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Robust Nonlinear Estimation of Varying Optical Phase

Obaid Ur Rehman, Ian R. Petersen, Hongbin Song and Elanor H. Huntington

Abstract— **This work presents a robust approach to estimate the continuously varying phase of an optical system. Here we extend the adaptive homodyne estimator scheme by using a nonlinear guaranteed cost robust filter in the feedback loop to improve the estimator performance. The nonlinear robust filter is designed by using a copy of the sector bounded nonlinear uncertainty present in the measurement of the adaptive homodyne estimator. Finally, closed loop simulations considering both linear and nonlinear uncertainty are performed and the results are compared with the standard Kalman filter results. Simulation results show a significant improvement using the nonlinear filter which provides a better choice for the time varying phase estimation problem.**

I. INTRODUCTION

The phase estimation problem has attracted considerable research interest since it plays important role in quantum computing, quantum communications, and high-precision measurement [1], [2], [3]. As phase cannot be directly measured, its measurement has to rely on the measurement of other quantities. Consequently, in addition to the existence of intrinsic uncertainty governed by the uncertainty principle in quantum mechanics, excess uncertainty may be induced. Therefore, one of the key issues in phase estimation is to reduce the introduced uncertainty.

Heterodyne measurement is a standard approach to approximately estimate a completely unknown phase by using simultaneously measured orthogonal quadratures. In contrast to heterodyne phase measurement, homodyne measurement offers a relative high sensitivity. However, the initial phase information has to be known since homodyne measurement is only able to estimate a constrained phase [4]. To keep the flexibility of heterodyne phase measurement and the increased sensitivity of homodyne measurement, the adaptive dyne technique was theoretically and experimentally proposed, in which a feedback loop was used to adjust the local oscillator phase in a homodyne measurement [1], [4], [5], [6], [7]. The estimated phase is assumed to be sufficiently close to the actual system phase so that the sinusoidal relationship between the measurement and phase is approximately linear. This assumption would be true if the phase estimation error is very small. However, as the error increases, the linear relation will no longer be valid.

In this paper, we consider a measurement which is a nonlinear function of the phase and propose an improved estimator which will be valid for an increased range of estimation error. This approach extends the adaptive homodyne design technique using a nonlinear robust estimator rather than a standard linear Kalman filter in the feedback loop (see Fig. 1). The robust nonlinear estimator will be robust to the initial phase error and also provides robustness to the uncertainties in the underlying parameters of the model. However, in this paper only nonlinear uncertainty in the measurement has been taken into account. The filter proposed here also provides a guaranteed bound on the associated cost function.

O. Rehman I. R. Petersen, H. Song and E. H. Huntington are with school of Engineering and Information Technology, University of New South Wales, Canberra, Australia. Emails are s.obaid.rehman@gmail.com, i.r.petersen@gmail.com, h.song@adfa.edu.au and e.huntington@adfa.edu.au.

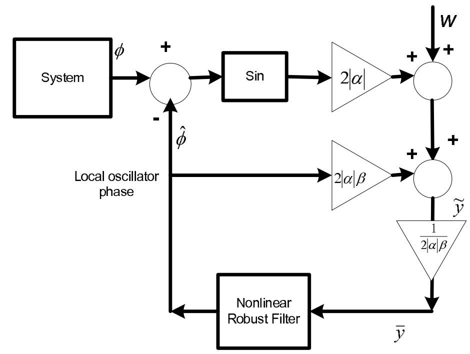

Fig. 1: Adaptive homodyne system with robust nonlinear filter.

The paper is organized as follows. Section II presents a description of a nonlinear optical system considered for the phase estimation problem. Derivation of uncertainty modeling for the given system is presented in Section III. A brief introduction to the procedure of robust nonlinear filter/estimator design is given in Section IV. The relevant phase estimation problem has been solved in Section V. Simulation results and comparison with the standard Kalman filter are presented in Section VI and the paper is concluded in Section VII.

II. SYSTEM DEFINITION

Let us consider a coherent optical beam with a varying phase $\phi(t)$ and having a constant amplitude of $|\alpha|$. The magnitude is scaled so that $|\alpha|^2$ is the photon flux. The governing equation for the this system can be written in the form of stochastic differential equation as follows:

$$d\phi(t) = -\lambda\phi(t)dt + \sqrt{\kappa}dV(t). \quad (1)$$

The process model for the above system is given by the equation below:

$$\dot{\phi}(t) = -\lambda\phi(t)dt + \sqrt{\kappa}v(t), \quad (2)$$

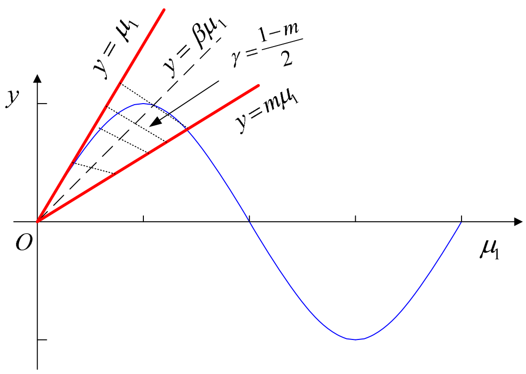

Fig. 2: Sector bound.

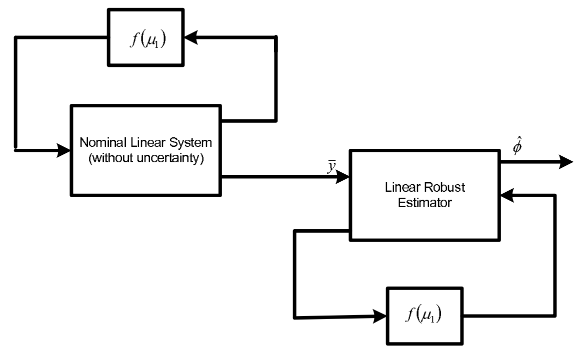

Fig. 3: Nonlinear system with nonlinear state estimator.

where $v(t) := \frac{dV}{dt}$ is a unit variance white noise process. Here $dV(t)$ is Wiener increment satisfying $(dV)^2 = dt$. The photon current for a continuous coherent beam is given by

$$I(t)dt = 2Re(\alpha e^{-i\Phi(t)})dt + dW(t), \tag{3}$$

where $\Phi(t) = \hat{\phi} + \pi/2$ is the phase of the local oscillator, and $dW(t)$ is Wiener increment independent of $dV(t)$. The homodyne photocurrent is given by the following relation:

$$I(t)dt = 2|\alpha| \sin[\phi(t) - \hat{\phi}(t)]dt + dW(t). \tag{4}$$

III. Uncertainty Modelling

The measurement equation (4) of the optical system is a nonlinear equation. However, we can approximate the model (1), (4) with a linear system having a sector bounded nonlinear uncertainty. Let us define $\mu_1(t) := \phi(t) - u(t)$ where $u(t) = \hat{\phi(t)}$ is controller/estimator output and $y(t) = f(\mu_1(t))$. Hence, we can write (4) using the following relation (see Fig. 2), we remove argument t from some of the variables:

$$I(t)dt = 2\alpha(f(\mu_1) + \beta\mu_1)dt + dW(t), \tag{5}$$

where β is the slope of the tangent on the curve at $\mu_1 = 0$. Also,

$$\begin{aligned} I(t)dt &= [2\alpha(f(\mu_1) + 2\alpha\beta(\phi(t) - u(t))]dt + dW(t) \\ I(t) + 2\alpha\beta u(t) &= 2\alpha f(\mu_1) + 2\alpha\beta\phi(t) + W(t) \end{aligned} \tag{6}$$

and

$$\begin{aligned} \tilde{y}(t) &= 2\alpha f(\mu_1) + 2\alpha\beta\phi(t) + W(t) \\ \frac{\tilde{y}(t)}{2\alpha\beta} &= \phi(t) + \frac{f(\mu_1)}{\beta} + \frac{1}{2\alpha\beta}W(t) \end{aligned} \tag{7}$$

where $\tilde{y} = I(t) + 2\alpha\beta u(t)$. Finally we can write a linear system model with sector bounded uncertainty as follows:

$$\begin{aligned} \dot{\phi} &= -\lambda\phi(t)dt + \sqrt{\kappa}v(t), \\ \bar{y}(t) &= \phi(t) + \frac{f(\mu_1)}{\beta} + \frac{1}{2\alpha\beta}W(t), \end{aligned} \tag{8}$$

where $\bar{y}(t) = \frac{\tilde{y}(t)}{2\alpha\beta}$. We assume that the sector is defined by the region $\gamma = \frac{1-m}{2}$ (see Fig. 2), where m is the lowest boundary of the sector. Also, $f^2(\mu_1) \leq \gamma^2\mu_1^2$. Let us assume that $\tilde{f}(\mu_1) = f(\mu_1)\gamma$. Then (8) can be written as

$$\begin{aligned} \dot{\phi} &= -\lambda\phi(t)dt + \sqrt{\kappa}dv(t), \\ \bar{y}(t) &= \phi(t) + \gamma\frac{\tilde{f}(\mu_1)}{\beta} + \frac{1}{2\alpha\beta}W(t). \end{aligned} \tag{9}$$

IV. Robust Nonlinear Estimation

In this section, we present an approach to estimate the phase $\phi(t)$ of the system in the presence of the nonlinear time varying uncertainty $\tilde{f}(\mu_1)$. The robust nonlinear state estimation technique used here is based on the minimax LQG control theory [8] which uses Integral Quadratic Constraints (IQCs) to exploit this nonlinear uncertainty [9]. This method provides a systematic methodology for constructing a robust nonlinear phase estimator for the system given above. The main idea behind this approach is to modify the standard IQC approach to robust state control and estimation by including a copy of the nonlinearity in the state estimator as shown in Fig. 3. This approach enables us to use minimax LQG control theory to construct the linear part of the estimator and then the nonlinear estimator is constructed by including a copy of the plant nonlinearity. Here, the method in [9] is used for the infinite horizon case where $t \to \infty$.

A. Methodology

In this subsection we present a summary of the method from [9] which will later be used to solve phase estimation problem.

Definition 1: Let (Ω, F, P) be complete probability space on which a p-dimensional standard Wiener process $W(\cdot)$ and a Gaussian random variable $x_0 : \Omega \to \Re^n$ are defined. The probability measure P is defined as the product of the probability measure

$$\mu(dx \times dy) = \frac{1}{(2\pi)^{n/2}|Y_0|} e^{-\frac{1}{2}(x-\check{x}_0)^T Y_0^{-1}(x-\check{x}_0)} dx \times \delta(y)dy \tag{10}$$

on $\Re^n \times \Re^l$ and the standard measure on $C([0,\infty], \Re^p)$. Here, $\check{x}_0$, $Y_0 > 0$ denote the mean and variance of the Gaussian variable x_0, and $\delta(y)$ denotes the deltafunction on $\Re^l$.

Let us consider a system on the probability space (Ω, F, P) driven by the noise input $W(\cdot)$ as follows:

$$
\begin{aligned}
dx(t) &= Ax(t)dt + [\sum_{i=1}^{g} \bar{B}_{1i}\mu_i(t) + \sum_{s=1}^{k} B_{1s}\xi_s(t)]dt \\
&\quad + B_1 dW(t); \quad x(0) = x_0, \\
w(t) &= C_0 x(t); \\
\zeta_1(t) &= C_{1,1}x(t); \\
&\vdots \\
\zeta_k(t) &= C_{1,k}x(t); \\
\nu_1(t) &= \bar{C}_{1,1}x(t); \\
&\vdots \\
\nu_g(t) &= \bar{C}_{1,g}x(t); \\
dy(t) &= C_2 x(t)dt + [\sum_{i=1}^{g} \bar{D}_{21,i}\mu_i(t) \\
&\quad + \sum_{s=1}^{k} D_{21,s}\xi_s(t)]dt + D_{21}dW(t); \quad y(0) = 0,
\end{aligned}
\tag{11}
$$

where $x(t) \in \Re^n$ is the state, $w(t) \in \Re^m$ is the estimated output, $\zeta_1(t) \in \Re^{h_1}, \cdots, \zeta_k(t) \in \Re^{h_k}$ are the uncertainty outputs, $\nu_1(t) \in \Re^{h_1}, \cdots, \nu_g(t) \in \Re$ are the nonlinearity outputs, $\xi_1(t) \in \Re^{r_1}, \cdots, \xi_k(t) \in \Re^{r_k}$ are the uncertainty inputs, $\nu_1(t) \in \Re, \cdots, \nu_g(t) \in \Re$ are nonlinearity inputs, and $y(t) \in \Re^l$ is the measured output. The nonlinearity inputs are related to the nonlinearity outputs by the following nonlinear relations

$$
\mu_i(t) = \psi_i(\nu_i(t)) \quad \forall i = 1, 2, \cdots, g, \tag{12}
$$

where the nonlinear functions $\psi_i(\cdots)$ are such that $\psi_i(0) = 0$ and satisfy the following global Lipschitz conditions:

$$
|\psi_i(\nu_i(t)) - \psi_i(\nu_i(t))|\mu_i(t) \le \beta_i|\nu_1 - \nu_2| \tag{13}
$$

for all ν_1, ν_2 and for all $i = 1, 2, \cdots, g$.

Definition 2: An uncertainty is an admissible uncertainty for the system (11), (12) if given any strong solution to the system (11), (12) then for all $s = 1, \cdots, k$ and S_s are given positive-definite matrices and $\|\cdot\|$ denotes the standard Euclidean norm.

The uncertainty in the system is described by the following Stochastic Integral Quadratic Constraint.

$$
E\int_0^\infty \|\xi_s(t)\|^2 dt \le E[\int_0^\infty \|\zeta_s(t)\|^2 dt + x(0)^T S_s x(0)]. \tag{14}
$$

The nonlinear dynamic state estimator is of the form as follows:

$$
\begin{aligned}
d\hat{x}(t) &= (A_c\hat{x}(t) + \sum_{i=1}^{g} \bar{G}_{ci}\tilde{\mu}_i(t))dt \\
&\quad + B_c dy; \quad \hat{x}(0) = \hat{x}_0, \\
\tilde{\nu}_1(t) &= \bar{K}_{c1}\hat{x}(t); \\
&\vdots \\
\tilde{\nu}_g(t) &= \bar{K}_{cg}\hat{x}(t); \\
\hat{w}(t) &= C_c\hat{x}(t),
\end{aligned}
\tag{15}
$$

where $\tilde{\mu}_i(t) = \psi_i(\tilde{\nu}_i(t))$ for $i = 1, 2, \cdots, g$. The nonlinear dynamic estimator (15) is designed such that it provides an upper bound on the following cost functional:

$$
J(\hat{w}(\cdot)) = E\big[\frac{1}{2}\int_0^\infty \|\hat{w}(t) - w(t)\|^2 dt\big]. \tag{16}
$$

The nonlinear estimator (15) can be written in compact form as follows:

$$
\begin{aligned}
d\hat{x}(t) &= A_c\hat{x}(t)dt + \tilde{B}_c d\tilde{y}(t), \\
\tilde{u}(t) &= \tilde{C}_c\hat{x}(t),
\end{aligned}
\tag{17}
$$

where

$$
\tilde{y}(t) \triangleq \begin{bmatrix} y(t) \\ \tilde{\mu}_1(t) \\ \vdots \\ \tilde{\mu}_g(t) \end{bmatrix}; \quad \tilde{u}(t) \triangleq \begin{bmatrix} \hat{w}(t) \\ \tilde{\nu}_1(t) \\ \vdots \\ \tilde{\nu}_g(t) \end{bmatrix};
$$

$$
\tilde{B}_c \triangleq \begin{bmatrix} B_c & \bar{G}_{c1} & \cdots & \bar{G}_{cg} \end{bmatrix}; \; \tilde{C}_c \triangleq \begin{bmatrix} C_c \\ \bar{K}_{c1} \\ \vdots \\ \bar{K}_{cg} \end{bmatrix}.
$$

The IQCs for the repeated nonlinearities can be written as follows (see [9] for detail):

$$
\begin{aligned}
E\int_0^\infty &[\mu_i(t) - \tilde{\mu}_i(t)]^2 dt \\
&\le E[\int_0^\infty \beta_i^2[\nu_i - \tilde{\nu}(t)]^2 dt + x(0)^T S_{1i}x(0)],
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
E\int_0^\infty &[\mu_i(t)]^2 dt \\
&\le E[\int_0^\infty \beta_i^2[\nu_i]^2 dt + x(0)^T S_{2i}x(0)],
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
E\int_0^\infty &[\tilde{\mu}_i(t)]^2 dt \\
&\le E[\int_0^\infty \beta_i^2[\tilde{\nu}(t)]^2 dt + x(0)^T S_{3i}x(0)]
\end{aligned}
\tag{20}
$$

for all $i = 1, \cdots, g$. Here the $\bar{S}_{1i}$, $\bar{S}_{2i}$, $\bar{S}_{3i}$ are any positive definite matrices. System (11) can be written in a compact form as follows:

$$
\begin{aligned}
dx(t) &= [Ax(t) + \tilde{B}_1\tilde{\xi}(t)]dt + B_1 dW(t);\\
w(t) &= C_0 x(t);\\
\tilde{\zeta}(t) &= \tilde{C}_1 x(t) + \tilde{D}_{12}\tilde{u}(t);\\
d\tilde{y}(t) &= [\tilde{C}_2 x(t) + \tilde{D}_{21}\tilde{\xi}(t)]dt + \bar{D}_{21} dW(t),
\end{aligned} \tag{21}
$$

where

$$
\tilde{\xi}(t) = \begin{bmatrix} \tilde{\xi}_1(t) \\ \vdots \\ \tilde{\xi}_{k+2g}(t) \end{bmatrix} \triangleq \begin{bmatrix} \xi_1(t) \\ \vdots \\ \xi_k(t) \\ \mu_1(t) \\ \vdots \\ \mu_g(t) \\ \tilde{\mu}_1(t) \\ \vdots \\ \tilde{\mu}_g(t) \end{bmatrix};
$$

$$
\tilde{\zeta}(t) = \begin{bmatrix} \tilde{\zeta}_1(t) \\ \vdots \\ \tilde{\zeta}_{k+2g}(t) \end{bmatrix} \triangleq \begin{bmatrix} \zeta_1(t) \\ \vdots \\ \zeta_k(t) \\ \mu_1(t) \\ \vdots \\ \mu_g(t) \\ \tilde{\nu}_1(t) \\ \vdots \\ \tilde{\nu}_g(t) \end{bmatrix};
$$

$$
\tilde{B}_1 = \begin{bmatrix} B_{1,1} & \cdots & B_{1,k} & \bar{B}_{1,1} & \cdots \bar{B}_{1,g} & 0_{n\times g} \end{bmatrix};
$$

$$
\tilde{C}_1 = \begin{bmatrix} C_{1,1} \\ \vdots \\ C_{1,k} \\ \bar{C}_{1,1} \\ \vdots \\ \bar{C}_{1,g} \\ 0_{g\times n} \end{bmatrix}; \ \tilde{D}_{12} = \begin{bmatrix} 0_{h_1\times m} & 0_{h_1\times g} \\ \vdots & \vdots \\ 0_{h_k\times m} & 0_{h_k\times g} \\ 0_{1\times m} & 0_{1\times g} \\ \vdots & \vdots \\ 0_{1\times m} & 0_{1\times g} \\ 0_{g\times m} & I_{g\times g} \end{bmatrix};
$$

$$
\tilde{C}_2 = \begin{bmatrix} C_2 \\ 0_{g\times n} \end{bmatrix}; \ \bar{D}_{21} = \begin{bmatrix} D_{21} \\ 0_{g\times(h+g)} \ J_{21} \end{bmatrix};
$$

$$
\tilde{D}_{21} = \begin{bmatrix} D_{21,1} & \cdots & D_{21,k} & \bar{D}_{21,1} & \cdots & \bar{D}_{21,g} & 0_{l\times g} \\ 0_{g\times r_1} & \cdots & 0_{g\times r_k} & 0_{g\times 1} & \cdots & 0_{g\times 1} & I_{g\times g} \end{bmatrix}.
$$

Also $h = \sum_{i=1}^{k} h_i$, $r = \sum_{i=1}^{k} r_i$, $p = h + 2g$ and J_{21} is any suitable matrix satisfying $J_{21} > 0$. Considering new variables, the IQCs (14), (18), (19), and (20) can be written as follows:

$$
E\int_0^\infty \tilde{\xi}^T(t)\tilde{M}(\lambda)\tilde{\xi}(t)dt \le E[\int_0^\infty \tilde{\zeta}^T(t)\tilde{N}\tilde{\zeta}(t) + x(0)^T\tilde{S}_i x(0)] \tag{22}
$$

for all $\lambda = [\lambda_1 \ \lambda_2, \cdots, \lambda_{\tilde{k}}]^T \in \Re^{\tilde{k}}$ where $\tilde{k} = k + 3g$. Also, $\tilde{M}(\lambda) = \sum_{i=1}^{\tilde{k}} = \lambda_i M_i \ge 0$ and $\tilde{N}(\lambda) = \sum_{i=1}^{\tilde{k}} = \lambda_i N_i \ge 0$ where $M_i = m_i^T m_i \ge 0$, $N_i = n_i^T n_i \ge 0$ and $\tilde{S}_i$ are positive-definite matrices. The constraint on the $\tilde{M}(\lambda)$ is defined as follows:

$$
\lambda \in \tilde{\Gamma} : \tilde{M}(\lambda)^{-1} \ge JJ^T \ \forall t, \tag{23}
$$

where $\tilde{\Gamma} = \{\lambda \in \Re^{\tilde{k}} : \lambda_i \ge 0 \ \forall i \tilde{M}(\lambda) > 0\}$.

Assumption 1: There exist a square matrix function J such that we can write

$$
\begin{bmatrix} B_1 \\ \bar{D}_{21} \end{bmatrix} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{D}_{21} \end{bmatrix} J. \tag{24}
$$

Assumption 2: There exist a constant $d_0 > 0$ such that

$$
\bar{D}_{21}\bar{D}_{21}^T = \tilde{D}_{21}JJ^T\tilde{D}_{21}^T \ge d_0 I \tag{25}
$$

for all t.

Assumption 3: There exist a constant $\tau > 0$ such that the following conditions hold:

1) The algebraic Riccati equation

$$
\begin{aligned}
&(A - \tilde{B}_1\tilde{M}(\lambda)^{-1}\tilde{D}_{21}^T E_\lambda^{-1}\tilde{C}_2)^T Y \\
&+ Y(A - \tilde{B}_1\tilde{M}(\lambda)^{-1}\tilde{D}_{21}^T E_\lambda^{-1}\tilde{C}_2) \\
&- Y(\tilde{C}_2^T E_\lambda^{-1}\tilde{C}_2 - \frac{1}{\tau}R_{\tau,\lambda})Y + \tilde{B}_1\tilde{M}(\lambda)^{-1}\tilde{B}_1^T \\
&- \tilde{B}_1\tilde{M}(\lambda)^{-1}\tilde{D}_{21}^T E_\lambda^{-1}\tilde{D}_{21}\tilde{M}(\lambda)^{-1}\tilde{B}_1^T = 0
\end{aligned} \tag{26}
$$

has a symmetric positive definite solution.

2) The algebraic Riccati equation

$$
\begin{aligned}
&XA + A^T X + (R_{\tau,\lambda} - \Gamma_{\tau,\lambda})^{-1}\Gamma_{\tau,\lambda}^T \\
&\qquad + \frac{1}{\tau}X\tilde{B}_1\tilde{M}(\lambda)^{-1}\tilde{B}_1^T X = 0
\end{aligned} \tag{27}
$$

has a symmetric nonnegative definite solution.

3) and $\rho(YX) < \tau$, where $\rho(\cdot)$ denotes the spectral radius of a matrix and

$$
R_{\tau,\lambda} = C_0^T C_0 + \tau\tilde{C}_1^T\tilde{N}(\lambda)\tilde{C}_1,
$$

$$
G_{\tau,\lambda} = \begin{bmatrix} I_{m\times m} & 0_{m\times g} \\ 0_{g\times m} & 0_{g\times g} \end{bmatrix} + \tau\tilde{D}_{12}^T\tilde{N}(\lambda)\tilde{D}_{12},
$$

$$
\Gamma_{\tau,\lambda} \triangleq -[C_0^T \quad 0_{n\times g}] + \tau\tilde{C}_1^T\tilde{N}(\lambda)\tilde{D}_{12}.
$$

Also, $\tilde{M}(\lambda) = \sum_{i=1}^{\tilde{k}} \lambda_i M_i \ge 0$ where $\lambda = [\lambda_1 \ \lambda_2 \ \cdots \lambda_{\tilde{k}}]^T \in \Re^{\tilde{k}}$.

If all the given assumptions are satisfied then parameters in (17) can be determined using the following relations.

$$
\begin{aligned}
A_c &= A + \frac{1}{\tau}YR_{\tau,\lambda} - (Y\tilde{C}_2^T + \tilde{B}_1\tilde{M}(\lambda)^{-1}\tilde{D}_{21}^T)E_\lambda^{-1}\tilde{C}_2 \\
&\quad - \frac{1}{\tau}Y\Gamma_{\tau,\lambda}G_{\tau,\lambda}^{-1}\Gamma_{\tau,\lambda}^T(I - \frac{1}{\tau}YX)^{-1}; \\
\tilde{B}_c &= (Y\tilde{C}_2^T + \tilde{B}_1\tilde{M}(\lambda)^{-1}\tilde{D}_{21}^T)E_\lambda^{-1}; \\
\tilde{C}_c &= -G_{\tau,\lambda}^{-1}\Gamma_{\tau,\lambda}^T[I - \frac{1}{\tau}YX]^{-1}.
\end{aligned} \tag{28}
$$

The guaranteed cost bound is given by the following relation.

$$
\begin{aligned}
V_\tau = \frac{1}{2}tr[&YR_{\tau,\lambda} + (Y\tilde{C}_2^T + \tilde{B}_1\tilde{M}(\lambda)^{-1}\tilde{D}_{21}^T)E_\lambda^{-1} \\
&\times(\tilde{C}_2 Y + \tilde{D}_{21}\tilde{M}(\lambda)^{-1}\tilde{B}_1^T) \times X(I - \frac{1}{\tau}YX)^{-1}].
\end{aligned} \tag{29}
$$

V. Phase Estimation

The uncertainty model of the original system under consideration in (9) can be written in the form (11) as follows:

$$\begin{aligned} d\phi(t) &= -\lambda\phi(t)d(t) + [0 \times \tilde{f}(\mu_1)(t) \\ &\quad + \sqrt{\kappa}\Delta_1\zeta_1]dt + [\sqrt{\kappa} \quad 0]dW(t), \\ \nu_1(t) &= \gamma 2\alpha\phi(t), \\ \zeta_1 &= 0 \times \phi(t). \end{aligned} \tag{30}$$

Also, the uncertainty $\tilde{f}(\mu_1)$ satisfies the IQC (14). A comparison of the above model with (11) gives the following model parameters:

$$\begin{aligned} &A = -\lambda,\ \bar{B}_{11} = 0,\ B_{11} = \sqrt{\kappa}, B_1 = [\sqrt{\kappa} \quad 0 \quad 0], \\ &C_2 = 1, \bar{D}_{21,1} = \frac{1}{2\alpha\beta},\ \bar{C}_{1,1} = \gamma 2\alpha,\ C_{1,1} = 0, \\ &D_{21,1} = 0, D_{21} = [0 \quad \frac{1}{2\alpha\beta}], g = 1,\ k = 1. \end{aligned}$$

Considering the above model parameters, the notation used in (21) can be defined as follows:

$$\tilde{B}_1 = [\sqrt{\kappa}\ 0\ 0],\ \tilde{C}_1 = [0\ \gamma 2\alpha\ 0]^T,\ \tilde{D}_{12} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\tilde{C}_2 = [1\ 0]^T, \bar{D}_{21} = \begin{pmatrix} 0 & \frac{1}{2\alpha\beta} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\tilde{D}_{21} = \begin{pmatrix} 0 & \frac{1}{2\alpha\beta} & 0 \\ 0 & 0 & 1 \end{pmatrix},\ h = 1,\ r = 1,\ p = 3,\ \tilde{k} = 4.$$

For the above definition of the model, the assumptions (1) and (2) are satisfied for $J(t) = I$ and $0 < d_0 \le 1$. From the definition of IQC (22) we have

$$\tilde{M}_\lambda(\lambda) = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 + \lambda_3 & -\lambda_2 \\ 0 & -\lambda_2 & \lambda_2 + \lambda_4 \end{pmatrix}$$

and its inverse is given by

$$\tilde{M}^{-1} = \begin{pmatrix} \frac{1}{\lambda_1} & 0 & 0 \\ 0 & \frac{\lambda_2+\lambda_4}{\lambda_3\lambda_4+\lambda_2(\lambda_3+\lambda_4)} & \frac{\lambda_2}{\lambda_3\lambda_4+\lambda_2(\lambda_3+\lambda_4)} \\ 0 & \frac{\lambda_2}{\lambda_3\lambda_4+\lambda_2(\lambda_3+\lambda_4)} & \frac{\lambda_2+\lambda_3}{\lambda_3\lambda_4+\lambda_2(\lambda_3+\lambda_4)} \end{pmatrix}.$$

Also,

$$\begin{aligned} G_{\tau,\lambda(t)} &= \begin{pmatrix} 1 & 0 \\ 0 & (\lambda_2 + \lambda_4)\tau \end{pmatrix}, \\ R_{\tau,\lambda(t)} &= 4\gamma^2\alpha^2\tau(\lambda_2 + \lambda_3) + 1, \\ \Upsilon_{\tau,\lambda(t)} &= \{-1 \quad -2\alpha\lambda_2\tau\gamma\}. \end{aligned}$$

The constraints on λ_i for $i = 1, 2, 3, 4$ due to (23) are given below.

$$\begin{aligned} &\lambda_1 > 0, \quad \lambda_2 > 0, \quad \lambda_3 > 0, \quad \lambda_4 > 0, \quad 0 < \lambda_1 \le 1, \\ &0 < \lambda_2 + \lambda_3 \le 1, \quad 0 < \lambda_2 + \lambda_4 \le 1, \\ &(1 - \lambda_2 - \lambda_3)(1 - \lambda_2 - \lambda_4) - \lambda_2^2 \ge 0. \end{aligned}$$

For the values of the parameters given in Table I, the minimum value of the bound (29) obtained using the 'Interior-point' numerical optimization method is $V_\tau = 0.1506$ at $\tau = 1.08 \times 10^{-6}$ and the values of the λ_i for $i = 1, 2, 3, 4$ are obtained as follows:

$$\lambda_1 = 0.9975,\ \lambda_2 = 0.4985,\ \lambda_3 = 0.0002,\ \lambda_4 = 0.0001.$$

Also, the solution of the Riccati equations at the optimum parameter values are obtained as $Y = 0.166, \quad X = 1.446 \times 10^{-8}$. The estimator parameters in (28) are calculated as follows:

$$A_c = -4.5725 \times 10^5,\quad \tilde{B}_c = [4.476 \times 10^5 \quad -192.56],$$
$$\tilde{C}_c = [1.002 \quad 931.405].$$

TABLE I: Parameters values for the optical system.

λ	9.14×10^3 rad/s	γ	0.4
κ	40000 rad/s	α	1162 s^{-1}
β	1		

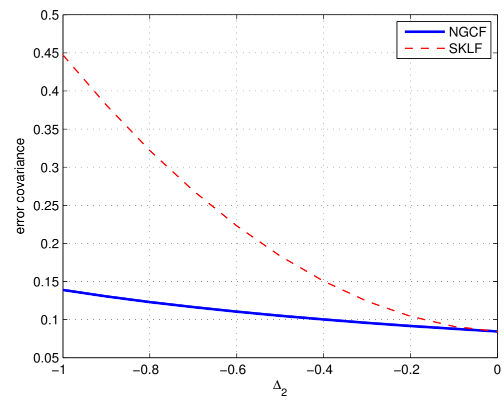

Fig. 4: Error covariance with uncertainty variation.

VI. Closed Loop Simulation

In this paper, simulations with both linear and nonlinear uncertainties have been performed. The linear simulation allows for the comparison with the standard Kalman filter (SKLF) which gives an insight into the performance of the nonlinear guaranteed cost filter (NGCF) for different values of the time varying uncertainty. The nature of the uncertainty in the given system is actually nonlinear therefore, a Monte-Carlo simulation using Simulink has been performed for the closed loop system.

A. Linear Simulation

In order to perform the simulation with linear uncertainty, a closed loop equation has been derived using the system and estimator dynamics. The system dynamics in (30) can be written in the form of (11) as follows:

$$\begin{aligned} d\phi(t) &= [A\phi(t) + \tilde{B}_1\tilde{\xi}(t)]d(t) + B_1 dW(t), \\ \tilde{\zeta} &= \tilde{C}_1\phi(t) + \tilde{D}_{12}(\tilde{C}_c\hat{\phi}(t)). \end{aligned} \tag{31}$$

Since $\tilde{\xi}(t) = \Delta\tilde{\zeta}(t)$ or

$$\tilde{\xi}(t) = \begin{bmatrix} \Delta_1 & 0 & 0 \\ 0 & \Delta_2 & 0 \\ 0 & 0 & \Delta_2 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \mu_1 \\ \tilde{\mu}_1 \end{bmatrix},$$

we can write above equation as given below:

$$d\phi(t) = [A\phi(t) + \tilde{B}_1\Delta(\tilde{C}_1)\phi + \tilde{D}_{12}\tilde{C}_c\hat{\phi}(t)]d(t) + B_1 dW(t),$$
$$\tilde{\zeta} = \tilde{C}_1\phi(t) + \tilde{D}_{12}(\tilde{C}_c\hat{\phi}(t)),$$

where $\Delta_1 = 0$ and $\Delta_2 = \tilde{f}(\mu_1)$.

$$d\phi(t) = [(A + \tilde{B}_1\Delta(\tilde{C}_1)\phi(t) + \tilde{B}_1\Delta\tilde{D}_{12}\tilde{C}_c\hat{\phi}(t)]d(t) + B_1 dW(t)$$
$$\tilde{\zeta} = \tilde{C}_1\phi(t) + \tilde{D}_{12}(\tilde{C}_c\hat{\phi}(t)).$$

Also,

$$d\hat{\phi}(t) = [A_c\hat{\phi}(t) + \tilde{B}_c(\tilde{C}_2\phi(t) + \tilde{D}_{21}\tilde{\xi}(t))]dt + \bar{D}_{21}(t)]dW(t)$$
$$d\hat{\phi}(t) = [A_c\hat{\phi}(t) + \tilde{B}_c\tilde{C}_2\phi(t) + \tilde{B}_c\tilde{D}_{21}\Delta(\tilde{C}_1)\phi(t) + \tilde{D}_{12}\tilde{C}_c\hat{\phi}(t)]dt + \tilde{B}_c\bar{D}_{21}(t)]dW(t). \quad (32)$$

Finally we get the simplified expression as follows:

$$d\hat{\phi} = [(A_c + \tilde{B}_c\tilde{D}_{21}\Delta\tilde{D}_{12}\tilde{C}_c)\hat{\phi} + (\tilde{B}_c\tilde{C}_2 + \tilde{B}_c\tilde{D}_{21}\Delta\tilde{C}_1)\phi(t)]dt \quad (33)$$

$$\text{or} \quad d\phi_c = A_s\phi_c dt + B_s dW, \quad (34)$$

where $d\phi_c = \begin{bmatrix} d\phi(t) \\ d\hat{\phi}(t) \end{bmatrix}, \quad \phi_c = \begin{bmatrix} \phi(t) \\ \hat{\phi}(t) \end{bmatrix},$

$$A_s = \begin{bmatrix} A + \tilde{B}_1\Delta(\tilde{C}_1 & \tilde{B}_1\Delta\tilde{D}_{12}\tilde{C}_c \\ \tilde{B}_c\tilde{C}_2 + \tilde{B}_c\tilde{D}_{21}\Delta\tilde{C}_1 & A_c + \tilde{B}_c\tilde{D}_{21}\Delta\tilde{D}_{12}\tilde{C}_c \end{bmatrix},$$
$$B_s = \begin{bmatrix} B_1 & 0 \\ 0 & \tilde{B}_c\bar{D}_{21} \end{bmatrix}.$$

The steady state covariance matrix P of the closed loop system is calculated by solving the following Lyapunov equation:

$$A_sP + PA_s^T + B_sB_s^T = 0 \quad (35)$$

where $P_s = E(\phi_c\phi_c^T)$ is the symmetric matrix. The error covariance then can be written as follows:

$$\sigma^2 = E(ee^T) = [1 \quad -1]E(\phi_c\phi_c^T)\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad (36)$$

where $e = \phi - \hat{\phi}$. Simulation with Δ_2 varying from 0 to -1 has been performed for both NGCF and standard Kalman filter. Fig. 4 shows the error covariance for each case. In the figure, $\Delta_2 = 0$ means zero uncertainty and $\Delta_2 = -1$ represents maximum uncertainty for which the estimator has been designed i.e. $\gamma = 0.4$. It is obvious that at zero uncertainty the error covariances of Kalman filter and NGCF are approximately equal. However, as the uncertainty increases, the error covariance increases for both the filters but NGCF error remains below the theoretical value of the error bound which is found to be 0.15. Hence, NGCF out performs the SKLF in the presence of uncertainty and proved to be a better design option for the phase estimation problem considered here.

B. Nonlinear simulation

Since original model of the system is nonlinear, a nonlinear simulation has also been performed. A nonlinear Monte-Carlo simulation has been performed by collecting error samples during $50,000$ runs using Simulink. The nonlinear uncertainty is of the form $\sin(\phi) - \phi$. The samples have been collected by running the simulation for a fixed time interval with randomly generated noise signals and the error signal is collected at the end of the simulation. The error covariance for the simulation with the NGCF is found to be 0.1031. In the case of nonlinear uncertainty a similar simulation with the SKLF in the feedback loop yields the error covariance of 0.1872 which is 81% larger than the error covariance obtained using NGCF.

VII. CONCLUSION

In this paper the problem of phase estimation has been considered for a quantum optical system. The scheme in this paper extended the design of an adaptive homodyne estimator by using a nonlinear robust estimator in the feedback loop. The scheme uses an uncertainty model of the system and provides a nonlinear robust estimator by considering a copy of nonlinear uncertainty in the estimator. Simulation results with both linear and nonlinear uncertainty shows that the scheme improves the error covariance significantly and works better than the standard Kalman filter in the case of large difference between initial guess of the phase and the original phase of the system. Further research will be directed toward implementing the scheme on the real hardware and perform experiments.

VIII. ACKNOWLEDGMENTS

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