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Optimal Synchronization of Circulant Networked Multi-Agent Systems[★]

Andrej Mosebach and Jan Lunze

Institute of Automation and Computer Control, Ruhr-Universität Bochum,
Universitätsstrasse 150, 44801 Bochum, Germany, (*e-mail:* mosebach@atp.rub.de)

***Abstract*— This paper investigates the synchronization problem of identical agents while minimizing an LQ-based performance index that reflects the behavior of the synchronization errors. For an objective function which penalizes the synchronization errors with a circulant structured weighting matrix, an optimal networked controller is proposed which has the same structure. The resulting communication graph is a linear combination of circulant subgraphs which couple all agents with all other agents.**

I. INTRODUCTION

Uncoupled agents can be synchronized by a networked controller (NC) only if the controller introduces couplings among the agents in which the input of every agent is influenced by every other agent directly or through a path of agents. Hence, using local controllers which are linked by the states of the agents, synchronization of the multi-agent systems can be achieved. Figure 1 illustrates the coupling of a multi-agent system by a NC over a communication network $\boldsymbol{L}$, which in this paper is assumed to couple all agents with all others directly.

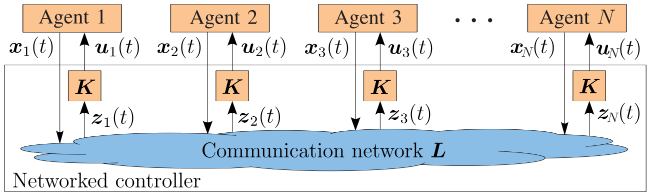

Fig. 1. Networked multi-agent system

The main contribution of this paper is a new design method of a synchronizing NC for identical linear agents. As an important issue, this design method chooses the controller parameters with respect to the transient behavior, whereas most methods found in literature only consider synchronization as an asymptotic property, where the closed-loop system has to satisfy the requirement

$$\lim_{t\to\infty} \|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\| = 0, \qquad i, j = 1, 2, \ldots, N. \tag{1}$$

Since asymptotic synchronization does not imply smooth synchronization with a fast transient behavior the method presented in this paper should, in addition to (1), minimize the following performance index

$$J_{\text{trans}} = \int_0^\infty \sum_{i,j=1}^{N} \left(\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\right)^T \boldsymbol{W}_{ij} \left(\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\right) \mathrm{d}t. \tag{2}$$

The objective function is chosen so as to penalize the differences $\boldsymbol{e}_{ij}(t) = \boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)$ between the agent states

$$J(\underline{\boldsymbol{u}}(t),\, \underline{\boldsymbol{x}}_0) = \int_0^\infty \sum_{\substack{i=1,\\ j=i+1}}^{N} \boldsymbol{e}_{ij}^T(t)\boldsymbol{Q}_{ij}\boldsymbol{e}_{ij}(t) + \boldsymbol{u}_i^T(t)\boldsymbol{R}\,\boldsymbol{u}_i(t)\,\mathrm{d}t. \tag{3}$$

For a weighting of the state differences according to a circulant structure, as explained in Section IV, it is shown that the resulting NC adopts the same circulant structure.

An important characteristic of synchronization is the fact that the NC should steer the agents towards one another, but should no longer affect the agents if the overall system is synchronized. This requirement implies that for identical initial states $\boldsymbol{x}_{0i}$ of the agents, the multi-agent system remains synchronized

$$\boldsymbol{x}_i(t) = \boldsymbol{x}_j(t) \;\; \Leftrightarrow \;\; \boldsymbol{x}_{0i} = \boldsymbol{x}_0, \;\; i, j = 1, 2, \ldots, N. \tag{4}$$

By the optimal control theory it is well known that the solution of the standard LQR-problem is a stabilizing state feedback if and only if unstable dynamics of the system are visible in the objective function [3]. Equation (4) shows that there exist initial conditions which do not influence the value of the objective function (3). Clearly, if the agents are stable, there are non-visible stable dynamics in the objective function (3) which does not have to be considered. This paper considers non-trivial solutions with unstable agents, where it is shown that the unstable behavior has explicitly to be considered as a constraint to the LQR-problem to obtain the synchronizing NC. Using decomposition methods, the overall LQR-problem is solved by solving subproblems of a single-agent order.

Literature survey. In recent years, increasing attention was given to the synchronization problem of linear multi-agent systems with identical subsystems [6], [7], [11]. This subject is motivated by many application examples, such as vehicle formation [1], [9] or flocking [12].

In [4], [5], [10], [13], [14] design methods based on the LQR-theory and linear matrix inequalities are used to obtain synchronizing NCs for linear multi-agent systems. In [13], [14] a NC was obtained by solving only one algebraic Riccati equation (ARE) of a single subsystem order. A similar idea has been also studied in [4] and an LMI-based method

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has been introduced. Performance of networked multi-agent systems was analyzed in [5]. The representation of the performance index in terms of the L_2 norm shows that the overall system performance depends primarily on the second smallest eigenvalue of the coupling matrix $\boldsymbol{L}$. In [10] LQR methods are used for the design of an optimal NC with a communication network, representing a ring structure to synchronize single-order systems.

In contrast to the design methods published in [4], [13], [14] this paper solves the synchronization problem not only by satisfying requirement (1), but also by considering the transient behavior, expressed by the performance index (2).

Structure of this paper. Section II presents the notation and important results from graph theory. Agent models, the considered network and conditions on synchronization are introduced in Section III. The main result is the optimal controller design given in Section IV. Section V illustrates the design by its application to an example.

II. PRELIMINARIES AND NOTATION

A. Notation

Scalars are represented by italic letters (a, x), vectors by bold letters ($\boldsymbol{x}$, $\boldsymbol{v}$) and matrices respectively by bold face upper-case letters ($\boldsymbol{A}$, $\boldsymbol{L}$). The i-th element of the j-th column of the matrix $\boldsymbol{A}$ is given by $(\boldsymbol{A})_{ij}$. The identity matrix with dimension N is symbolized by $\boldsymbol{I}_N$. Sets are denoted by calligraphic letters ($\mathcal{P}$, $\mathcal{N}$). The expression $\boldsymbol{A} \succ 0$ describes the positive definiteness of the Matrix $\boldsymbol{A}$. $\operatorname{diag}_{i=1}^{p}(\lambda_i)$ is a diagonal matrix with the diagonal entries $\lambda_1, \ldots, \lambda_p$. $\lambda_i\{\boldsymbol{A}\}$ denotes the i-th eigenvalue of the matrix $\boldsymbol{A}$. The set of real numbers is represented by $\mathbb{R}$ and the set of integers by $\mathbb{Z}$. The Kronecker product of two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ will be denoted by $\boldsymbol{A} \otimes \boldsymbol{B}$. $\mathbb{1}_N = (1 \ldots 1)^T$ is the N-dimensional one vector, $\boldsymbol{0}_N = (0 \ldots 0)^T$ the N-dimensional zero vector.

A circulant matrix $\boldsymbol{C}$ is a matrix where each row vector is rotated one element to the right relative to the preceding row vector. Hence the information of the first row is sufficient to represent the matrix, $\operatorname{circ}\begin{pmatrix} c_0 & c_1 & \cdots & c_{N-1} \end{pmatrix} := \boldsymbol{C}$.

B. Graph theory

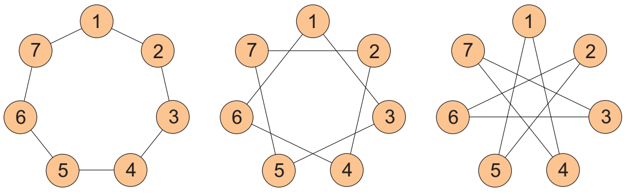
Consider an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ and edge set $\mathcal{E} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_M\}$. It is assumed that each edge $\boldsymbol{e}_j = \{v_i, v_k\} \in \mathcal{E}$ is defined by a set of two distinct vertices. The adjacency matrix $\boldsymbol{A}_{\mathcal{G}} = [a_{ij}]$ of the graph $\mathcal{G}$ has the entries $a_{ij} = 1$ if $\{v_i, v_j\} \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The Laplacian matrix $\boldsymbol{L}_{\mathcal{G}}$ of the graph $\mathcal{G}$ is defined by

$$(\boldsymbol{L}_{\mathcal{G}})_{ij} = \begin{cases} \sum\limits_{j=1}^{N} a_{ij} & \text{if } i = j \\ -a_{ij} & \text{if } i \neq j. \end{cases}$$

Due to this definition, all row sums of the Laplacian vanish and, therefore, the Laplacian matrix always has a zero eigenvalue with the right eigenvector $\boldsymbol{t}_1 = \mathbb{1}_N$.

C. Circulant graphs

A complete graph $\mathcal{K}$ is a graph, where every pair of vertices is connected by an edge. The complete graph $\mathcal{K}$ can be decomposed into circulant subgraphs $\mathcal{G}_{\mathrm{c},k}$, $k \in \mathcal{N}_p := \{1, 2, \ldots, p\}$, where all subgraphs have the vertex set $\mathcal{V}_{\mathcal{G}_{\mathrm{c},k}}$ and edge set $\mathcal{E}_{\mathcal{G}_{\mathrm{c},k}}$. When the vertices are arranged in a ring, the first subgraph $\mathcal{G}_{\mathrm{c},1}$ describes the connection between neighboring vertices (Fig. 2(a)), the second one $\mathcal{G}_{\mathrm{c},2}$ between their adjacent vertices (Fig. 2(b)), and so on. Hence, the complete graph $\mathcal{K}$ with N vertices v_i can be decomposed into $N/2$ subgraphs for even N, and $(N-1)/2$ subgraphs for odd N.



(a) Subgraph $\mathcal{G}_{\mathrm{c},1}$ (b) Subgraph $\mathcal{G}_{\mathrm{c},2}$ (c) Subgraph $\mathcal{G}_{\mathrm{c},3}$

Fig. 2. Decomposed complete graph $\mathcal{K}$ for $N = 7$ vertices

For reasons of simplicity, only the case of an odd number N of vertices is considered. Henceforth, the Laplacian matrix of the k-th circulant subgraph $\mathcal{G}_{c,k}$ will be denoted by $\boldsymbol{L}_{\mathrm{c},k}$, $k \in \mathcal{N}_p$ and $p = \frac{N-1}{2}$:

$$\begin{aligned} \boldsymbol{L}_{\mathrm{c},1} &= \operatorname{circ}\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 \end{pmatrix} \\ \boldsymbol{L}_{\mathrm{c},2} &= \operatorname{circ}\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & -1 & 0 \end{pmatrix} \\ &\vdots \\ \boldsymbol{L}_{\mathrm{c},p} &= \operatorname{circ}\begin{pmatrix} 2 & 0 & \cdots & 0 & -1 & -1 & 0 & \cdots & 0 \end{pmatrix}. \end{aligned}$$

The i-th eigenvalue of the k-th circulant Laplacian matrix $\boldsymbol{L}_{\mathrm{c},k}$ is characterized by

$$\lambda_{i-1}\{\boldsymbol{L}_{\mathrm{c},k}\} = 2 - \left(\mathrm{e}^{-\mathrm{j}2\pi\frac{(i-1)k}{N}} + \mathrm{e}^{\mathrm{j}2\pi\frac{(i-1)k}{N}}\right), \tag{5}$$

where $k \in \mathcal{N}_p$ and $i \in \mathcal{N} := \{1, 2, \ldots, N\}$ (cf. [2]).

***Lemma** 1: All circulant Laplacian matrices $\boldsymbol{L}_{\mathrm{c},k}$, $k \in \mathcal{N}_p$ have an eigenvalue $\lambda_0\{\boldsymbol{L}_{\mathrm{c},k}\} = 0$ and p symmetric ordered eigenvalues, such that the following holds:*

$$\lambda_\alpha\{\boldsymbol{L}_{\mathrm{c},k}\} = \lambda_{N-\alpha}\{\boldsymbol{L}_{\mathrm{c},k}\}, \quad \alpha \in \mathcal{N}_p. \tag{6}$$

Proof: Equation (6) is proved by substitution of eqn. (5) into eqn. (6). Expanding the terms to

$$\mathrm{e}^{-\mathrm{j}2\pi\frac{\alpha k}{N}} + \mathrm{e}^{\mathrm{j}2\pi\frac{\alpha k}{N}} = \mathrm{e}^{-\mathrm{j}2\pi k}\mathrm{e}^{\mathrm{j}2\pi\frac{\alpha k}{N}} + \mathrm{e}^{\mathrm{j}2\pi k}\mathrm{e}^{-\mathrm{j}2\pi\frac{\alpha k}{N}}, \ \alpha, k \in \mathcal{N}_p$$

and considering that $\mathrm{e}^{\mathrm{j}2\pi k} = 1, \forall k \in \mathbb{Z}$ it is obvious that eqn. (6) holds. ■

There always exists an orthogonal transformation matrix $\boldsymbol{T}$ that simultaneously diagonalizes the circulant Laplacian matrices $\boldsymbol{L}_{\mathrm{c},k}$, $k \in \mathcal{N}_p$ (cf. [2]):

$$\boldsymbol{T}^T = \boldsymbol{T}^{-1} : \boldsymbol{T}^{-1}\boldsymbol{L}_{\mathrm{c},k}\boldsymbol{T} = \operatorname*{diag}_{i=0}^{N-1}\left(\lambda_i\{\boldsymbol{L}_{\mathrm{c},k}\}\right), \forall k \in \mathcal{N}_p. \tag{7}$$

As a consequence of Lemma 1 the Laplacian matrices $\boldsymbol{L}_{\mathrm{c},k}$, $k \in \mathcal{N}_p$ have not more then $p+1$ distinct eigenvalues.

D. Review of the linear quadratic regulator problem

Consider the dynamical system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{8}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ denotes the system state and $\boldsymbol{u}(t) \in \mathbb{R}^m$ the control input. The linear quadratic regulator problem deals with finding of a control input $\boldsymbol{u}^*(t)$ such that the following objective function is minimized

$$J(\boldsymbol{u}(t), \boldsymbol{x}_0) = \int_0^\infty \boldsymbol{x}^T(t)\bar{\boldsymbol{Q}}^T\bar{\boldsymbol{Q}}\boldsymbol{x}(t) + \boldsymbol{u}^T(t)\boldsymbol{R}\,\boldsymbol{u}(t)\mathrm{d}t. \tag{9}$$

A stabilizing optimal control law

$$\boldsymbol{u}^*(t) = -\boldsymbol{K}^*\boldsymbol{x}(t) \tag{10}$$

can be obtained if and only if the weighting matrix $\boldsymbol{R}$ is symmetric positive definite and all unstable eigenvalues of $\boldsymbol{A}$ are observable by the pair $\left(\boldsymbol{A}, \bar{\boldsymbol{Q}}\right)$ and controllable by the pair $(\boldsymbol{A}, \boldsymbol{B})$. The solution of the minimization problem is the optimal state feedback (10), where

$$\boldsymbol{K}^* = \boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P} \tag{11}$$

$$\boldsymbol{P} = \boldsymbol{P}^T \succeq 0 : \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P} + \bar{\boldsymbol{Q}}^T\bar{\boldsymbol{Q}} = \boldsymbol{O}. \tag{12}$$

The solution of the ARE (12) is equal to

$$\boldsymbol{P} = \int_0^\infty \mathbf{e}^{\hat{\boldsymbol{A}}^T t}\left(\bar{\boldsymbol{Q}}^T\bar{\boldsymbol{Q}} + \boldsymbol{K}^{*T}\boldsymbol{R}\boldsymbol{K}^*\right)\mathbf{e}^{\hat{\boldsymbol{A}}t}\mathrm{d}t,$$

where $\hat{\boldsymbol{A}} = \boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}^*$ and $J^*(\boldsymbol{x}_0) = \boldsymbol{x}_0^T\boldsymbol{P}\,\boldsymbol{x}_0$. Hence, the minimization problem can be rewritten as follows:

***Corollary** 1: The stabilizing solution of the LQR-problem*

$$\min_{\boldsymbol{K}} \ \boldsymbol{x}_0^T\boldsymbol{P}\,\boldsymbol{x}_0, \qquad \forall \boldsymbol{x}_0 \in \mathbb{R}^n$$

$$\boldsymbol{P} = \int_0^\infty \mathbf{e}^{\hat{\boldsymbol{A}}^T t}\left(\bar{\boldsymbol{Q}}^T\bar{\boldsymbol{Q}} + \boldsymbol{K}^T\boldsymbol{R}\boldsymbol{K}\right)\mathbf{e}^{\hat{\boldsymbol{A}}t}\mathrm{d}t$$

is given by eqn. (11) and the ARE (12).

III. MODELS AND SYNCHRONIZATION CONDITION

A. Agent model and networked controller

The multi-agent system consists of N identical agents

$$\dot{\underline{\boldsymbol{x}}}(t) = \underline{\boldsymbol{A}}\,\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{B}}\,\underline{\boldsymbol{u}}(t), \qquad \underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0, \tag{13}$$

where $\underline{\boldsymbol{x}}^T(t) = \left(\boldsymbol{x}_1^T(t), \ldots, \boldsymbol{x}_N^T(t)\right)$ and $\underline{\boldsymbol{u}}^T(t) = \left(\boldsymbol{u}_1^T(t), \ldots, \boldsymbol{u}_N^T(t)\right)$ are vectors and $\underline{\boldsymbol{A}} = (\boldsymbol{I}_N \otimes \boldsymbol{A})$ and $\underline{\boldsymbol{B}} = (\boldsymbol{I}_N \otimes \boldsymbol{B})$ are block-diagonal matrices.

As shown in Fig. 1 the NC consists of the local controllers $\boldsymbol{K}$ and the communication network $\boldsymbol{L}$. The topology of the communication network can be described by an undirected graph $\mathcal{G}$, where the i-th and j-th agents are said to be connected if and only if $(\boldsymbol{L}_{\mathcal{G}})_{ij} \neq 0$.

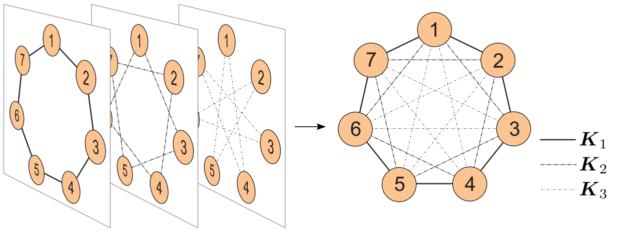


Fig. 3. Networked controller for $N = 7$ agents

This paper is concerned with the design of a NC with a circulant, multi-layer communication network

$$\boldsymbol{u}_i(t) = -\sum_{k=1}^{p}\boldsymbol{K}_k\,\boldsymbol{z}_{ik}(t) \tag{14}$$

with coupling signals

$$\boldsymbol{z}_{ik}(t) = -\sum_{j=1}^{N}\left(\boldsymbol{L}_{\mathrm{c},k}\right)_{ij}\left(\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\right), \ k \in \mathcal{N}_p. \tag{15}$$

This means that the communication graph is decomposed into subgraphs, where the interconnection between agents belonging to every subgraph is labeled by different local controllers. The topology of the k-th layer is described by the circulant Laplacian matrix $\boldsymbol{L}_{\mathrm{c},k}$ and the associated local controller matrix $\boldsymbol{K}_k$. Figure 3 illustrates the topology of the NC (14) – (15) for $N = 7$ agents. Equations (14) – (15) can be rewritten as the overall controller

$$\underline{\boldsymbol{u}}(t) = -\tilde{\boldsymbol{K}}\underline{\boldsymbol{x}}(t) \tag{16}$$

with $\tilde{\boldsymbol{K}} = \sum_{k=1}^{p}\left(\boldsymbol{L}_{\mathrm{c},k} \otimes \boldsymbol{K}_k\right)$. The all-to-all couplings of the NC (16) can be seen by considering the sum of the Laplacian matrices $\boldsymbol{L}_{\mathrm{c},k}$, $k \in \mathcal{N}_p$, which is the Laplacian of the complete graph $\mathcal{K}$.

B. Model of the overall closed-loop system

The networked multi-agent system is obtained by substitution of the control law (16) into (13):

$$\begin{aligned}\dot{\underline{\boldsymbol{x}}}(t) &= (\boldsymbol{I}_N \otimes \boldsymbol{A})\,\underline{\boldsymbol{x}}(t) - (\boldsymbol{I}_N \otimes \boldsymbol{B})\sum_{k=1}^{p}\left(\boldsymbol{L}_{\mathrm{c},k} \otimes \boldsymbol{K}_k\right)\underline{\boldsymbol{x}}(t)\\ &= \tilde{\boldsymbol{A}}\,\underline{\boldsymbol{x}}(t), \quad \underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0,\end{aligned} \tag{17}$$

where $\tilde{\boldsymbol{A}} = (\boldsymbol{I}_N \otimes \boldsymbol{A}) - \sum_{k=1}^{p}\left(\boldsymbol{L}_{\mathrm{c},k} \otimes \boldsymbol{B}\boldsymbol{K}_k\right)$. The solution of the system is described by

$$\underline{\boldsymbol{x}}(t) = \mathbf{e}^{\tilde{\boldsymbol{A}}t}\underline{\boldsymbol{x}}_0. \tag{18}$$

C. Synchronization condition

Conditions on synchronization of networked multi-agent systems were first investigated in [8]. It was shown that synchronization of networked agents can be proved by stability analysis of transformed subsystems. The following theorem provides a similar result for multi-agents system (17), which takes into account the multi-layer communication topology.

***Theorem** 1: All agents of the networked multi-agent system (17) are synchronized if and only if the matrices*

$$\tilde{\boldsymbol{A}}_i = \boldsymbol{A} - \sum_{k=1}^{p} \lambda_i \left\{ \boldsymbol{L}_{\mathrm{c},k} \right\} \boldsymbol{B} \boldsymbol{K}_k, \;\; i \in \mathcal{N}_p \tag{19}$$

are asymptotically stable.

Proof: According to requirement (1), the agents are synchronized if and only if every entry of the error vector

$$\Delta \boldsymbol{x}(t) = \begin{pmatrix} \boldsymbol{x}_1(t) - \boldsymbol{x}_2(t) \\ \boldsymbol{x}_2(t) - \boldsymbol{x}_3(t) \\ \vdots \\ \boldsymbol{x}_{N-1}(t) - \boldsymbol{x}_N(t) \end{pmatrix} = \left(\boldsymbol{V} \otimes \boldsymbol{I}_n \right) \underline{\boldsymbol{x}}(t) \tag{20}$$

or any invertible linear transformation of this vector converges asymptotically to zero. $\boldsymbol{V}$ is a $(N-1 \times N)$-matrix which by construction has a row sum equal to zero.

Using the state transformation

$$\tilde{\underline{\boldsymbol{x}}}(t) = \tilde{\boldsymbol{T}}^{-1} \underline{\boldsymbol{x}}(t) \tag{21}$$

with $\tilde{\underline{\boldsymbol{x}}}^T(t) = \left(\tilde{\boldsymbol{x}}_1^T(t), \ldots, \tilde{\boldsymbol{x}}_N^T(t) \right)$, $\tilde{\boldsymbol{T}} = (\boldsymbol{T} \otimes \boldsymbol{I}_n)$ and $\boldsymbol{T}$ given in (7) the solution (18) gets the following form:

$$\begin{aligned} \tilde{\underline{\boldsymbol{x}}}(t) &= \tilde{\boldsymbol{T}}^{-1} \mathbf{e}^{\tilde{\underline{\boldsymbol{A}}} t} \, \tilde{\boldsymbol{T}} \tilde{\boldsymbol{T}}^{-1} \underline{\boldsymbol{x}}_0 & (22) \\ &= \mathbf{e}^{\left(\boldsymbol{T}^{-1} \otimes \boldsymbol{I}_n \right) \tilde{\underline{\boldsymbol{A}}} \left(\boldsymbol{T} \otimes \boldsymbol{I}_n \right) t} \, \tilde{\underline{\boldsymbol{x}}}(0) \\ &= \mathbf{e}^{\left[\underline{\boldsymbol{A}} - \sum_{k=1}^{p} \left(\boldsymbol{T}^{-1} \boldsymbol{L}_{\mathrm{c},k} \boldsymbol{T} \otimes \boldsymbol{B} \boldsymbol{K}_k \right) \right] t} \, \tilde{\underline{\boldsymbol{x}}}(0) \\ &= \mathbf{e}^{\left[\underline{\boldsymbol{A}} - \sum_{k=1}^{p} \left(\operatorname*{diag}_{i=0}^{N-1} \left(\lambda_i \left\{ \boldsymbol{L}_{\mathrm{c},k} \right\} \right) \otimes \boldsymbol{B} \boldsymbol{K}_k \right) \right] t} \, \tilde{\underline{\boldsymbol{x}}}(0) \\ &= \operatorname*{diag}_{i=0}^{N-1} \left(\mathbf{e}^{\tilde{\boldsymbol{A}}_i t} \right) \tilde{\underline{\boldsymbol{x}}}(0) & (23) \end{aligned}$$

with $\tilde{\boldsymbol{A}}_0 = \boldsymbol{A}$ and $\tilde{\boldsymbol{A}}_i$, $i \in \mathcal{N}_p$ given in (19). By virtue of Lemma 1, $\tilde{\boldsymbol{A}}_{N-i} = \tilde{\boldsymbol{A}}_i$, for all $i \in \mathcal{N}_p$. Since the first eigenvector of all Laplacian matrices $\boldsymbol{L}_{\mathrm{c},k}$, $k \in \mathcal{N}_p$ is given by the one vector, the relation

$$\boldsymbol{V} \boldsymbol{T} = \boldsymbol{V} \left(\mathbb{1}_N \;\; \boldsymbol{t}_2 \;\; \cdots \;\; \boldsymbol{t}_N \right) = \left(\; \boldsymbol{0}_{N-1} \; \middle| \; \hat{\boldsymbol{V}} \; \right) \tag{24}$$

holds. $\hat{\boldsymbol{V}}$ is a square invertible $(N-1 \times N-1)$-matrix. Applying the transformation (21) with eqns. (23) and (24) to the error vector (20), yields

$$\begin{aligned} \Delta \boldsymbol{x}(t) &= \left(\boldsymbol{V} \otimes \boldsymbol{I}_n \right) \tilde{\boldsymbol{T}} \tilde{\boldsymbol{T}}^{-1} \underline{\boldsymbol{x}}(t) = \left(\boldsymbol{V} \boldsymbol{T} \otimes \boldsymbol{I}_n \right) \tilde{\underline{\boldsymbol{x}}}(t) \\ &= \left[\left(\; \boldsymbol{0}_{N-1} \; \middle| \; \hat{\boldsymbol{V}} \; \right) \otimes \boldsymbol{I}_n \right] \operatorname*{diag}_{i=0}^{N-1} \left(\mathbf{e}^{\tilde{\boldsymbol{A}}_i t} \right) \tilde{\underline{\boldsymbol{x}}}(0) \\ &= \left(\hat{\boldsymbol{V}} \otimes \boldsymbol{I}_n \right) \begin{pmatrix} \mathbf{e}^{\tilde{\boldsymbol{A}}_1 t} \tilde{\boldsymbol{x}}_2(0) \\ \vdots \\ \mathbf{e}^{\tilde{\boldsymbol{A}}_{N-1} t} \tilde{\boldsymbol{x}}_N(0) \end{pmatrix}. \end{aligned}$$

Considering the last investigation, the networked multi-agent system (17) is synchronized if and only if the matrices $\tilde{\boldsymbol{A}}_i$, $i \in \mathcal{N}_p$ are stable. ■

IV. LQR-BASED CONTROLLER DESIGN

Considering the multi-agent system (13), the optimization problem concerned for the design of the NC is given by

$$\min_{\boldsymbol{K}_k} \;\; J(\underline{\boldsymbol{u}}(t), \, \underline{\boldsymbol{x}}_0) \tag{25}$$

$$s.t. \;\; \dot{\underline{\boldsymbol{x}}}(t) = \underline{\boldsymbol{A}} \, \underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{B}} \, \underline{\boldsymbol{u}}(t), \;\; \underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0 \tag{26}$$

$$\underline{\boldsymbol{u}}(t) = - \sum_{k=1}^{p} \left(\boldsymbol{L}_{\mathrm{c},k} \otimes \boldsymbol{K}_k \right) \underline{\boldsymbol{x}}(t), \tag{27}$$

where the objective function (3) weights the synchronization errors $\boldsymbol{e}_{ij}(t)$ according to the following circulant structure:

$$\boldsymbol{Q}_{ij} = \sum_{k=1}^{p} \left| \left(\boldsymbol{L}_{\mathrm{c},k} \right)_{ij} \right| \bar{\boldsymbol{Q}}_k^T \bar{\boldsymbol{Q}}_k. \tag{28}$$

The overall representation of the objective function (3) with weighting matrices (28) and $\underline{\boldsymbol{R}} = (\boldsymbol{I}_N \otimes \boldsymbol{R})$ is given by

$$\begin{aligned} J \left(\underline{\boldsymbol{u}}(t), \, \underline{\boldsymbol{x}}_0 \right) = \int_0^\infty \underline{\boldsymbol{x}}^T(t) \sum_{k=1}^{p} \left(\boldsymbol{L}_{\mathrm{c},k} \otimes \bar{\boldsymbol{Q}}_k^T \bar{\boldsymbol{Q}}_k \right) \underline{\boldsymbol{x}}(t) + \\ \underline{\boldsymbol{u}}^T(t) \, \underline{\boldsymbol{R}} \, \underline{\boldsymbol{u}}(t) \mathrm{d}t. \end{aligned} \tag{29}$$

To ensure that the objective function (29) is influenced by the unstable modes of the synchronization errors $\boldsymbol{e}_{ij}(t)$, the following requirements on the weighting matrices $\boldsymbol{Q}_k = \bar{\boldsymbol{Q}}_k^T \bar{\boldsymbol{Q}}_k$ and $\boldsymbol{R}$ have to be made:

(R1) the pair $\left(\boldsymbol{A}, \bar{\boldsymbol{Q}}_1 \right)$ is completely observable,
(R2) $\boldsymbol{R} = \boldsymbol{R}^T \succ 0$.

The following theorem is the main result of this paper.

***Theorem** 2: The solution of the optimization problem (25) – (27) with objective function (29) is given by*

$$\boldsymbol{K}_k^* = \boldsymbol{R}^{-1} \boldsymbol{B}^T \boldsymbol{P}_k^*, \quad k \in \mathcal{N}_p \tag{30}$$

with

$$\left(\boldsymbol{P}_1^* \;\; \boldsymbol{P}_2^* \;\; \cdots \;\; \boldsymbol{P}_p^* \right) = \left(\tilde{\boldsymbol{P}}_1 \;\; \tilde{\boldsymbol{P}}_2 \;\; \cdots \;\; \tilde{\boldsymbol{P}}_p \right) \left(\Lambda^{-1} \otimes \boldsymbol{I}_n \right). \tag{31}$$

The matrices $\tilde{\boldsymbol{P}}_i$, $i \in \mathcal{N}_p$ are the symmetric positive definite solutions of the following AREs

$$\boldsymbol{A}^T \tilde{\boldsymbol{P}}_i + \tilde{\boldsymbol{P}}_i \boldsymbol{A} - \tilde{\boldsymbol{P}}_i \boldsymbol{B} \boldsymbol{R}^{-1} \boldsymbol{B}^T \tilde{\boldsymbol{P}}_i + \sum_{k=1}^{p} \lambda_i \left\{ \boldsymbol{L}_{\mathrm{c},k} \right\} \boldsymbol{Q}_k = \boldsymbol{O}. \tag{32}$$

$\Lambda \in \mathbb{R}^{p \times p}$ has the entries $(\Lambda)_{ik} = \lambda_i \left\{ \boldsymbol{L}_{\mathrm{c},k} \right\}$, given in (5).

Proof: The proof is divided into three steps:

1. Combining the objective function and the constraints. Substituting the constraints (26) – (27) into the objective function (29), reduces the optimization problem (25) – (27) to the following representation

$$\min_{\boldsymbol{K}_k} \;\; \underline{\boldsymbol{x}}_0^T \int_0^\infty \mathbf{e}^{\tilde{\underline{\boldsymbol{A}}}^T t} \tilde{\underline{\boldsymbol{Q}}} \, \mathbf{e}^{\tilde{\underline{\boldsymbol{A}}} t} \mathrm{d}t \, \underline{\boldsymbol{x}}_0 \tag{33}$$

with

$$\tilde{\underline{\boldsymbol{Q}}} = \sum_{k=1}^{p} \left(\boldsymbol{L}_{\mathrm{c},k} \otimes \boldsymbol{Q}_k \right) + \tilde{\underline{\boldsymbol{K}}}^T \underline{\boldsymbol{R}} \, \tilde{\underline{\boldsymbol{K}}}. \tag{34}$$

2. Decomposition of the overall optimization problem. Using the state transformation

$$\tilde{\boldsymbol{x}}(t) = \tilde{\boldsymbol{T}}^{-1}\underline{\boldsymbol{x}}(t)$$

with transformation matrix $\tilde{\boldsymbol{T}} = (\boldsymbol{T} \otimes \boldsymbol{I}_n)$ and the unitary matrix $\boldsymbol{T}^T = \boldsymbol{T}^{-1}$ given in (7), the optimization problem (33) gets the following form:

$$\min_{\boldsymbol{K}_k} \; \underline{\boldsymbol{x}}_0^T \tilde{\boldsymbol{T}} \int\limits_0^\infty \tilde{\boldsymbol{T}}^{-1} \mathrm{e}^{\tilde{\boldsymbol{A}}^T t} \, \tilde{\boldsymbol{T}} \tilde{\boldsymbol{T}}^{-1} \tilde{\boldsymbol{Q}} \tilde{\boldsymbol{T}} \tilde{\boldsymbol{T}}^{-1} \, \mathrm{e}^{\tilde{\boldsymbol{A}} t} \, \tilde{\boldsymbol{T}} \mathrm{d}t \, \tilde{\boldsymbol{T}}^{-1} \underline{\boldsymbol{x}}_0.$$

The transformation of the exponential function was already performed in eqns. (22) – (23). The transformed weighting matrix is block diagonal, since

$$\begin{aligned} \tilde{\boldsymbol{T}}^{-1} \tilde{\boldsymbol{Q}} \tilde{\boldsymbol{T}} = & \sum_{k=1}^{p} \left(\boldsymbol{T}^{-1} \boldsymbol{L}_{\mathrm{c},k} \boldsymbol{T} \otimes \boldsymbol{Q}_k \right) + \\ & \sum_{k=1}^{p} \left(\boldsymbol{T}^{-1} \boldsymbol{L}_{\mathrm{c},k} \boldsymbol{T} \otimes \boldsymbol{K}_k^T \right) \underline{\boldsymbol{R}} \sum_{k=1}^{p} \left(\boldsymbol{T}^{-1} \boldsymbol{L}_{\mathrm{c},k} \boldsymbol{T} \otimes \boldsymbol{K}_k \right) \\ = & \operatorname*{diag}_{i=0}^{N-1} \left(\tilde{\boldsymbol{Q}}_i \right) \end{aligned}$$

with diagonal entries $\tilde{\boldsymbol{Q}}_i$ given by

$$\tilde{\boldsymbol{Q}}_i = \sum_{k=1}^{p} \lambda_i \left\{ \boldsymbol{L}_{\mathrm{c},k} \right\} \boldsymbol{Q}_k + \hat{\boldsymbol{K}}_i^T \boldsymbol{R} \hat{\boldsymbol{K}}_i$$

and

$$\hat{\boldsymbol{K}}_i = \sum_{k=1}^{p} \lambda_i \left\{ \boldsymbol{L}_{\mathrm{c},k} \right\} \boldsymbol{K}_k. \tag{35}$$

As all the matrices in the objective function are shown to be block diagonal, the optimization problem can be rewritten as a block diagonal representation too:

$$\min_{\boldsymbol{K}_k} \; \tilde{\boldsymbol{x}}_0^T \operatorname*{diag}_{i=0}^{N-1} \left(\int\limits_0^\infty \mathrm{e}^{\tilde{\boldsymbol{A}}_i^T t} \, \tilde{\boldsymbol{Q}}_i \, \mathrm{e}^{\tilde{\boldsymbol{A}}_i t} \mathrm{d}t \right) \tilde{\boldsymbol{x}}_0.$$

To show that the optimization problem can now be solved by considering subproblems of a single-agent order, notice that the minimization over $\boldsymbol{K}_k$ can, by substitution, be transfered to the minimization over $\hat{\boldsymbol{K}}_i$:

$$\begin{aligned} & \min_{\hat{\boldsymbol{K}}_i} \; \sum_{i=0}^{N-1} J_i(\hat{\boldsymbol{K}}_i, \, \tilde{\boldsymbol{x}}_{0i}), \\ & J_i(\hat{\boldsymbol{K}}_i, \, \tilde{\boldsymbol{x}}_{0i}) = \tilde{\boldsymbol{x}}_{0i}^T \int\limits_0^\infty \mathrm{e}^{\tilde{\boldsymbol{A}}_i^T t} \, \tilde{\boldsymbol{Q}}_i \, \mathrm{e}^{\tilde{\boldsymbol{A}}_i t} \mathrm{d}t \, \tilde{\boldsymbol{x}}_{0i}. \end{aligned} \tag{36}$$

As the objective functions given in eqn. (36) distinguish upon the eigenvalues $\lambda_i \left\{ \boldsymbol{L}_{\mathrm{c},k} \right\}$, a consequence of Lemma 1 is that the objective functions (36) are pairwise equal:

$$J_\alpha(\hat{\boldsymbol{K}}_i, \, \tilde{\boldsymbol{x}}_{0i}) = J_{N-\alpha}(\hat{\boldsymbol{K}}_i, \, \tilde{\boldsymbol{x}}_{0i}), \qquad \alpha \in \mathcal{N}_p. \tag{37}$$

The first cost function J_0 vanishes, since all first eigenvalues of the Laplacian matrices $\boldsymbol{L}_{\mathrm{c},k}$, $k \in \mathcal{N}_p$ are zero:

$$J_0(\hat{\boldsymbol{K}}_i, \, \tilde{\boldsymbol{x}}_{0i}) = 0. \tag{38}$$

Considering the last two investigations (37) – (38) and the property that the minimization of a sum of positive functions can be obtained by minimizing the functions independently, the solution of the optimization problem (25) – (27) is separable, and hence can be determined by solving the subproblems

$$\min_{\hat{\boldsymbol{K}}_i} \; J_i(\hat{\boldsymbol{K}}_i, \, \tilde{\boldsymbol{x}}_{0i}), \qquad i \in \mathcal{N}_p. \tag{39}$$

3. Solution of the subproblems. The solutions of the optimization problems (39) with objective functions (36) are

$$\hat{\boldsymbol{K}}_i^* = \boldsymbol{R}^{-1} \boldsymbol{B}^T \tilde{\boldsymbol{P}}_i, \quad i \in \mathcal{N}_p \tag{40}$$

(cf. Corollary 1), where $\tilde{\boldsymbol{P}}_i$ are the symmetric positive definite solutions of AREs

$$\boldsymbol{A}^T \tilde{\boldsymbol{P}}_i + \tilde{\boldsymbol{P}}_i \boldsymbol{A} - \tilde{\boldsymbol{P}}_i \boldsymbol{B} \boldsymbol{R}^{-1} \boldsymbol{B}^T \tilde{\boldsymbol{P}}_i + \sum_{k=1}^{p} \lambda_i \left\{ \boldsymbol{L}_{\mathrm{c},k} \right\} \boldsymbol{Q}_k = \boldsymbol{O}. \tag{41}$$

Considering the sum in eqn. (35) for every $i \in \mathcal{N}_p$ as a matrix multiplication shows that the NC (16) can be obtained by applying the transformation

$$(\, \hat{\boldsymbol{K}}_1^* \;\, \hat{\boldsymbol{K}}_2^* \;\, \cdots \;\, \hat{\boldsymbol{K}}_p^* \,) = (\, \boldsymbol{K}_1^* \;\, \boldsymbol{K}_2^* \;\, \cdots \;\, \boldsymbol{K}_p^* \,) \, (\Lambda \otimes \boldsymbol{I}_n) \tag{42}$$

with $(\Lambda)_{ij} = \lambda_j \left\{ \boldsymbol{L}_{\mathrm{c},i} \right\} = \lambda_i \left\{ \boldsymbol{L}_{\mathrm{c},j} \right\}$. Hence, by combining eqns. (40) and (42), the controllers $\boldsymbol{K}_k^*$ are given by

$$\begin{aligned} (\, \boldsymbol{K}_1^* \;\, \boldsymbol{K}_2^* \;\, \cdots \;\, \boldsymbol{K}_p^* \,) &= (\, \hat{\boldsymbol{K}}_1^* \;\, \hat{\boldsymbol{K}}_2^* \;\, \cdots \;\, \hat{\boldsymbol{K}}_p^* \,) \left(\Lambda^{-1} \otimes \boldsymbol{I}_n \right), \\ &= \boldsymbol{R}^{-1} \boldsymbol{B}^T (\, \boldsymbol{P}_1^* \;\, \boldsymbol{P}_2^* \;\, \cdots \;\, \boldsymbol{P}_p^* \,), \end{aligned} \tag{43}$$

where

$$(\boldsymbol{P}_1^* \;\, \boldsymbol{P}_2^* \;\, \cdots \;\, \boldsymbol{P}_p^*) = (\tilde{\boldsymbol{P}}_1 \;\, \tilde{\boldsymbol{P}}_2 \;\, \cdots \;\, \tilde{\boldsymbol{P}}_p) \left(\Lambda^{-1} \otimes \boldsymbol{I}_n \right). \tag{44}$$

The theorem is proved by summarizing eqns. (41) – (44). ■

Comparing eqns. (39) and (36) as a representation of the optimization problem (25) – (27) to Corollary 1 shows that Theorem 1 is fulfilled. Thus, the NC provided by Theorem 2 synchronizes the agents asymptotically.

Theorem 2 provides a solution of the optimization problem (25) – (27), which synchronizes the agents by minimizing the weighted synchronization errors $\boldsymbol{e}_{ij}(t)$, $i, j, \in \mathcal{N}$. The associated objective function (29) can easily be rewritten as performance index (2). Hence, the provided design method leads to a NC for which the networked agents do not only satisfy requirement (1) but also minimize the performance index (2). The quality of the transient behavior can be influenced by the choice of the weighting matrices $\boldsymbol{Q}_k$, $k \in \mathcal{N}_p$.

V. EXAMPLE

As an example, angle synchronization of $N = 9$ rolls of a drying section of a paper converting machine (Fig. 4) is considered. In a paper converting machine paper is used to produce another paper products. For the processing of the paper it has to be transported by rolls. To avoid a tearing of the paper the rolls have to be synchronized.

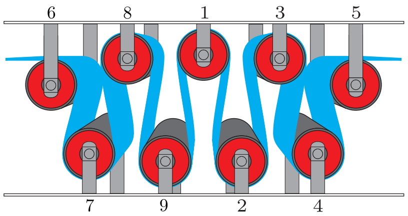


Fig. 4. Drying section of a paper converting machine with $N = 9$ rolls

The model of the i-th agent is given by

$$\dot{\boldsymbol{x}}_i(t) = \boldsymbol{A}\boldsymbol{x}_i(t) + \boldsymbol{b}\,u_i(t),\ \boldsymbol{x}_i(0) = \boldsymbol{x}_{0i},\ (i = 1,\,2,\,\ldots,\,9)$$
$$y_i(t) = \boldsymbol{c}^T\boldsymbol{x}_i(t),$$
$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -0.01 & 0.2 \\ 0 & 0 & -125 \end{pmatrix},\ \boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \\ 20 \end{pmatrix},\ \boldsymbol{c} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Theorem 2 shows that the optimal NC can be designed by solving $p = 4$ AREs (32) of a single-agent order. The objective function (29) of the optimization problem (25) – (27) with positive definite weighting matrices chosen as

$$\boldsymbol{Q}_1 = \begin{pmatrix} \alpha\,30 & 0 & 0 \\ 0 & 80 & 0 \\ 0 & 0 & 0.1 \end{pmatrix},\ \boldsymbol{Q}_2 = \boldsymbol{Q}_3 = \boldsymbol{Q}_4 = \boldsymbol{O},\ \boldsymbol{R} = 1$$

is for $\alpha = \{1,\,10\}$ minimized by the following two controller parameter sets:

α	1	10
$\boldsymbol{k}_1^T$	$(\,2.40\ \ 6.03\ \ 0.02\,)$	$(\,7.58\ \ 9.23\ \ 0.02\,)$
$\boldsymbol{k}_2^T$	$(\,0.54\ \ 2.19\ \ 0.00\,)$	$(\,1.72\ \ 3.75\ \ 0.01\,)$
$\boldsymbol{k}_3^T$	$(\,0.29\ \ 1.53\ \ 0.00\,)$	$(\,0.91\ \ 2.68\ \ 0.00\,)$
$\boldsymbol{k}_4^T$	$(\,0.22\ \ 1.33\ \ 0.00\,)$	$(\,0.70\ \ 2.35\ \ 0.00\,)$

The NC results from the linear combination with Laplacian matrices $\boldsymbol{L}_{\mathrm{c},k}$, $(k = 1,\,2,\,3,\,4)$:

$$\underline{\boldsymbol{K}}^{*} = (\boldsymbol{L}_{\mathrm{c},1}\otimes\boldsymbol{k}_1^T) + (\boldsymbol{L}_{\mathrm{c},2}\otimes\boldsymbol{k}_2^T) + (\boldsymbol{L}_{\mathrm{c},3}\otimes\boldsymbol{k}_3^T) + (\boldsymbol{L}_{\mathrm{c},4}\otimes\boldsymbol{k}_4^T).$$

Each of the sum terms represents a layer of the NC.

In order to reveal the effect of the weighting matrix $\boldsymbol{Q}_1$ on the transient behavior of the networked agents, the initial state of only the first agent is picked to be different from zero

$$\boldsymbol{x}_{01}^T = \begin{pmatrix} 5 & 0 & 0 \end{pmatrix},\ \boldsymbol{x}_{0i} = \boldsymbol{0}_3,\ i = 2,\,3,\,\ldots,\,9.$$

Figure 5 shows the transient behavior of the networked agents, where the upper simulations show the case for $\alpha = 1$ and the lower for $\alpha = 10$. The parameter α influences the weighting of output differences $y_i(t) - y_j(t)$, $i \neq j \in \{1,\,2,\,\ldots,\,9\}$. Increasing the parameter α provides a faster synchronization (Fig. 5(a)). Figure 5(b) shows the synchronization errors between agents . The agents synchronize after a transition time of $t = 15s$ for $\alpha = 1$ and $t = 10s$ for $\alpha = 10$.

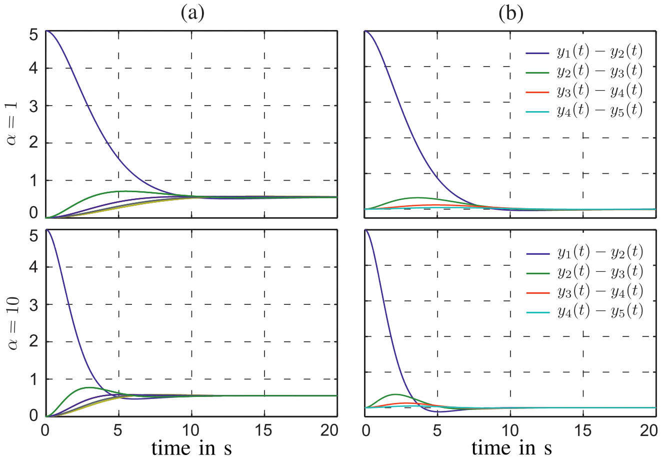


Fig. 5. (a) Angles of the agents during the transient behavior, (b) Transient behavior of the synchronization errors

VI. CONCLUSION

This paper studies the optimal design of a NC with a communication graph, which is set up by layers of circulant subgraphs. For the networked multi-agent system which consists of identical agents, conditions for the synchronization have been obtained. The main contribution of this paper is the introduced LQR-based design approach. Since the value of the objective function depends only on the synchronization errors and the control input, the resulting NC leads to an optimal transient behavior.

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