [12] assumes that the field is known up to finitely many uncertain steady parameters, which may exceed the real level of knowledge in some applications.

The common feature of the previous research is that it dealt with only steady fields. However in real world, environmental fields are almost never steady and often cannot be well approximated by steady fields, whereas the theory of extremum seeking for dynamic fields lies in the uncharted territory. As a particular case, this topic includes the problem of navigation and guidance of a mobile robot towards an unknowingly maneuvering target based on a single measurement that decays as the sensor goes away from the target, like the strength of the infrared, acoustic, or electromagnetic signal, or minus the distance to the target. Such navigation is of interest in many areas [1], [10], [17]; it carries a potential to reduce the hardware complexity and cost and improve target pursuit reliability. A solution for such problem in the very special case of the unsteady field — minus the distance to an unknowingly moving Dubins-like target — was proposed and justified in [17]. However the results of [17] are not applicable to more general dynamic fields.

Contrary to the previous research, this paper addresses the source seeking problem for general dynamic fields. In this context, it justifies the new kinematic control paradigm that offers to keep the velocity orientation angle proportional to the discrepancy between the field value and a given linear ascending function of time, as opposed to conventionally trying to align the velocity vector with the gradient. This control law is free from evaluation of any field-derivative data, uses only finite gains instead of switching control, and demands only minor memory and computational robot's capacities, being reactive in its nature. Mathematically rigorous justification of

convergence and performance of this control law is offered in the case of general dynamic field. The applicability and performance of the control law are confirmed via extensive computer simulations.

An algorithm similar to ours can be found in [3], which however uses the estimated time-derivative of the measurement. Another difference is that in [3], only a steady harmonic field was examined, the performance during the transient and the behavior after reaching a vicinity of the maximizer were not addressed even for general harmonic fields, and the convergence conditions were partly implicit by giving no explicit bound on some entities that were assumed sufficiently large. The focus of this paper is on general dynamic fields, and we offer study of the entire maneuver with explicit conditions for convergence.

The proofs of the main results will be given in the full version of this paper and are available upon request.

## II. SYSTEM DESCRIPTION AND PROBLEM SETUP

We consider a planar point-mass robot traveling in a two-dimensional workspace. The robot is controlled by the time-varying linear velocity  $\vec{v}$  whose magnitude does not exceed a given constant  $v$ . The workspace hosts an unknown scalar time-varying field  $D(t, \mathbf{r})$ , where  $t$  is time,  $\mathbf{r} := (x, y)^\top$ , and  $x, y$  are the absolute Cartesian coordinates in the plane  $\mathbb{R}^2$ . The objective is to drive the robot to the point  $\mathbf{r}^0(t)$  where  $D(t, \mathbf{r})$  attains its maximum over  $\mathbf{r}$  and then to keep it in a vicinity of  $\mathbf{r}^0(t)$ , thus displaying the approximate location of  $\mathbf{r}^0(t)$ . The on-board control system has access only to the field value  $d(t) := D[t, \mathbf{r}(t)]$  at the robot current location  $\mathbf{r}(t) = [x(t), y(t)]^\top$ . No data about the derivatives of  $D$  are available; in particular, the robot is aware of neither the partial derivatives of  $D(\cdot)$  nor the time-derivative  $\dot{d}$  of the measurement  $d$ .

The kinematic model of the robot is as follows:

$$\dot{\mathbf{r}} = \vec{v}, \quad \mathbf{r}(0) = \mathbf{r}_{\text{in}} \quad \|\vec{v}\| \leq v, \quad (2.1)$$

where  $\|\cdot\|$  is the standard Euclidian norm. The problem is to design a controller that drives the robot into the desired vicinity  $V_*(t)$  of the time-varying maximizer  $\mathbf{r}^0(t)$  in a finite time  $t_0$  and then  $t \geq t_0$  keeps the robot within  $V_*(t)$ .

In this paper, we examine the following control algorithm:

$$\vec{v}(t) = v\vec{e}\left\{\mu[d(t) - v_*t] + \theta_0\right\}, \quad \text{where } \vec{e}(\theta) := \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad (2.2)$$

and  $v_*, \mu > 0, \theta_0 \in \mathbb{R}$  are tunable parameters.<sup>3</sup>

Throughout the paper, we employ the following notations and quantities characterizing the moving field  $D(\cdot)$ :

- $\langle \cdot; \cdot \rangle$  — the standard inner product in the plane;
- $\Phi_\beta = \begin{pmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{pmatrix}$  — the matrix of counter-clockwise rotation through angle  $\beta$ ;
- $\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$  — the spatial gradient;
- $D''$  — the spatial Hessian, i.e., the matrix of the second derivatives with respect to  $x$  and  $y$ ;

<sup>3</sup>Implementation of this algorithm requires access to an absolute direction via, e.g., a compass.

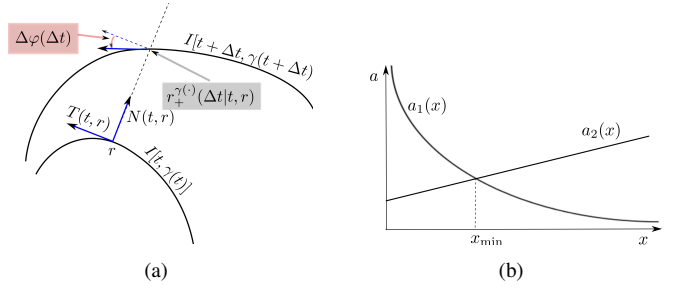


Fig. 1. (a) Two close isolines; (b) The graphs of  $a_i(x)$

- $I(t, \gamma) := \{\mathbf{r} : D(t, \mathbf{r}) = \gamma\}$  — the spatial isoline, i.e., the level curve of  $D(t, \cdot)$  with the field level  $\gamma$ ;
- $[T, N] = [T(t, \mathbf{r}), N(t, \mathbf{r})]$  — the Frenet frame of the spatial isoline  $I[t, \gamma]$  with  $\gamma := D(t, \mathbf{r})$  at the point  $\mathbf{r}$ :

$$N(t, \mathbf{r}) = \frac{\nabla D(t, \mathbf{r})}{\|\nabla D(t, \mathbf{r})\|}$$

and the unit tangent vector  $T(t, \mathbf{r})$  is oriented so that when traveling on  $I[t, \gamma]$  one has the domain of greater values  $\{\mathbf{r}' : D(t, \mathbf{r}') > \gamma\}$  to the left;

- $\varkappa$  — the signed curvature of the isoline;
- $\mathbf{r}_+^{\gamma(\cdot)}(\Delta t | t, \mathbf{r})$  — the nearest (to  $\mathbf{r}$ ) point of intersection between the ordinate axis of the Frenet frame  $[T(t, \mathbf{r}), N(t, \mathbf{r})]$  and the displaced isoline  $I[t + \Delta t, \gamma(t + \Delta t)]$ , where the given smooth function  $\gamma(\cdot)$  is such that  $\gamma(t) = D(t, \mathbf{r})$ ; see Fig. 1(a); if  $\gamma(\cdot) \equiv \text{const} (= D(t, \mathbf{r}))$ , the upper index  $\gamma(\cdot)$  is dropped;
- $\lambda^{\gamma(\cdot)}(t, \mathbf{r})$  — the front velocity of the spatial isoline:

$$\lambda^{\gamma(\cdot)} = \lambda^{\gamma(\cdot)}(t, \mathbf{r}) := \lim_{\Delta t \rightarrow 0} \frac{p^{\gamma(\cdot)}(\Delta t | t, \mathbf{r})}{\Delta t},$$

where  $p^{\gamma(\cdot)}(\Delta t | t, \mathbf{r})$  is the ordinate of  $\mathbf{r}_+^{\gamma(\cdot)}(\Delta t | t, \mathbf{r})$ ; if  $\gamma(\cdot) \equiv D(t, \mathbf{r})$ , the index  $\gamma(\cdot)$  is dropped;

- $\alpha(t, \mathbf{r})$  — the front acceleration of the spatial isoline  $I[t, \gamma], \gamma := D(t, \mathbf{r})$  at time  $t$  at the point  $\mathbf{r}$ :

$$\alpha = \alpha(t, \mathbf{r}) := \lim_{\Delta t \rightarrow 0} \frac{\lambda[t + \Delta t, \mathbf{r}_+(\Delta t)] - \lambda[t, \mathbf{r}]}{\Delta t},$$

$$\mathbf{r}_+(\Delta t) := \mathbf{r}_+(\Delta t | t, \mathbf{r});$$

- $\omega(t, \mathbf{r})$  — the angular velocity of rotation of the spatial isoline  $I[t, \gamma], \gamma := D(t, \mathbf{r})$  at time  $t$  at the point  $\mathbf{r}$ :

$$\omega = \omega(t, \mathbf{r}) := \lim_{\Delta t \rightarrow 0} \frac{\Delta\varphi(\Delta t)}{\Delta t},$$

where  $\Delta\varphi(\Delta t)$  is the angular displacement of  $T[t + \Delta t, \mathbf{r}_+(\Delta t)]$  with respect to  $T[t, \mathbf{r}]$ ; see Fig. 1(a);

- $\rho(t, \mathbf{r})$  — the density of isolines at time  $t$  at point  $\mathbf{r}$ :

$$\rho = \rho(t, \mathbf{r}) := \lim_{\Delta\gamma \rightarrow 0} \frac{\Delta\gamma}{q(\Delta\gamma | t, \mathbf{r})},$$

where  $q(\Delta\gamma | t, \mathbf{r})$  is the ordinate of the nearest (to  $\mathbf{r}$ ) point of intersection between the ordinate axis of the Frenet frame  $[T(t, \mathbf{r}), N(t, \mathbf{r})]$  and the close isoline  $I(t | t, \gamma + \Delta\gamma), \gamma := D(t, \mathbf{r})$ ; <sup>4</sup>

<sup>4</sup>This density characterizes the "number" of isolines within the unit distance from the basic one  $I(t, d_0)$ , where the "number" is evaluated by the discrepancy in the values of  $D(\cdot)$  observed on these isolines.

- $v_\rho(t, \mathbf{r})$  — the evolutionary (proportional) growth rate of the above density at time  $t$  at point  $\mathbf{r}$ :

$$v_\rho(t, \mathbf{r}) := \frac{1}{\rho(t, \mathbf{r})} \lim_{\Delta t \rightarrow 0} \frac{\rho[t + \Delta t, \mathbf{r}_+(\Delta t)] - \rho(t, \mathbf{r})}{\Delta t}$$

- $\tau_\rho(t, \mathbf{r})$  — the tangential (proportional) growth rate of the isolines density at time  $t$  at point  $\mathbf{r}$ :

$$\tau_\rho(t, \mathbf{r}) := \frac{1}{\rho(t, \mathbf{r})} \lim_{\Delta s \rightarrow 0} \frac{\ln \rho(t, \mathbf{r} + T\Delta s) - \ln \rho(t, \mathbf{r})}{\Delta s}$$

- $n_\rho(t, \mathbf{r})$  — the normal (proportional) growth rate of the isolines density at time  $t$  at point  $\mathbf{r}$ :

$$n_\rho(t, \mathbf{r}) := \frac{1}{\rho(t, \mathbf{r})} \lim_{\Delta s \rightarrow 0} \frac{\ln \rho(t, \mathbf{r} + N\Delta s) - \ln \rho(t, \mathbf{r})}{\Delta s}$$

- $\omega_\nabla(t, \mathbf{r})$  — the angular velocity of the gradient  $\nabla D$  rotation over time at point  $\mathbf{r}$  at time  $t$ .

### III. BASIC ASSUMPTIONS

In general setting, the problem at hand comprises problems of global optimization. In the presence of local extrema, NP-hardness was established for even the simplest classes of such problems [13]. However it is not realistic enough to assume complete absence of local extrema. Field perturbations are very likely to give rise to local extrema in the regions where the basic field is flat enough. By the Fermat theorem, one such region surrounds the maximizer  $\mathbf{r}^0(t)$ . For most physical fields, another region lies far enough from  $\mathbf{r}^0(t)$ , where the energy of the basic field becomes distributed over large areas.

In view of this, we adopt a compromising assumption. Local extrema are allowed but only in a "vicinity"  $Z_{\text{near}}$  of the maximizer and at the outskirts  $Z_{\text{far}}$ , where the field is not assumed even smooth. In the intermediary  $Z_{\text{reg}}$ , the field is smooth and has no critical points and thus local extrema. To make the problem tractable,  $Z_{\text{near}}$  is assumed to lie within the desired vicinity  $V_\star$  of the maximizer, whereas the initial location is in  $Z_{\text{reg}}$ . The controller should keep the robot in  $Z_{\text{reg}}$  until the control objective fulfillment, thus avoiding detrimental effects of local extrema. To reduce the amount of technicalities, we also assume that the zones  $Z_{\text{near}}$ ,  $Z_{\text{far}}$ ,  $Z_{\text{reg}}$ , and  $V_\star$  are separated by isolines. This requirement can usually be met by properly reducing  $Z_{\text{reg}}$  and  $V_\star$ , if necessary. To encompass theoretical distributions like  $D(\mathbf{r}) = c/\|\mathbf{r} - \mathbf{r}_\star(t)\|$  or  $D(\mathbf{r}) = -c \ln \|\mathbf{r} - \mathbf{r}_\star(t)\|$  and their sums, the field is allowed to be undefined at finitely many moving exceptional points. Summarizing, we arrive at the following.

*Assumption 3.1:* There exist smooth  $\gamma_-(t) < \gamma_\star(t) < \gamma_+(t) \in \mathbb{R}$  and continuous  $\mathbf{r}_1(t), \dots, \mathbf{r}_k(t) \in \mathbb{R}^2$  functions of time  $t$  such that the following statements hold:

- On  $Z_{\text{reg}} := \{(t, \mathbf{r}) : \gamma_-(t) \leq D(t, \mathbf{r}) \leq \gamma_+(t)\}$ , the distribution  $D(\cdot)$  is identical to a  $C^2$ -smooth function defined on a larger and open set, and  $\nabla D(t, \mathbf{r}) \neq 0$ ;
- At any time  $t$ , the spatial isoline  $I[t, \gamma_-(t)]$  is a Jordan curve that encircles  $I[t, \gamma_+(t)]$ ;
- At any time  $t$ , the points  $\mathbf{r}_j(t)$  lie inside  $I[t, \gamma_+(t)]$ ;
- In  $Z_{\text{near}} := \{(t, \mathbf{r}) : \mathbf{r} \text{ is inside } I[t, \gamma_+(t)] \text{ and } \mathbf{r} \neq \mathbf{r}_j(t) \forall j\}$ , the field  $D(\cdot)$  takes values greater than  $\gamma_+(t)$ ,

converges to a finite or infinite limit  $L_j(t)$  as  $\mathbf{r} \rightarrow \mathbf{r}_j(t)$  for any  $t$  and  $j$ , and is continuous in both  $Z_{\text{near}}$  and its outer boundary  $\{(t, \mathbf{r}) : \mathbf{r} \in I[t, \gamma_+(t)]\}$ ;

- In the domain  $Z_{\text{far}} := \{(t, \mathbf{r}) : \mathbf{r} \text{ is outside } I[t, \gamma_-(t)]\}$ , the field  $D(\cdot)$  is everywhere defined, takes values lesser than  $\gamma_-(t)$ , and is continuous in both  $Z_{\text{far}}$  and its boundary  $\{(t, \mathbf{r}) : \mathbf{r} \in I[t, \gamma_-(t)]\}$ ;

- The desired vicinity of the maximizer has the form  $V_\star(t) = \{\mathbf{r} : D[t, \mathbf{r}(t)] \geq \gamma_\star(t)\}$ ;

- For  $t = 0$ , the initial location  $\mathbf{r}_{\text{in}}$  lies in the domain of  $D(\cdot)$  and inside  $I[0, \gamma_-(0)]$  (i.e.,  $D[0, \mathbf{r}_{\text{in}}] > \gamma_-(0)$ ).

Since the maneuver is started at  $t = 0$ , we consider only  $t \geq 0$  here and throughout. It is feasible that  $k = 0$  and so the points  $\mathbf{r}_j(t)$  and concerned claims are everywhere omitted. Due to Assumption 3.1,  $\sup_{\mathbf{r} \in \mathbb{R}^2} D(t, \mathbf{r})$  (whether finite or infinite) is attained and all maximizers lie in  $V_\star(t)$ .

*Remark 3.1:* Assumption 3.1 holds if  $D(\cdot)$  is defined and  $C^2$ -smooth on the entire three-dimensional space of  $(t, \mathbf{r})$ , and there exist continuous functions  $\mathbf{r}^0(t)$  and  $\gamma_{\text{inf}}(t) < D[t, \mathbf{r}^0(t)]$  such that  $\nabla D(t, \mathbf{r}) \neq 0 \forall \mathbf{r} \neq \mathbf{r}^0(t)$  and  $D(t, \mathbf{r}) \rightarrow \gamma_{\text{inf}}(t)$  as  $\|\mathbf{r}\| \rightarrow \infty$ . Let the objective be to drive the robot to the  $R_\star$ -neighborhood  $\{\mathbf{r} : \|\mathbf{r} - \mathbf{r}^0(t)\| \leq R_\star\}$  of the maximizer. In this case,  $\gamma_\star(t)$  is chosen so that the desired vicinity  $V_\star(t)$  is inside this neighborhood  $\max_{\mathbf{r} : \|\mathbf{r} - \mathbf{r}^0(t)\| = R_\star} D[t, \mathbf{r}] < \gamma_\star(t) < D[t, \mathbf{r}^0(t)]$ . The functions  $\gamma_-(t), \gamma_+(t)$  can be arbitrarily chosen subject to  $\gamma_{\text{inf}}(t) < \gamma_-(t) < \gamma_\star(t) < \gamma_+(t) < D[t, \mathbf{r}^0(t)]$ .

The next assumption is typically fulfilled in real world, where physical quantities take bounded values.

*Assumption 3.2:* There exists constants  $b_\omega^\nabla, b_\varkappa$  for  $\varkappa = \rho, \lambda, \omega, \varkappa, v, \alpha, n, \tau$  and  $\gamma_+^0, \bar{\gamma}$  such that

$$\begin{aligned} |\rho| &\leq b_\rho, |\lambda| \leq b_\lambda, |\omega| \leq b_\omega, |\omega_\nabla| \leq b_\omega^\nabla, |\varkappa| \leq b_\varkappa, \\ |v_\rho| &\leq b_v, |\alpha| \leq b_\alpha, |\tau_\rho| \leq b_\tau, |n_\rho| \leq b_n \quad \forall (t, \mathbf{r}) \in Z_{\text{reg}} \\ \gamma_+(t) &\leq \gamma_+^0, |\dot{\gamma}_i(t)| \leq \bar{\gamma} \quad \forall t, i = \pm, \star. \end{aligned}$$

The only assumption about the robot capacity with respect to the field is that the mobility of the former exceeds that of the latter: everywhere in the main operational zone  $Z_{\text{reg}}$ , the speed of the robot is greater than the front speed of the concerned isoline  $v > |\lambda(t, \mathbf{r})| \forall (t, \mathbf{r}) \in Z_{\text{reg}}$ . Moreover, if the level  $\gamma_-(\cdot)$  or  $\gamma_\star(\cdot)$  is not constant, the robot is capable to remain inside the moving isoline  $I[t, \gamma_-(t)]$  and inside  $I[t, \gamma_\star(t)]$ , i.e.,  $v > |\lambda + \rho^{-1}\dot{\gamma}_i|$  for  $i = -, \star$ . Though these inequalities should hold on the respective isolines, we extend them on the entire  $Z_{\text{reg}}$  and enhance by putting  $|\lambda| + \rho^{-1}\bar{\gamma}$  in place of  $|\lambda + \rho^{-1}\dot{\gamma}_i|$  to simplify the matters and formulas.

All afore-mentioned strict inequalities are protected from degradation as time progresses. To flesh out this, we use the notation  $f \oslash g$  in  $Z$  to express that  $\exists \varepsilon > 0 : f(t, \mathbf{r}) \geq g(t, \mathbf{r}) + \varepsilon \forall (t, \mathbf{r}) \in Z$ ; the relation  $\oslash$  is defined likewise.

*Assumption 3.3:* The following inequalities hold:

$$\begin{aligned} \rho(t, \mathbf{r}) &\oslash 0, \quad v \oslash |\lambda(t, \mathbf{r})| + \rho^{-1}\bar{\gamma}, \quad \text{in } Z_{\text{reg}}; \\ \gamma_-(t) &\oslash \gamma_\star(t) \oslash \gamma_+(t) \quad \text{in } [0, \infty). \end{aligned}$$

If the isoline levels are constant and so  $\bar{\gamma} = 0$ , the second relation takes the form  $v \oslash |\lambda(t, \mathbf{r})|$  in  $Z_{\text{reg}}$ .

#### IV. MAIN RESULTS

The first theorem shows that the control objective can always be achieved by means of the control law (2.2).

*Theorem 4.1:* Suppose that Assumptions 3.1—3.3 hold. Then there exist parameters  $v_*, \mu, \theta_0$  of the controller (2.2) such that the following claim holds.

- (i) The controller (2.2) brings the robot to the desired vicinity of a maximizer in a finite time  $t_0$  and keeps it there afterwards:  $\mathbf{r}(t) \in V_*(t)$  for  $t \geq t_0$ .

Moreover, for any compact domain  $D \subset \mathbf{int}\{\mathbf{r} : (0, \mathbf{r}) \in Z_{\text{reg}}\}$ , there exist common values of the parameters for which (i) holds whenever the initial location  $\mathbf{r}_{\text{in}} \in D$ .

The remainder of the section is devoted to discussion of the controller parameters tuning. The parameters  $\theta_0, v_*$  in (2.2) are chosen prior to  $\mu$ . Whereas  $\theta_0$  is arbitrary,  $v_*$  is chosen so that

$$v \oslash |\lambda| + \rho^{-1}v_*, \quad \text{in } Z_{\text{reg}}, \quad v_* > \bar{\gamma}, \quad (4.1)$$

which is possible thanks to (3.3). The choice of  $\mu$  is prefaced by picking constant  $\Delta_{\nabla} > 0, \Delta_{\gamma} > 0$  such that

$$\begin{aligned} \rho(t, \mathbf{r}) &\geq \Delta_{\nabla} \quad \forall (t, \mathbf{r}) \in Z_{\text{reg}} \\ \gamma_-(t) + \Delta_{\gamma} &\leq \gamma_*(t) \leq \gamma_+(t) - \Delta_{\gamma}, \\ D[0, \mathbf{r}_{\text{in}}] &\geq \gamma_-(0) + \Delta_{\gamma}/2 \end{aligned} \quad (4.2)$$

which is possible by the same argument. The parameter  $\mu$  is chosen so that for some  $k = 1, 2, \dots$ ,

$$\mu(v_* - \bar{\gamma}) \oslash 2\omega + \varkappa v_T - 2\frac{\tau_{\rho}\bar{\gamma}}{\rho} - 2\frac{v_{\rho}\bar{\gamma}}{v_T} + \frac{\alpha}{v_T} - \frac{n_{\rho}\bar{\gamma}^2}{v_T\rho^2} \quad (4.3)$$

where  $v_T := \pm\sqrt{v^2 - [\lambda + \rho^{-1}\bar{\gamma}]^2}$ , in  $Z_{\text{reg}}$ ;

$$\mu \oslash \frac{\omega + \tau_{\rho}(\lambda - v)}{\rho(v - \lambda) - v_*} \quad \text{in } Z_{\text{reg}}; \quad (4.4)$$

$$\mu v_* > a_1(k) := 2b_{\omega}^{\nabla} \left[1 + \frac{1}{k}\right] + 2v\sqrt{b_{\varkappa}^2 + b_{\tau}^2} \left[2 + \frac{1}{k}\right], \quad (4.5)$$

$$\begin{aligned} \mu v_* > a_2(k) := \\ 2b_{\rho} \frac{(2k+1)v + 2\pi(k+1)(b_{\lambda} + \Delta_{\nabla}^{-1}\bar{\gamma})}{\Delta_{\gamma}}. \end{aligned} \quad (4.6)$$

Relations (4.5) are evidently true for some natural  $k$  if and only if  $\mu v_* > \min_{k=1,2,\dots} \max\{a_1(k), a_2(k)\}$ . Here  $a_1(x)$  and  $a_2(x)$  are respectively, descending and ascending functions of  $x > 0$ , see Fig. 1(b). So the minimum is attained at the integer either ceiling  $\lceil x_{\text{min}} \rceil$  or floor  $\lfloor x_{\text{min}} \rfloor$  of the positive root  $x_{\text{min}}$  of the quadratic equation  $a_1(x) = a_2(x)$ .

Choice of  $v_*, \mu$  satisfying (4.1), (4.3), (4.4) is typically based on upper estimation of the respective right-hand sides.

*Theorem 4.2:* Suppose that Assumptions 3.1—3.3 hold and the controller parameters satisfy (4.1) and (4.3)—(4.5). Then (i) from Theorem 4.1 is true.

#### V. ILLUSTRATIVE EXAMPLES

Now we illustrate the discussion of the previous section in simple yet instructive cases.

#### A. Linear Field

Since any smooth field is well approximated by a linear one in sufficiently small (and sometimes not so small) areas, the focus on linear fields permits us to disclose some basic behavioral primitives that underly, more or less, the closed-loop behavior in general fields. We show that for properly tuned controller parameters, these primitives are monotonic, not oscillatory despite the fact that (2.2) contains expressions  $\cos[\mu(d - v_*t) + \theta_*]$  and  $\sin[\mu(d - v_*t) + \theta_*]$ . This study also illuminates the effects of various controller parameters, thus elucidating the recommendations on their choice.

In this section, we examine the behavior of the system (2.1) driven by the controller (2.2) in the linear scalar field

$$D(x, y) = n(x \cos \varphi + y \sin \varphi) + D_0,$$

where  $n > 0, \varphi, D_0 \in \mathbb{R}$  are given. We observe that the closed-loop system (2.1), (2.2) is described by autonomous ordinary differential equations (ode) with respect to  $\mathbf{r}, \theta$

$$\dot{\mathbf{r}} = v\vec{e}(\theta), \quad \dot{\theta} = \mu[\dot{d} - v_*], \quad \dot{d} = v \langle \nabla D(\mathbf{r}); \vec{e}(\theta) \rangle \quad (5.1)$$

and address the phase portrait of this system of ode.

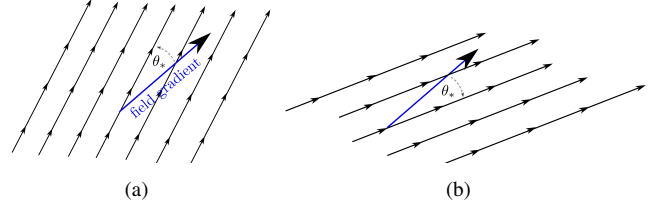


Fig. 2. (a) Stable and (b) unstable series of motions.

*Lemma 5.1:* For  $\mu, v_* > 0$ , the following statements hold:

- i) Let  $v_*/v < n$  and  $\theta_* := \arccos \frac{v_*}{vn}$ . The system of ode (5.1) has two subsets of solutions, each constituted by motions along parallel straight lines with climbing the gradient at the constant rate  $v_*$  (see Fig. 2):

$$\mathbf{r}(t) = v\vec{e}(\pm\theta_* + \varphi)t + \mathbf{r}_{\text{in}}, \quad \theta \equiv \pm\theta_* + \varphi, \quad \dot{d} \equiv v_*.$$

The solutions with  $-$  taken in  $\pm$  are locally exponentially unstable. Any solution outside the “ $-$ ” subset exponentially converges to a solution from the “ $+$ ” set. In doing so,  $\dot{d}(t) \rightarrow v_*$  as  $t \rightarrow \infty$  and the robot does not oscillate or perform full turns: its orientation angle  $\theta$  monotonically evolves over an interval whose length  $< 2\pi$  (see Fig. 3(a)).

- ii) If  $v_*/v = n$  and so  $\theta_* = 0$ , the two subsets coincide, which may be viewed as if the  $+$  subset survives but the  $-$  subset becomes empty. Modulo this, claim i) remains true, and the straight motions from the  $+$  subset are in exactly the gradient direction.
- iii) If  $v_*/v > n$ , any solution is a periodic motion subjected to the drift in the gradient direction  $\vec{n}_0 := \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}$  at a constant rate. Moreover, there exist  $\tau > 0, \eta > 0$  and  $\tau$ -periodic functions  $\theta_{\oslash}(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$  and  $\mathbf{r}_{\oslash}(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^2$

such that the solutions of (5.1) have the form

$$\begin{aligned} \mathbf{r}(t) &= \mathbf{r}_\cup(t - t_0) + \mathbf{r}_* + (t - t_0)\eta\vec{n}_0, \\ \theta(t) &= \theta_\cup(t - t_0) - \frac{2\pi}{\tau}(t - t_0), \end{aligned}$$

where the constants  $t_0, \mathbf{r}_*$  are determined by the initial data. For any solution, the orientation vector  $\vec{e}(\theta)$  constantly rotates clockwise:  $\dot{\theta} < 0$ ; see Fig. 3(b).

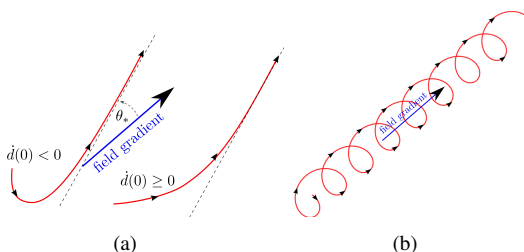


Fig. 3. Typical closed-loop behavior (a)  $v_*/v \leq n$  (b)  $v_*/v > n$ .

Though the pattern from iii) ensures overall gradient climbing, it involves systematic side and backward maneuvers and so is ineffective. Scenario ii) is most attractive but requires the exact knowledge of the gradient length  $n$  to tune the controller parameters  $v_*/v = n$ , which is unrealistic for unknown fields. Scenario i) combines a rather effective pattern of gradient climbing with a potential for real-world implementation since this requires knowledge of only a lower estimate  $\underline{n} \leq n$  of the gradient  $v_*/v < \underline{n}$ . This scenario holds if  $v_*/v < n = \|\nabla D\| = \rho \Leftrightarrow v > \rho^{-1}v_*$ , which is identical to the first requirement in (4.1) (where now  $\lambda = 0$  since the field is steady). For unsteady fields, the addend  $|\lambda|$  on the right accounts for the field motion:  $v$  is replaced by the relative speed  $v \pm \lambda$  in the above condition  $v > \rho^{-1}v_*$ .

### B. Radial Field with a Moving Source

We consider a scalar field of the form  $D(t, \mathbf{r}) = cf(\|\mathbf{r} - \mathbf{r}^0(t)\|)$ , where  $c > 0$  and  $\mathbf{r}^0(\cdot)$  are unknown, whereas the twice continuously differentiable function  $f : (0, \infty) \rightarrow \mathbb{R}$  is known, strictly decaying and convex  $f'(z) < 0, f''(z) \geq 0 \forall z > 0$ . An example of such field is minus the distance to a moving target  $D(t, \mathbf{r}) = -\|\mathbf{r} - \mathbf{r}^0(t)\|$ , in which case  $c = 1, f(z) = -z$  and the target  $\mathbf{r}^0(t)$  can be viewed as the "source" of the field and is its maximizer.

The objective is to display the unknown location  $\mathbf{r}^0(t)$  of the source by bringing the robot to its  $R_*$ -neighborhood  $\{\mathbf{r} : \|\mathbf{r} - \mathbf{r}^0(t)\| \leq R_*\}$  on the basis of the following known estimates

$$\begin{aligned} \|\mathbf{r}_{\text{in}} - \mathbf{r}^0(0)\| &\leq R_{\text{in}}, \quad \|\dot{\mathbf{r}}^0(t)\| \leq v_0, \\ \|\ddot{\mathbf{r}}^0(t)\| &\leq a_0, \quad \forall t \geq 0, \quad 0 < c_- \leq c \leq c_+. \end{aligned} \quad (5.2)$$

We assume that  $R_{\text{in}} > R_*$  to simplify the formulas. An elementary analysis of assumptions and requirements from Theorem 4.2 implies the following.

*Proposition 5.1:* Suppose that the robot is faster than the field source  $v > v_0$ . Then there exist parameters  $v_*, \mu, \theta_0$  such that whenever (5.2) holds, the controller (2.2) brings the

robot to the desired  $R_*$ -neighborhood of the source  $\mathbf{r}^0(t)$  in a finite time  $t_0$  and keeps it there afterwards  $\|\mathbf{r}(t) - \mathbf{r}^0(t)\| \leq R_* \forall t \geq t_0$ . Specifically, this claim is true if the parameters satisfy the following relations with some  $R_- > R_{\text{in}}, 0 < R_+ < R_*$

$$\begin{aligned} 0 < v_* &< c_-|f'(R_-)|(v - v_0), \\ \mu v_* &> \max_{R \in [R_-, R_+]} \left[ \frac{4 \frac{f''(R)}{|f'(R)|} v_0^2 + a_0}{\sqrt{v^2 - v_0^2}} + \frac{v + 2v_0}{R} \right], \\ \mu &> \frac{1}{R_+} \frac{v_0}{c_-|f'(R_-)|(v - v_0) - v_*}, \\ \mu v_* &> \frac{4v_0 + 6v}{R_+}, \\ \mu v_* &> 2 \frac{c_+}{c_-} |f'(R_+)| \frac{3v + 4\pi v_0}{a}. \\ a &:= \min \left\{ f(R_*) - f(R_-); f(R_+) - f(R_*) \right. \\ &\quad \left. 2[f(R_{\text{in}}) - f(R_-)] \right\} \end{aligned}$$

## VI. SIMULATION TESTS

Simulations were carried out with the point-mass robot (2.1) driven by the control law (2.2). The numerical values of the parameters used for simulations are shown in Table I (where  $u_d$  is the unit of measurement of  $d = D(\mathbf{r})$ ). The control was updated with a sampling time of 0.1s.

$v$	1m/s	$\mu$	0.8rad/ $u_d$
$v_*$	0.299 $u_d$ /s	$\theta_0$	1.5rad

TABLE I

NUMERICAL VALUES OF THE PARAMETERS USED FOR SIMULATION.

Fig. 4 displays the results of tests in a linear field with the orientation angle of 0.5 rad and the ascension rate of  $0.3m^{-1}$ . The steady state angular error may be computed on the basis of Lemma 5.1 and is  $\approx 0.082rad$ . Fig. 4 shows that the heading converges to approximately  $0.582rad$ , as is expected.

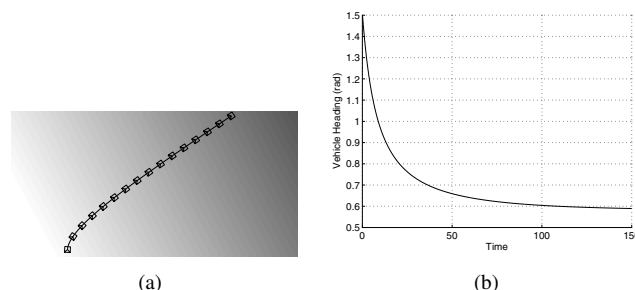


Fig. 4. Behavior in a linear field: (a) Path (b) Robot's orientation

Fig. 5 presents the results of tests in an unsteady field with a moving source. The source  $\mathbf{r}^0(t)$  moves to the right at the speed of  $0.3ms^{-1}$ , the field  $D(t, \mathbf{r})$  is given by

$$D(t, \mathbf{r}) = -0.8 \cdot \|\mathbf{r} - \mathbf{r}^0(t)\| + 5 \cdot [\sin(0.05 \cdot x) + \sin(0.05 \cdot y)].$$

As can be seen, the robot converges to the source (whose path is depicted by the solid black line) and then wheels around it in a close proximity, thus displaying its current location and highlighting its displacement. "Wheeling" commences when the robot achieves the desired vicinity of the source and is unavoidable since the robot's speed exceeds that of the source.

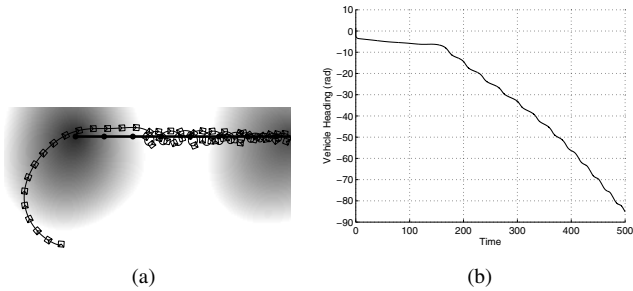


Fig. 5. Seeking a moving source: (a) Path (b) Robot's orientation

In Fig. 6, the same simulation setup was used, except measurement noise and kinematic constraints were added. The robot's heading was not allowed to change faster than  $0.5 \text{ rads}^{-1}$ , and the field readings were corrupted by a random additive noise uniformly distributed over the interval  $[-2.5, 2.5]$ . It may be seen that nearly the same behavior is observed.

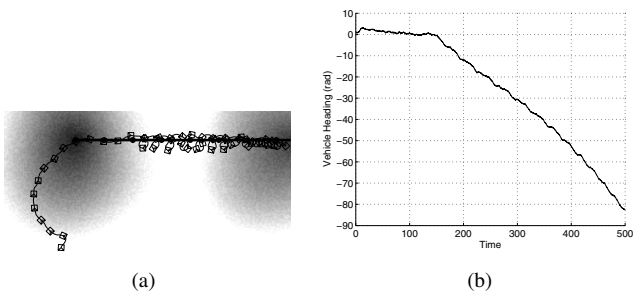


Fig. 6. Seeking a moving source under measurement noise and kinematic constraints: (a) Path (b) Robot's orientation

## VII. SUMMARY

This paper addresses a method for environmental extremum seeking by kinematically controlled mobile robots. The method is suitable for unknown dynamic field distributions, and only requires a single sensor for the current distribution value with no gradient or time derivative calculations being required. Mathematically rigorous analysis and justification of convergence to the field maximum and performance are provided. Simulation results confirm the applicability and performance of the proposed guidance approach. Future work will investigate experimental applications of the proposed control system for realistic fields.

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