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Stability and Stabilization of Discrete-Time Descriptor Systems with Several Extensions

Izumi Masubuchi*‡ and Yuzo Ohta*

Abstract— **This paper is concerned with stability criterion of discrete-time descriptor systems in terms of LMIs. Though there have been proposed several LMIs for stability of discrete-time descriptor systems, their use is seemingly limited in comparison with LMIs for continuous-time descriptor systems. A remedy of those existing results is that they are not able to be linearized with respect to the coefficient matrices of the descriptor equation. This makes it difficult to apply them to synthesis problems, while LMIs for analysis of state-space systems or those for continuous-time descriptor systems enjoy change-of-variable methods to obtain LMIs for synthesis. In this paper, a new LMI is presented for admissibility of discrete-time descriptor systems that has such a useful feature of affineness of coefficient matrices in the inequality. After showing the main result with the new LMI, we apply it to several problems on discrete-time descriptor systems, namely, state-feedback synthesis, stability analysis of δ-operator systems and $\mathcal{D}$-stability problems generalized to descriptor systems, and a generalization of quadratic stabilization. Some of these are illustrated via numerical examples.**

I. INTRODUCTION

Descriptor systems have been receiving a great deal of attention for many decades as a representation of dynamical systems [1], [2], [3]. As development of theory of Lyapunov equations and LMI-based methods for state-space systems, also generalized Lyapunov equations and LMIs have been proposed for linear descriptor systems for stability or admissibility, H_∞ and H_2 norm conditions, $\mathcal{D}$-stability and so forth. For continuous-time systems, the Lyapunov and Riccati equations shown by Takaba et. al. [5], [6], [7] prompted LMIs for descriptor systems [24], which enjoys convexification results for state-feedback and output-feedback synthesis problems for continuous-time descriptor systems.

Also for discrete-time descriptor systems ($Ex_{k+1} = Ax_k$), several LMIs have been proposed that provide a necessary and sufficient condition of the admissibility. Similarly to the Lyapunov inequality for state-space systems, they have a term quadratic to A. In the state-space case, such quadratic dependence is removed in an equivalent inequality derived via Schur complement, and affineness with respect to A leads to convexification of LMIs for constant state-feedback synthesis problems with simple change-of-variables involving the feedback gain. However, unlike the state-space case, those LMIs proposed for discrete-time descriptor systems do not have this property due to indefinite sign of matrices in the term quadratic in A [12], [14], [15], [13], [20], [21], [16], [22], [10] or a slack variable that is linked to the feedback gain [19], [21], [23], [11]. It is therefore difficult to linearize matrix inequalities for synthesis shown in these existing results.

In this paper, we propose a new LMIs for admissibility of discrete-time descriptor systems in a similar form to those in [24], [25] for continuous-time descriptor systems. These LMIs are also affine with respect to A as well as linear in the unknown variable of the LMIs. For state (descriptor variable) feedback control problem, via a simple change-of-variables performed on the terms of product of the unknown variable and the state-feedback gain, we derive easily an LMI for synthesis. The results are extended to descriptor systems with δ-operator and to analysis problem of $\mathcal{D}$-admissibility [22] for descriptor systems. Numerical examples are provided that support the results, as well as showing an LMI for a generalization of quadratic stabilization, which is also derived easily from the basic result of this paper.

The rest of the paper is organized as follows. In Section II, we provide the main result with an new LMI after brief preliminaries. Then an LMI for synthesis is presented. Section III extends the main result to stability analysis of δ-operator systems in the descriptor form and $\mathcal{D}$-admissibility, which is a generalization of $\mathcal{D}$-stability of state-space systems. In Section IV, we illustrate methods based on the main result by numerical examples, where we show an application of the methodology of quadratic stabilization to discrete-time descriptor systems, deriving an LMI to guarantee robust stability of discrete-time descriptor systems under norm-bounded uncertainty. Lastly, the paper is concluded in Section V.

Notation: Symbols $\mathbf{R}$, $\mathbf{C}$, $\mathbf{Z}$ represent real, complex numbers and integers, respectively. $\mathbf{R}^n$, $\mathbf{R}^{m\times n}$ are the set of n-dimensional real vectors and the set of real matrices of the size of $m \times n$, respectively. M^T stands for the transpose of a matrix M and $\mathsf{sym}\, M = M + M^\mathsf{T}$. Blocks of symmetric matrices are abbreviated with '$*$' if they are known from the other block, such as

$$\begin{bmatrix} A & B \\ B^\mathsf{T} & C \end{bmatrix} = \begin{bmatrix} A & B \\ * & C \end{bmatrix} = \begin{bmatrix} A & * \\ B^\mathsf{T} & C \end{bmatrix}.$$

A square matrix is said to be Schur stable if all of its eigenvalues belong to the open unit disk of the complex plane. For a matrix E that has n columns with $\operatorname{rank} E = r$, $E^\perp \in \mathbf{R}^{n\times n-r}$ is a full-rank matrix whose kernel coincides with the range space of E. The pseudo inverse of a matrix E is denoted by $E^\dagger$. $\bar{\sigma}(M)$ is the largest singular value of matrix M. I_n stands for the $n\times n$ identity matrix. The degree of a polynomial $f(z)$ is denoted by $\deg f(z)$.

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*The authors are with Graduate School of System Informatics, Kobe University, 1-1 Rokkodai, Nada, Kobe 657-8501, Japan.

‡Corresponding author. E-mail: msb@harbor.kobe-u.ac.jp

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II. LMIS FOR STABILITY AND STABILIZATION

A. Preliminaries

Let us consider the following discrete-time descriptor system:

$$Ex_{k+1} = Ax_k, \tag{1}$$

where $x_k \in \mathbf{R}^n$ is the descriptor variable, with $k \in \mathbf{Z}$. The matrices E and A are square and $\operatorname{rank} E = r$, where r is typically less than n in this paper.

Definition 1: (Dai [3]) The system (1), or the pair (E, A), is said to be

1) *regular* if $\det(zE - A)$ is not identically zero,
2) *causal* if $\deg\det(zE - A) = \operatorname{rank} E$ for all $z \in \mathbf{C}$,
3) *stable* if all the finite eigenvalues λ of (E, A) satisfy $|\lambda| < 1$, and
4) *admissible* if it is regular, causal and stable.

Let $U, V \in \mathbf{R}^{n\times n}$ be orthogonal matrices that give a singular-value decomposition (SVD) of E as

$$E = U \begin{bmatrix} \Sigma_0 & 0 \\ 0 & 0 \end{bmatrix} V^{\mathsf{T}}, \tag{2}$$

where $\Sigma_0 \in \mathbf{R}^{r\times r}$ is a diagonal matrix with positive diagonal entries. Define A_{ij} by

$$A = U \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} V^{\mathsf{T}}, \quad A_{11} \in \mathbf{R}^{r\times r}. \tag{3}$$

Lemma 1: (Dai [3]) The system (1) is admissible if and only if A_{22} is invertible and

$$A_0 := \Sigma_0^{-1}(A_{11} - A_{12}A_{22}^{-1}A_{21}) \tag{4}$$

is Schur stable.

Several stability criteria have been proposed for admissibility of discrete-time descriptor systems in terms of matrix inequalities. They are generalizations of the widely-known Lyapunov inequality for state-space systems. A key for such generalization is to guarantee nonsingularity of A_{22} in the SVD form. This is reduced to positive definiteness of certain matrices, in several different ways. The following Lemma summarizes some of them [12], [19], [21]:

Lemma 2: The following statements are equivalent:

1) (E, A) is admissible.
2) There exists a symmetric matrix $P \in \mathbf{R}^{n\times n}$ that satisfies
$$E^{\mathsf{T}}PE \geq 0, \quad A^{\mathsf{T}}PA < EPE^{\mathsf{T}}. \tag{5}$$
3) There exist a positive definite matrix $P \in \mathbf{R}^{n\times n}$ and a symmetric matrix $Q \in \mathbf{R}^{(n-r)\times(n-r)}$ such that
$$A^{\mathsf{T}}(P - E^{\perp\mathsf{T}}QE^{\perp})A - E^{\mathsf{T}}PE < 0. \tag{6}$$
4) There exist a positive definite matrix $P \in \mathbf{R}^{n\times n}$ and a matrix $Q \in \mathbf{R}^{n\times(n-r)}$ such that
$$A^{\mathsf{T}}PA - E^{\mathsf{T}}PE + QE^{\perp}A + A^{\mathsf{T}}E^{\perp\mathsf{T}}Q^{\mathsf{T}} < 0. \tag{7}$$

These criteria reduce to the LMI for state-space systems:

$$P = P^{\mathsf{T}} > 0, \quad P > A^{\mathsf{T}}PA \tag{8}$$

if $E = I_n$.

The condition 2) is utilized in [12], [14], [15], [13], [20], [16], [22], [10], where the inequality (5) implies sign-indefiniteness of P. Though the conditions 3) [21], [11] and 4) [19], [23] are stated with positive definite matrix P, the matrix inequalities include a slack variable Q which is multiplied with A. Because of such sign-indefiniteness or a slack variable, it is not as straightforward as in the state-space case to apply such inequalities to derive an LMI for state-feedback synthesis. Some nonconvexity is remained or only partial convexification limited to e.g. synthesis feedback of dynamical part of the descriptor variable is available.

B. Main result

In the following theorem, we show a new LMI that provides a criterion of admissibility of the discrete-time descriptor system (1). This is a discrete-time version of the one in [24] for continuous-time systems. Let $N := I_n - E^{\dagger}E$.

Theorem 1: The pair (E, A) is admissible if and only if there exist a matrix $X \in \mathbf{R}^{n\times n}$ and a scalar α that satisfy

$$\begin{cases} E^{\mathsf{T}}X = X^{\mathsf{T}}E \geq 0, \\ \begin{bmatrix} \mathsf{sym}\{(E-A)^{\mathsf{T}}X\} & (E-A)^{\mathsf{T}}X \\ X^{\mathsf{T}}(E-A) & E^{\mathsf{T}}X + \alpha N \end{bmatrix} > 0. \end{cases} \tag{9}$$

Remark 1: The second inequality of (9) implies $\alpha > 0$ if $\operatorname{rank} E < n$.

Proof: (Sufficiency.) Suppose that (9) holds. Using the representations of E and A in (2) and (3), the condition $E^{\mathsf{T}}X = X^{\mathsf{T}}E \geq 0$ implies

$$X = U \begin{bmatrix} \Sigma_0^{-1}P_0 & 0 \\ X_{12} & X_{22} \end{bmatrix} V^{\mathsf{T}}, \tag{10}$$

where $P_0 \in \mathbf{R}^{r\times r}$ and $P_0 = P_0^{\mathsf{T}} \geq 0$. Then the $(1,1)$-block of (9) is represented as

$$V\,\mathsf{sym}\left(\begin{bmatrix} \Sigma_0 - A_{11}^{\mathsf{T}} & -A_{21}^{\mathsf{T}} \\ -A_{12}^{\mathsf{T}} & -A_{22}^{\mathsf{T}} \end{bmatrix}\begin{bmatrix} \Sigma_0^{-1}P_0 & 0 \\ X_{12} & X_{22} \end{bmatrix}\right)V^{\mathsf{T}},$$

which is positive definite. In particular, we have $\mathsf{sym}(A_{22}^{\mathsf{T}}X_{22}) < 0$, which implies that A_{22} is nonsingular. Setting

$$\begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \end{bmatrix} = \Sigma_0^{-1}\begin{bmatrix} A_{11} & A_{12} \end{bmatrix},$$

we see

$$\begin{bmatrix} \mathsf{sym}\left(\begin{bmatrix} I_r - \hat{A}_{11}^{\mathsf{T}} & -A_{21}^{\mathsf{T}} \\ -\hat{A}_{12}^{\mathsf{T}} & -A_{22}^{\mathsf{T}} \end{bmatrix}\begin{bmatrix} P_0 & 0 \\ X_{12} & X_{22} \end{bmatrix}\right) & \begin{bmatrix} I_r - \hat{A}_{11}^{\mathsf{T}} & -A_{21}^{\mathsf{T}} \\ -\hat{A}_{12}^{\mathsf{T}} & -A_{22}^{\mathsf{T}} \end{bmatrix}\begin{bmatrix} P_0 & 0 \\ X_{12} & X_{22} \end{bmatrix} \\ * & \begin{bmatrix} P_0 & 0 \\ 0 & \alpha I_{n-r} \end{bmatrix} \end{bmatrix} > 0 \tag{11}$$

from (9). Multiplying to this inequality

$$\begin{bmatrix} I_r & 0 \\ -A_{22}^{-1}A_{21} & 0 \\ 0 & I_{n-r} \\ 0 & 0 \end{bmatrix}$$

from right and its transpose from left, we obtain

$$\begin{bmatrix} \mathsf{sym}\{(I_r - A_0)^{\mathsf{T}} P_0\} & (I_r - A_0)^{\mathsf{T}} P_0 \\ * & P_0 \end{bmatrix} > 0,$$

which reduces to $P_0 > 0$ and $P_0 > A_0^{\mathsf{T}} P_0 A_0$. Thus A_0 is Schur stable and the proof of the sufficiency is completed.

(Necessity.) From Lemma 1, A_{22} in the SVD form (3) is nonsingular and A_0 in (4) is Schur stable if (E, A) is admissible. Hence, we see that again (9) is equivalent to (11), with $P_0 > 0$ being a solution to the Lyapunov inequality $P_0 > A_0^{\mathsf{T}} P_0 A_0$. Substitute

$$X_{22} = -\alpha A_{22}^{-\mathsf{T}}, \tag{12}$$

$$X_{12} = X_{22} A_{12}^{-1} A_{21} - A_{22}^{-\mathsf{T}} \hat{A}_{12}^{\mathsf{T}} P_0, \tag{13}$$

which are particular choices of components of X, to LHS of (11) and multiply

$$\begin{bmatrix} I_r & 0 & 0 & 0 \\ -A_{22}^{-1}A_{21} & I_{n-r} & 0 & 0 \\ 0 & 0 & I_r & 0 \\ 0 & 0 & -A_{22}^{-1}A_{21} & I_{n-r} \end{bmatrix}$$

from right and its transpose from left. Then LHS of (11) is congruent to

$$\begin{bmatrix} \begin{bmatrix} \mathsf{sym}\{(I_r - A_0)^{\mathsf{T}} P_0\} & 0 \\ 0 & 2\alpha I_{n-r} \end{bmatrix} & \begin{bmatrix} (I_r - A_0)^{\mathsf{T}} P_0 & 0 \\ 0 & \alpha I_{n-r} \end{bmatrix} \\ * & \begin{bmatrix} P_0 + \alpha M^{\mathsf{T}} M & \alpha M^{\mathsf{T}} \\ \alpha M & \alpha I_{n-r} \end{bmatrix} \end{bmatrix}, \tag{14}$$

where $M = -A_{22}^{-1} A_{21}$. Define

$$Q_0 = \begin{bmatrix} \mathsf{sym}\{(I_r - A_0)^{\mathsf{T}} P_0\} & (I_r - A_0)^{\mathsf{T}} P_0 \\ * & P_0 \end{bmatrix}, \tag{15}$$

which is positive definite from $P_0 > A_0^{\mathsf{T}} P_0 A_0$, $P_0 > 0$. Via Schur complement, we see that the matrix in (14) is positive definite iff

$$Q_0 - \begin{bmatrix} 0 & 0 \\ 0 & \alpha M^{\mathsf{T}} M \end{bmatrix} > 0, \quad \alpha > 0,$$

which holds for sufficiently small $\alpha > 0$ (or, since P_0 is a solution to the Lyapunov inequality, which is affine with respect to P_0, we can take P_0 such that Q_0 in (15) has a large enough minimum eigenvalue). Thus a solution X and α to (9) is shown, which proves the necessity. ■

The LMI condition (9) is similar to that of state-space systems shown in [24], namely, the admissibility of a continuous-time descriptor systems $E\dot{x} = Ax$ is equivalent to the existence of X satisfying

$$E^{\mathsf{T}} X = X^{\mathsf{T}} E \geq 0, \quad A^{\mathsf{T}} X + X^{\mathsf{T}} A < 0. \tag{16}$$

LMI (9) also involves the variable X only as the terms of $E^{\mathsf{T}} X$ and $A^{\mathsf{T}} X$, which is exactly in the same manner as (16). This easily leads to convex formulation of sate (descriptor variable) feedback stabilization.

The LMI (9) involves equality constraint $E^{\mathsf{T}} X = X^{\mathsf{T}} E$. Such X is parameterized in terms of SVD of E as in (10). Then (9) holds if and only if $P_0 > 0$ and the second inequality of (9) with X in (10) substituted. Then the inequalities are both strict. It is easy to see that without loss of generality the condition $P_0 \geq 0$, which the first inequality of (9) implies, is replaced with the strict one. LMIs shown below can be obviously modified in terms of only strict inequalities. Note that the strict LMI approach by [8] can be also applicable.

C. Stabilization

Let us consider here state-feedback stabilization. For this purpose, let us first show a dual LMI to (9). Since the (E, A) is admissible if and only if $(E^{\mathsf{T}}, A^{\mathsf{T}})$ is admissible, the following result is in order. Let $S := I_n - EE^{\dagger}$.

Corollary 1: The pair (E, A) is admissible if and only if there exist a matrix $Y \in \mathbf{R}^{n \times n}$ and a scalar β that satisfy

$$\begin{cases} EY^{\mathsf{T}} = YE^{\mathsf{T}} \geq 0, \\ \begin{bmatrix} \mathsf{sym}\{(E - A)Y^{\mathsf{T}}\} & (E - A)Y^{\mathsf{T}} \\ Y(E - A)^{\mathsf{T}} & EY^{\mathsf{T}} + \beta S \end{bmatrix} > 0. \end{cases} \tag{17}$$

Now let us consider a discrete-time descriptor system with input $u_k \in \mathbf{R}^m$:

$$Ex_{k+1} = Ax_k + Bu_k \tag{18}$$

and control input

$$u_k = Fx_k, \quad F \in \mathbf{R}^{m \times n}. \tag{19}$$

The closed-loop system is

$$Ex_{k+1} = (A + BF)x_k. \tag{20}$$

Proposition 1: There exists a gain F such that the closed-loop system (20) is admissible if and only if there exist matrices $Y \in \mathbf{R}^{n \times n}$, $W \in \mathbf{R}^{n \times m}$ and a scalar β that satisfy

$$\begin{cases} EY^{\mathsf{T}} = YE^{\mathsf{T}} \geq 0, \\ \begin{bmatrix} \mathsf{sym}\{(E - A)Y^{\mathsf{T}} - BW^{\mathsf{T}}\} & * \\ Y(E - A)^{\mathsf{T}} + WB^{\mathsf{T}} & EY^{\mathsf{T}} + \beta S \end{bmatrix} > 0. \end{cases} \tag{21}$$

If (21) is true, there exists Y which is also nonsingular and satisfies (21). A stabilizing feedback gain F is given by

$$F = W^{\mathsf{T}} Y^{-\mathsf{T}}. \tag{22}$$

Proof: The necessity is clear from Corollary 1. For sufficiency, we note that one can always find a nonsingular solution Y (see [24]), which is ready to drive the stabilizing gain in (22). ■

Feedback of the dynamic part of the state variable is formulated as

$$u_k = KEx_k \tag{23}$$

with a gain $K \in \mathbf{R}^{m\times n}$. It has been shown by [15] that the problem of computing a static feedback of the dynamic part alone is convexified. The following is a presentation of such an LMI condition in our formulation.

Proposition 2: There exists a gain K such that the closed-loop system by the input (23) is admissible if and only if there exist matrices $Y \in \mathbf{R}^{n\times n}$, $W \in \mathbf{R}^{n\times m}$ and a scalar β that satisfy

$$\left\{\begin{array}{l} EY^\mathsf{T} = YE^\mathsf{T} \geq 0, \\ \begin{bmatrix} \mathsf{sym}\{(E-A)Y^\mathsf{T} - BW^\mathsf{T}E^\mathsf{T}\} & * \\ Y(E-A)^\mathsf{T} + WB^\mathsf{T} & EY^\mathsf{T} + \beta S \end{bmatrix} > 0. \end{array}\right. \tag{24}$$

If (24) is true, a stabilizing gain K is given by

$$K = W^\mathsf{T} Y^\dagger \tag{25}$$

Proof: Replace A in (9) with $A + BKE$, which is the coefficient of x_k of the closed-loop system. Then the term involved with K is manipulated as

$$BKEY^\mathsf{T} = BKYE^\mathsf{T} = BW^\mathsf{T}E^\mathsf{T}$$

to obtain (24), where $W^\mathsf{T} := KY$. For the sufficiency of (24), we note that, if $K = W^\mathsf{T}Y^\dagger$ and $EY^\mathsf{T} = YE^\mathsf{T} \geq 0$,

$$BW^\mathsf{T}E^\mathsf{T} = BW^\mathsf{T}(Y^\dagger Y)E^\mathsf{T} = B(W^\mathsf{T}Y^\dagger)EY^\mathsf{T},$$

which deduces

$$\left\{\begin{array}{l} EY^\mathsf{T} = YE^\mathsf{T} \geq 0, \\ \begin{bmatrix} \mathsf{sym}\{E(Y-(A+BKE))^\mathsf{T}\} & * \\ (E-(A+BKE))Y^\mathsf{T} & EY^\mathsf{T} + \beta S \end{bmatrix} > 0. \end{array}\right.$$

from (24).

the input (23) depends only on the image of E and hence (25) is enough to give a stabilizing gain. ■

III. ADMISSIBILITY FOR δ-OPERATOR SYSTEMS AND $\mathcal{D}$-ADMISSIBILITY

A. *δ-operator systems*

As consequences of Theorem 1, in this section we show LMIs for different problems. First, let us consider the following δ-operator system in the descriptor form:

$$E\frac{x_{k+1} - x_k}{T} = Ax_k, \tag{26}$$

where T, a positive constant, is the sampling time to derive the δ-operator system (26) from a continuous-time system $E\dot{x} = Ax$. System (26) is admissible if $(E, E + AT)$ is admissible as a discrete-time descriptor system defined in Definition 1.

Proposition 3: The δ-operator descriptor system admissible if and only if there exist a matrix $X \in \mathbf{R}^{n\times n}$ and a scalar α that satisfy

$$\left\{\begin{array}{l} E^\mathsf{T}X = X^\mathsf{T}E \geq 0, \\ \begin{bmatrix} A^\mathsf{T}X + X^\mathsf{T}A & A^\mathsf{T}X \\ X^\mathsf{T}A & -(\frac{1}{T}E^\mathsf{T}X + \alpha N) \end{bmatrix} < 0. \end{array}\right. \tag{27}$$

As $T \to \infty$, the $(2,2)$-block of the second inequality of (27) can be arbitrarily large, since it is congruent to $\mathrm{diag}\{\frac{1}{T}P_0, \alpha I\}$. Hence, for small sampling time T, the condition (27) gets close to (16) for continuous-time system.

B. *$\mathcal{D}$-admissibility*

Let $\mathcal{D}$ be a region in complex plain. Then a square matrix A, or a state-space continuous-time or discrete-time system whose coefficient on the state variable is A, is said to be $\mathcal{D}$-stable if all the eigenvalues of A belong to $\mathcal{D}$. As a generalization of this, we define the following:

Definition 2: A pair (E, A) is $\mathcal{D}$-admissible if it is regular and causal and all the finite eigenvalue of (E, A) belong to $\mathcal{D}$.

Consider a region $\mathcal{D}$ represented as

$$\mathcal{D} = \{z \in \mathbf{C} : a + 2b\,\mathrm{Re}\,z + c|z|^2 < 0\}, \tag{28}$$

where a, b, c are real scalars. The following LMI condition for $\mathcal{D}$-stability is widely-known:

Lemma 3: A square matrix $A \in \mathbf{R}^{n\times n}$ is $\mathcal{D}$-stable if and only if there exists a symmetric matrix P such that

$$P > 0, \quad aP + b(PA + A^\mathsf{T}P) + cA^\mathsf{T}PA < 0. \tag{29}$$

The LMI (29) is quadratic in A and, via Schur complement, one can derive an LMI which is also affine in A if $c > 0$:

$$\begin{bmatrix} aP + b(A^\mathsf{T}P + PA) & A^\mathsf{T}P \\ PA & -\frac{1}{c}P \end{bmatrix} < 0. \tag{30}$$

A merit of being affine with respect to A is that for controller synthesis gains contained in A appears affinely itself in the dual LMI and change-of-variable can be performed to linearize the inequality.

In the previous section, we concentrated on the case of $b = 0$, where we introduced an LMI which brings the term of $A^\mathsf{T}X + X^\mathsf{T}A$ (in (9)) 'artificially' in the $(1,1)$-block so that the inequality can guarantee nonsingularity of A. If $b \neq 0$, this is not a problem in view of the inequality (30) for state-space systems, which has the term of $A^\mathsf{T}P + PA$ in the $(1,1)$-block.

Proposition 4: Consider a region (28) with $b \neq 0$, $c > 0$. Then (E, A) is $\mathcal{D}$-admissible if and only if there exist a matrix $X \in \mathbf{R}^{n\times n}$ and a scalar α that satisfy

$$\left\{\begin{array}{l} E^\mathsf{T}X = X^\mathsf{T}E \geq 0, \\ \begin{bmatrix} -aE^\mathsf{T}X - b\,\mathsf{sym}(A^\mathsf{T}X) & A^\mathsf{T}X \\ X^\mathsf{T}A & \frac{1}{c}E^\mathsf{T}X + \alpha N \end{bmatrix} > 0. \end{array}\right. \tag{31}$$

Proof: The proof is similar to that of Theorem 1 except for the choice of X_{22} in (12) in the proof of necessity, which we set $X_{22} = -b\,\alpha A_{22}^{-\mathsf{T}}$, depending on b. ■

IV. NUMERICAL EXAMPLES WITH FURTHER EXTENSIONS

In this section, we provide three numerical examples to illustrate the new LMI criterion and its application. In particular, in Subsection IV-C we show a generalization of quadratic stabilization for discrete-time descriptor systems based on the LMIs shown so far.

A. Admissibility

Consider the following coefficient matrices:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$A = \begin{bmatrix} 0.8 & -0.4 & 0.2 \\ -0.5 & -0.7 & 0.2 \\ 0.1 & 0.2 & 0.4 \end{bmatrix},$$

for which the discrete-time descriptor system (1) is admissible with finite eigenvalues 0.9107, -0.9607. The LMI (9), implemented with X in the form of (10), is feasible for this pair (E, A) and a solution is:

$$X = \begin{bmatrix} 135.66 & -16.331 & 0 \\ -16.331 & 77.581 & 0 \\ -46.306 & 28.076 & -106.26 \end{bmatrix},$$
$$a = 127.9861.$$

If the $(3,3)$-element of A is set 0.311 the finite eigenvalues of (E, A) is 0.7075 and -1.0004 and the LMI (9) is infeasible, while if the $(3,3)$-element is 0.312 then (9) is feasible, where (E, A)'s finite eigenvalues are 0.9075 and -1.004.

B. $\mathcal{D}$-admissibility by state-feedback

Let us consider the following coefficient matrices for the system (18):

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$A = \begin{bmatrix} 0.8 & -0.4 & 0.2 \\ -0.5 & -0.7 & 0.2 \\ 0.1 & 0.2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

and a region (28) with $a = 5/36$, $b = 1/2$ and $c = 1$, for which $\mathcal{D}$ is a disk with center $-1/2 + 0j$ and radius $1/3$. Note that this system is not admissible with $u_k = 0$. To find a state-feedback control (19), we can formulate the following synthesis version of Proposition 4:

Proposition 5: Consider a region (28) with $b \neq 0$, $c > 0$. Then $(E, A + BF)$ is $\mathcal{D}$-admissible for some $F \in \mathbf{R}^{m \times n}$ if and only if there there exist a matrix $Y \in \mathbf{R}^{n \times n}$, $W \in \mathbf{R}^{n \times m}$ and a scalar β that satisfy

$$\begin{cases} EY^{\mathsf{T}} = YE^{\mathsf{T}} \geq 0, \\ \begin{bmatrix} -aEY^{\mathsf{T}} - b \,\mathsf{sym}\, M_A & M_A \\ M_A^{\mathsf{T}} & \frac{1}{c} EY^{\mathsf{T}} + \beta N \end{bmatrix} > 0, \end{cases} \tag{32}$$

where $M_A = AY^{\mathsf{T}} + BW^{\mathsf{T}}$. If (32) holds, there exists Y which is also nonsingular and satisfies (32). A stabilizing feedback gain F is given by (22).

A solution to the LMI (32) is obtained as

$$Y = \begin{bmatrix} 0.18193 & 0.63436 & 1.283 \\ 0.6343 & 3.3111 & 7.2220 \\ 0 & 0 & 1.6583 \end{bmatrix},$$
$$W = \begin{bmatrix} -0.22571 \\ -0.84270 \\ -0.26216 \end{bmatrix}.$$

where Y is nonsingular. A gain F for which $(E, A + BF)$ is $\mathcal{D}$-admissible is given by (22) as

$$F = \begin{bmatrix} -1.3273 & 0.3446 & -0.1581 \end{bmatrix}.$$

One can verify that $A + BF$ is regular and causal and its two finite eigenvalues are $-0.4760 \pm 0.1315j$, which belong to the disk $\mathcal{D}$.

C. Robust admissibility by state-feedback

Let us consider the following system:

$$Ex_{k+1} = (A + H\Delta L_A)x_k + (B + H\Delta L_B)u_k,$$

where $\Delta \in \mathbf{R}^{m_\delta \times p_\delta}$ is a time-varying uncertainty satisfying $\bar{\sigma}(\Delta) \leq 1$. The closed-loop system with input $u_k = Fx_k$ is

$$\begin{aligned} &Ex_{k+1} = A_{cl}(\Delta)x_k, \\ &A_{cl}(\Delta) = \{\underbrace{(A + BF)}_{A_{cl}} + H\Delta \underbrace{(L_A + L_B F)}_{L_{cl}}\}x_k. \end{aligned}$$

Let us compute F so that the closed-loop system satisfies a 'quadratic stability condition' implied form (9). Namely, we find F for which there exist Y and β that satisfy

$$\begin{cases} EY^{\mathsf{T}} = YE^{\mathsf{T}} \geq 0, \\ \begin{bmatrix} \mathsf{sym}\{(E - A_{cl}(\Delta))Y^{\mathsf{T}}\} & * \\ Y(E - A_{cl}(\Delta))^{\mathsf{T}} & EY^{\mathsf{T}} + \beta S \end{bmatrix} > 0 \end{cases} \tag{33}$$

for all Δ. This is guaranteed by the following LMI:

Proposition 6: Denote by $Q(Y, W, \beta)$ the LHS of the second inequality in (24) and suppose that the following LMI holds for $Y \in \mathbf{R}^{n \times n}$, $W \in \mathbf{R}^{n \times m}$ and scalars β and ε:

$$\begin{cases} EY^{\mathsf{T}} = YE^{\mathsf{T}} \geq 0, \\ \begin{bmatrix} Q(Y, W, \beta) - \varepsilon \begin{bmatrix} H \\ 0 \end{bmatrix} \begin{bmatrix} H \\ 0 \end{bmatrix}^{\mathsf{T}} & * \\ (L_A Y^{\mathsf{T}} + L_B W^{\mathsf{T}}) \begin{bmatrix} I_n & I_n \end{bmatrix} & \varepsilon I_{p_\delta} \end{bmatrix} > 0. \end{cases} \tag{34}$$

If (21) is true, there exists Y which is also nonsingular and satisfies (21). A stabilizing feedback gain F is given by (22).

Proof: The proof follows from the standard quadratic stability result [4]. ∎

Let us consider a numerical example for the following coefficient matrices:

$$E = \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -0.2 & 0.2 & 1.2 \\ 0 & 1.5 & 2 \\ 1 & 0 & 0 \end{bmatrix},$$
$$B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad H = \begin{bmatrix} 0.18 & 0 \\ 0.18 & -0.36 \\ -0.18 & 0 \end{bmatrix},$$
$$L_A = \begin{bmatrix} 0.2 & 0.1 & 0 \\ 0.3 & 0.8 & 0.3 \end{bmatrix}, \quad L_B = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}.$$

A solution $(Y, W, \beta, \varepsilon)$ is obtained as

$$Y = \begin{bmatrix} 407.31 & -127.86 & 103.15 \\ -17.670 & 19.04 & -14.078 \\ -5.889 & -11.77 & 11.773 \end{bmatrix},$$
$$W = \begin{bmatrix} -265.36 \\ 1.3264 \\ -113.33 \end{bmatrix}, \quad \beta = 368.54, \quad \varepsilon = 21.70.$$

A robust stabilizing controller is obtained as

$$F = \left[\begin{array}{ccc} 0.176318 & -26.1538 & -35.6879 \end{array}\right].$$

Fig. 1 shows the finite eigenvalues of $(E, A_{cl}(\Delta))$ for sampled $\Delta \in \mathbf{R}^{2\times 2}$ from those satisfying $\bar{\sigma}(\Delta) \le 1$. The eigenvalues are thus included in the unit disk on the complex plane.

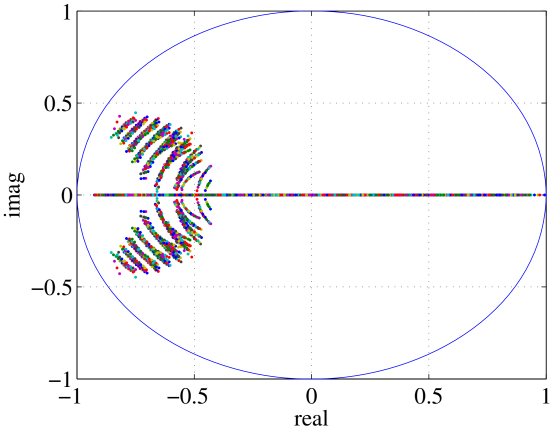

Fig. 1. Finite eigenvalues of $(E, A_{cl}(\Delta))$

V. CONCLUSION

In this paper, we proposed a criterion for admissibility of discrete-time descriptor systems in terms of LMIs and showed several applications of the new LMI for state-feedback synthesis, admissibility of descriptor systems with δ-operator, $\mathcal{D}$-admissibility and a robust stabilization based on quadratic stability. A new LMIs are affine not only in the decision variables but also in coefficient matrices. This allows us to derive such LMIs as for synthesis and robustness analysis by applying simple change-of-variables.

Future work based on the results of the paper would include performance analysis of such as H_∞ (dissipativity), H_2 for descriptor systems and application of them to control.

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