2013 European Control Conference (ECC)
July 17-19, 2013, Zürich, Switzerland.

Inversion-based Feedforward Design for Constrained Fractional Control Systems

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Abstract— **In this paper we propose an input-output inversion-based methodology for the synthesis of the feedforward action for a fractional control system in order to achieve a predefined process variable transition from a steady-state value to another. In particular, the feedforward action is implemented either as a signal to be added to the feedback control variable or as a command signal to be applied (instead of the typical step signal) to the closed-loop system. The method allows the minimization of the transition time by taking explicitly into account constraints on the process input an output and their derivatives. Simulation results show the effectiveness of the technique.**

Index Terms— **Fractional systems, constrained set-point regulation, input-output inversion, feedforward control, optimization.**

I. INTRODUCTION

Fractional systems and fractional control have received a great attention recently, both from an academic and industrial viewpoint, because of their increased flexibility (with respect to integer-order systems) which allows a more accurate modelling of complex systems and the achievement of more challenging control requirements [1], [2], [3], [4], [5], [6].

Regarding the design of control systems, many contributions are related to the synthesis of robust control systems (see, for example, [7], [8], [9], [10], [11], [12]) and to the synthesis of fractional-order PID controllers (see, for example, [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24]). Among the other relevant research topics, it is worth mentioning the approximation of a fractional system with an integer one [25], [26], [27] and the stability of fractional (control) systems [28], [29], [30], [31].

In this paper we consider the problem of the design of a feedforward (model-based) control law based on the input-output inversion concept for a standard unity feedback linear control system. In fact, it is well-known that the main purpose of a feedback controller is to compensate for external disturbances and to reduce the effect of modelling uncertainties and the addition of a feedforward control law can improve the set-point following performance of a control system [32]. In this context, we address the problem of achieving a process variable transition from one steady-state value to another one. We consider two different cases: the synthesis of a feedforward signal to be added to the feedback control variable and the synthesis of a command signal to be applied to the closed-loop control system instead of the typical set-point step signal. Basically, the method consists in defining a suitable smooth output function [33] and to determine the corresponding system input by applying an input-output inversion procedure. In this context, the transition time can be minimized subject to constraints on the process input and output variables and their derivatives until an arbitrary order.

The paper is organized as follows. In Section 2 the problem is formalized. The design of the desired output function is reviewed in Section 3. The input-output inversion procedure and the solution of the minimum-time optimization problem are given in Section 4. The synthesis of the feedforward signal to be added to the feedback control variable is presented in Section 5 while the synthesis of the command input to be applied to the closed-loop control system is explained in Section 6. An illustrative example is given in Section 7 and finally conclusions are drawn in Section 8.

Notation. We denote by $C^{(i)}$ the space of the scalar real functions which are continuous till the i-th time derivative. D^i denotes the i-th derivative operator. Finally $[x]$ with $x \in \mathbb{R}$ is the integer part of x, *i.e.*, the biggest integer lower than x.

II. PROBLEM FORMULATION

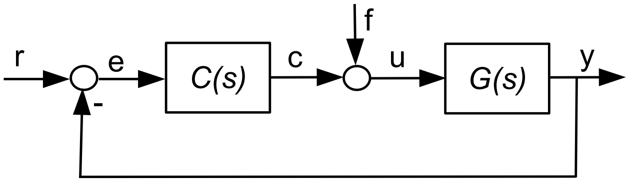

Fig. 1. The unity feedback control scheme.

Consider the unity feedback control scheme of Figure 1, where $C(s)$ is a linear time invariant commensurate fractional controller and

$$G(s) = \bar{G}(s)e^{-Ls} \tag{1}$$

is a linear time invariant strictly proper fractional system $G(s)$, where L is a possible delay term and $\bar{G}(s)$ is minimum-phase.

The closed loop transfer function is

$$T(s) = \frac{C(s)G(s)}{1 + C(s)G(s)} \tag{2}$$

and is assumed to be strictly proper. Note that the controller can be also improper from a theoretical point of view in order to make the approach suitable for a wider range of

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regulators (*e.g.*, PID controllers). In fact, in many cases the controller can be effectively treated and designed as an improper system, even if in practice it has to be made proper in its implementation (possibly by adding high frequency poles that can be neglected at the design phase).

It is also assumed that the controller has been designed in order to make the considered feedback loop internally stable. The aim of this work is to find suitable signals for the loop command signal $r(t)$ or for the feedforward signal $f(t)$ in order to obtain a perfect tracking of a desired output which allows a transition from an initial steady-state value to a new one in a finite time interval τ, given a set of bounds on the control and process variables and their derivatives.

The problem can be formalized as follows: starting from null initial conditions and given a steady-state output value y_e, design a a "sufficiently smooth" τ-parameterized desired output $\bar{y}(\cdot;\tau)$ such that $\bar{y}(0;\tau)=0$ and $\bar{y}(t;\tau)=y_e \ \forall t \geq \tau$, and $\bar{y}(\cdot;\tau) \in C^{(k)}$ for some $k \in \mathbb{N}$. Then, find

1) (feedforward signal generation) $f(\cdot;\tau)$ and $r(\cdot;\tau)$ such that, for the τ-parameterized functions $f(\cdot;\tau), r(\cdot;\tau)$ and $\bar{y}(\cdot;\tau)$, it holds that

$$\mathcal{L}[\bar{y}(t-L;\tau)] = G(s)\mathcal{L}[f(t;\tau))] \tag{3}$$

and $y(t) = \bar{y}(t-L;\tau) \ \forall t \geq 0$; or

2) (command signal generation) $r(\cdot;\tau)$ such that, for the $\tau-$parameterized couple $r(\cdot;\tau), \bar{y}(\cdot;\tau)$, it holds that

$$\mathcal{L}[\bar{y}(t-L;\tau)] = T(s)\mathcal{L}[r(t;\tau))]. \tag{4}$$

Moreover, in both cases, determine the minimum time τ^* such that $u(t;\tau^*)$ and the first $l \in \mathbb{N}_0$ ($v \in \mathbb{N}$, respectively) derivatives of $u(t;\tau^*)$ ($\bar{y}(t;\tau^*)$), are bounded:

$$\begin{array}{l} |D^i u(t;\tau^*)| < u_M^i, \ \forall t>0, \ i=0,1,\ldots,l; \\ |D^i \bar{y}(t;\tau^*)| < y_M^i, \ \forall t>0, \ i=1,2,\ldots,v. \end{array} \tag{5}$$

It is worth stressing that the requirement of null initial conditions is without loss of generality in view of the system linearity.

Note that in the first case, since a perfect tracking is obtained, the controller output is null and the feedforward signal $f(t;\tau)$ coincides with the control signal $u(t)$, whereas in the second case the control variable $u(t)$ and the controller output $c(t)$ coincide since no feedforward action is considered.

III. OUTPUT FUNCTION DESIGN

Although different function bases could be used to design the output function, the simple and computationally efficient $\tau-$parameterized transition polynomial proposed in [33] is chosen. It has the nice property of being monotonic, which implies that neither overshoots nor undershoots occur. For the sake of simplicity and without loss of generality in view of the linearity of the system, we consider $y_e = 1$. The output function is therefore selected as:

$$\bar{y}(t;\tau) := \begin{cases} 0 & \text{if} \quad t<0 \\ \frac{(2n+1)!}{n!\tau^{2n+1}} & \\ \times \sum_{r=0}^{n} \frac{(-1)^{n-r}\tau^r t^{2n-r+1}}{r!(n-r)!(2n-r+1)} & \text{if} \quad 0 \leq t \leq \tau \\ 1 & \text{if} \quad t>\tau. \end{cases} \tag{6}$$

Note that $\bar{y}(t;\tau)$ allows an arbitrarily smooth transition between 0 and 1; indeed, it is possible to show that $\bar{y}(t;\tau) \in C^{(n)}$ [33]. Moreover it can be shown that the following lemma holds.

Lemma 1: Let be given the transition signal $\bar{y}(\cdot;\tau) \in C^{(n)}$ defined in (6). Then there exits constants $c_i \in \mathbb{R}_+$, $i=1,\ldots,n$ such that

$$\max_{t\in[0,\tau]} |D^i \bar{y}(t;\tau)| = \frac{c_i}{\tau^i}. \tag{7}$$

In fact, the previous lemma means that the transition polynomial becomes flatter when τ increases, that is, it is always possible to increase the value of τ until the second condition of (5) is satisfied, provided that $n \geq v$ and $y_M^i > 0$, $i=1,\ldots,v$.

IV. CONSTRAINED INPUT-OUTPUT INVERSION

In this section the main tools needed for the solution of the proposed problems are given.

A. Input-Output Inversion

Consider a general minimum-phase stable commensurate fractional linear system Σ whose transfer function is

$$H(s) = \frac{b(s)}{a(s)} = \frac{\sum_{k=0}^{m} b_k s^{k\nu}}{s^{p\nu} + \sum_{k=0}^{p-1} a_k s^{k\nu}}, \tag{8}$$

where ν is the commensurate order of the system and $p > m$. The integer order system associated to (8) is

$$\tilde{H}(w) = \frac{\tilde{b}(w)}{\tilde{a}(w)} = \frac{\sum_{k=0}^{m} b_k w^k}{w^p + \sum_{k=0}^{p-1} a_k w^k}, \tag{9}$$

where $\tilde{a}(s)$ and $\tilde{b}(s)$ are coprime polynomials. We define as $\rho := (p-m)\nu$ the relative order of the system Σ. Once the transition polynomial (6) has been defined, we can compute the corresponding input $u(t;\tau)$ by inverting the system through Laplace transform, that is

$$U(s;\tau) = H^{-1}(s)\bar{Y}(s;\tau), \tag{10}$$

where $\bar{Y}(s;\tau) = \mathcal{L}[\bar{y}(t;\tau)]$ denotes the Laplace transform of the desired output. By polynomial division on $\tilde{H}(w)$ and by means of the substitution $w = s^\nu$ we obtain

$$H^{-1}(s) = \gamma_{n-m} s^{\rho} + \gamma_{n-m-1} s^{\rho-\nu} + \cdots + \gamma_1 s^{\nu} + \gamma_0 + H_0(s) \tag{11}$$

where $H_0(s)$ is a strictly proper transfer function that describes the zero dynamics of Σ.

Defining $\eta_0(t) := \mathcal{L}^{-1}[H_0(s)]$ and considering the Laplace transform properties, we can derive the following proposition [34]:

Proposition 1: Consider $\bar{y}(t;\tau)$ defined in (6). If $n \geq [\rho]+1$ then

$$\begin{aligned} u(t;\tau) &= \gamma_{n-m} D^{\rho}\bar{y}(t;\tau) + \gamma_{n-m-1} D^{\rho-\nu}\bar{y}(t;\tau) + \cdots \\ &+\gamma_1 D^{\nu}\bar{y}(t;\tau) + \gamma_0\bar{y}(t;\tau) + \int_0^t \eta_0(t-\xi)\bar{y}(\xi;\tau)d\xi. \end{aligned} \tag{12}$$

In view of the minimum-phase assumption, the input (12) is bounded; hence, by means of (12), the input-output inversion problem is completely solved.

In order to compute (12), let us introduce the following functions: the two parameters Mittag-Leffler function [35]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \ \alpha > 0, \beta > 0 \tag{13}$$

and the function introduced by Podlubny in [35]

$$\varepsilon_k(t,\lambda;\alpha,\beta) := t^{k\alpha+\beta-1} \frac{d^k}{d(\lambda t^{\alpha})^k} E_{\alpha,\beta}(\lambda t^{\alpha}). \tag{14}$$

It is possible to show [35], [36] that the following property holds:

$$\mathcal{L}^{-1}\left[\frac{k! s^{\alpha-\beta}}{(s^{\alpha} \pm \lambda)^{k+1}}\right] = \varepsilon_k(t, \mp\lambda; \alpha, \beta). \tag{15}$$

Applying on $H_0(s)$ the same reasoning applied on $H(s)$, it is easy to see that it can be factorized via partial fraction expansion and represented as the summation of simple terms:

$$H_0(s) = \sum_{i=1}^{m} \frac{g_i}{(s^{\nu} - \lambda_i)^{k_i+1}}, \tag{16}$$

where λ_i and g_i can be either real or complex (in the latter case they always appear in conjugate pairs so that the time response is always real) and k_i is a nonnegative integer. Moreover, in view of the minimum-phase assumption on Σ, $H_0(s)$ is stable.

Considering (15) and (16) it is immediate to see that

$$\eta_0(t) = \sum_{i=1}^{m} \frac{g_i}{k_i!} \varepsilon_{k_i}(t, \lambda_i; \nu, \nu). \tag{17}$$

Now, by substituting the previous equation together with (6) into the convolution integral appearing in (12) we obtain, after some calculation [34], that

$$\begin{aligned} &\int_0^t \eta_0(t-\xi)\bar{y}(\xi;\tau)d\xi = \\ &\sum_{i=1}^{m} \tfrac{g_i}{k_i!} \tfrac{(2n+1)!}{n!\tau^{2n+1}} \sum_{r=0}^{n} \tfrac{(-1)^{n-r}\tau^r}{r!(n-r)!(2n-r+1)} \\ &\times \begin{cases} (2n-r+1)!\varepsilon_{k_i}(t,\lambda_i;\nu,2n-r+2+\nu) & \text{if} \quad t \leq \tau \\ \int_0^{\tau} \varepsilon_{k_i}(t-\xi,\lambda_i;\nu,\nu)\xi^{2n-r+1}d\xi & \\ + \int_{\tau}^{t} \varepsilon_{k_i}(t-\xi,\lambda_i;\nu,\nu)d\xi & \text{if} \quad t > \tau. \end{cases} \end{aligned} \tag{18}$$

Note that in the interval $[0,\tau]$, considering that the Laplace transform of the convolution integrals equals the product of the Laplace transforms and that $\mathcal{L}[t^{\alpha}] = \Gamma(\alpha+1)\frac{1}{s^{\alpha+1}}$, by means of (15) we can explicitly derive, in term of Podlubny functions, the convolution integral appearing in (12). Conversely, for $t > \tau$ a similar result is not achievable because the input of $H_0(s)$ (*i.e.*, $\bar{y}(\cdot;\tau)$) is no longer a polynomial.

Remark 1. The computation of (12), by means of (18), requires: (i) the computation of the Mittag-Leffler function, that is widely treated in the literature (see, for example, [35], [37]); (ii) a quadrature formula (see, for example, [38]); (iii) the numerical fractional differentiation of $\bar{y}(t;\tau)$. The last point in $[0,\tau]$ can be approached analytically considering that $\bar{y}(t;\tau)$ is a polynomial in $[0,\tau]$. However, the effect of the fractional derivative, because of the operator memory, still lasts when $t > \tau$. In this case a numerical approach can be used, *e.g.*, an explicit computation using the Grünwald-Letnikov definition (see [39]).

B. Constrained Minimum-Time Transition

Consider now the following constrained minimum-time transition problem: given the system (8), determine the minimum time τ^* such that $u(t;\tau^*)$ and the first $l \in \mathbb{N}$ derivatives of $u(t;\tau^*)$ are bounded, that is,

$$|D^i u(t;\tau^*)| < u_M^i, \ \forall t > 0, \ i = 0,1,\ldots,l. \tag{19}$$

We show here the condition under which the problem admits a solution. First consider the following lemma, quoted without proof (see [34] for details).

Lemma 2: Assume that $n \geq [\rho]+1+l$. The input-output pair defined by (6) and (12) satisfies the following limits

$$\begin{aligned} &\lim_{\tau\to\infty} \|H(0)u(\cdot;\tau) - \bar{y}(\cdot;\tau)\|_{\infty} = 0; \\ &\lim_{\tau\to\infty} \|D^i u(\cdot;\tau)\|_{\infty} = 0 \ \ i = 1,\ldots,l. \end{aligned} \tag{20}$$

Finally, the following theorem [34] states that the problem admits a solution under very mild conditions.

Theorem 1: The minimum constrained transition time problem admits a solution provided that

$$u_M^0 > \frac{1}{H(0)}, \ \ u_M^i > 0, \ \ i = 1,\ldots,l. \tag{21}$$

Roughly speaking, the previous theorem states that it is just required that the constraints on the maximum input do not prevent the input to keep the system output constant at the desired steady-state value.

In order to compute the minimum transition time τ^* such that the input constraints are satisfied, a simple and effective bisection algorithm can be employed [33].

V. FEEDFORWARD SIGNAL SYNTHESIS

In this section we address the problem of feedforward regulation. Consider the (possibly unstable) process (1). First we focus on the delay-free part $\bar{G}(s)$. Given a suitable transition polynomial $\bar{y}(t;\tau)$ we can easily compute, via Proposition 1 together with (12) and (18), the corresponding feedforward signal $f(t;\tau)$, provided that

$$n \geq [\rho_{\bar{G}}] + 1, \tag{22}$$

being $\rho_{\bar{G}}$ the relative order of $\bar{G}(s)$.

It is immediate to see that (3) is satisfied. Indeed, equation (3) can be rewritten as

$$\bar{Y}(s;\tau)e^{-Ls} = G(s)F(s;\tau) \tag{23}$$

and, by substituting (1) into the previous equation and eliminating the delay terms from both sides, we obtain

$$\bar{Y}(s;\tau) = \bar{G}(s)F(s;\tau). \tag{24}$$

The previous equation holds by construction because $f(t;\tau)$ is computed via input-output inversion of $\bar{G}(s)$, thus (3) is satisfied.

Note that the inversion-based signal does not excites the system modes, since it is computed as the convolution of the transition polynomial (6) with the inverse system $\bar{G}(s)^{-1}$. Thus, it is enough to have null control error to obtain the perfect tracking condition $y(t) = \bar{y}(t - L;\tau)$. Considering that (3) holds, it is enough to use the desired output as the closed-loop reference signal r in order to obtain a null error, that is, the perfect tracking is obtained by imposing:

$$r(t;\tau) = \bar{y}(t - L;\tau). \tag{25}$$

Now let us focus on the constraints (19). Lemma 1 shows that it is enough to select $n = v$ in (6) and to increase the transition time τ in order to decrease the maximum value of the first v derivatives of the transition polynomial and, in view of the perfect tracking condition, of the output signal $y(t)$ too. In other words, it is enough to increase τ until the second condition of (5) is accomplished.

Analogously, Theorem 1 states that the first of (5) can be satisfied, provided that $n \geq [\rho] + 1 + l$. Thus, in order to satisfy all the constraints (5) the desired output function can be selected by choosing

$$\begin{cases} n \geq \max\{v; [\rho_{\bar{G}}] + 1 + l\} \\ \tau \geq \max\{\tau_i^*; \tau_o^*\}, \end{cases} \tag{26}$$

where τ_o^* is the minimum transition time satisfying the output constraints (which, as already mentioned in Section III, can be computed explicitly following the techniques proposed in [33]), whereas τ_i^* is minimum transition time such that the input constraints are satisfied for each $\tau \geq \tau_i^*$. Also in order to compute τ_i^*, a simple and effective bisection algorithm can be employed. Note that (26) automatically implies (22), since l is non negative.

VI. COMMAND INPUT SYNTHESIS

An alternative approach to the one proposed in the previous paragraph is to compute a suitable command signal $r(t)$ to be applied to the closed-loop system in order to obtain a perfect tracking of the desired output. In this context a double strategy has been developed, one for the case where no dead time is present in the process dynamics and on for processes with time delay.

Consider first the delay-free case. The closed loop transfer function $T(s)$, accordingly to (2), can be expressed as a fractional transfer function. Hence, the input-output procedure proposed in Section IV can be straightforwardly applied to $T(s)$ obtaining a suitable command signal $r(t;\tau)$ that satisfies (4), provided that

$$n \geq [\rho_T] + 1, \tag{27}$$

where ρ_T is the relative order of $T(s)$. It is the product of the relative degree of $\bar{G}(s)$ and the (possibly negative) relative degree ρ_C of the controller $C(s)$. The point here is that the existence of a suitable command signal does not guarantee the existence of a bounded control signal. Indeed, when the controller is not proper, *i.e.,* when $\rho_C < 0$, we must also consider the following condition for the existence of the control signal:

$$n \geq [\rho_{\bar{G}}] + 1. \tag{28}$$

Conversely, when the controller is strictly proper, it may happen that the user is forced to use an over-regularized control signal $u(t)$ in order to have a feasible command signal $r(t;\tau)$. Note that, thank to (27), condition (28) is always satisfied when $\rho_C \geq 0$. Also note that it is enough to consider the command signal in order to check the existence of the control signal. Indeed, the feedback signal is $y(t) = \bar{y}(t;\tau)$ and its regularity is greater than the one of $r(t;\tau)$ because of the properness of $T(s)$.

Regarding the constraints (5), it is sufficient to note that, since in view of (4) a perfect tracking condition is obtained, this also implies that $u(t) = \bar{G}(s)^{-1}\bar{y}(t;\tau)$ and the same reasoning of the previous paragraph can be applied, leading again to (26), where $u(t) = u(t;\tau)$ can be computed via input-output inversion of $\bar{G}(s)$. It is worth stressing that constraints accomplishments automatically implies (28).

Finally, let us consider the case related to processes with dead time. When a delay term is present, the closed loop system cannot be inverted because of the exponential term appearing in the denominator of the transfer function. In this case the command signal is calculated as the summation of two different terms. First, consider the open-loop transfer function. Neglecting the delay term, we have $C(s)\bar{G}(s)$, that can be inverted by using the same procedure employed to invert $T(s)$ in the delay-free case, yielding the signal $r_{ol}(t;\tau)$. Then, a correction term $r_c(t;\tau) = \bar{y}(t - L;\tau)$ must be considered, so that the command signal is

$$r(t;\tau) = r_{ol}(t;\tau) + r_c(t;\tau). \tag{29}$$

It is easy to see that command signal (29) satisfies (4), indeed we have

$$\begin{aligned} Y(s) &= T(s)r(t;\tau) \\ &= \tfrac{C(s)\bar{G}(s)e^{-Ls}}{1+C(s)\bar{G}(s)e^{-Ls}}(R_{ol}(s;\tau) + R_c(s;\tau)), \end{aligned} \tag{30}$$

that is,

$$\begin{aligned} &Y(s)\left(1 + C(s)\bar{G}(s)e^{-Ls}\right) = \\ &C(s)\bar{G}(s)e^{-Ls}\left((C(s)\bar{G}(s))^{-1} + e^{-Ls}\right)\bar{Y}(s;\tau). \end{aligned} \tag{31}$$

Simplifying and applying the inverse Laplace transform to the previous expression, it is immediate to see that $y(t) = y(t - L;\tau)$, that evidently implies (4).

Regarding the constraints, considering that the relative order of the open-loop transfer function is the same of the closed-loop one, the reasoning used in the delay-free case holds straightforwardly and the problem of the command signal synthesis is completely solved.

Remark 2. Note that here both the reference signal and the control signal (in order to find τ_i^*) must be computed. Nevertheless the computational weight increment is minimal because, once a suitable value of n has been chosen, only the control signal must be computed repeatedly. Indeed the command signal has to be computed just once, when τ_i^* and τ_o^* have been already found.

VII. ILLUSTRATIVE EXAMPLE

As an illustrative example consider an unstable fractional system with the following transfer function:

$$G(s) = \frac{3s^{0.5}+1}{s^{1.5}-1}e^{-0.1s}, \tag{32}$$

whose commensurate order is evidently 0.5. A very simple stabilizing controller can be used, indeed the achievement of the perfect tracking is independent from the chosen controller. We choose here a proportional controller $C(s) = 2$. We control requirement is to obtain a smooth transition of the output from 0 to 1 with constraints on both the amplitude and the slew rate of control and process variables (note that these are common requirements in practical applications). Accordingly, considering that the relative order of the system $\bar{G}(s)$ is $\rho = 1$, we choose $n = 3$ in order to satisfy the hypothesis of Lemma 2, and we compute the transition polynomial $\bar{y}(t;\tau)$ via (6):

$$\bar{y}(t;\tau) = -\frac{20}{\tau^7}t^7 + \frac{70}{\tau^6}t^6 - \frac{84}{\tau^5}t^5 + \frac{35}{\tau^4}t^4. \tag{33}$$

Then, we apply the technique proposed in Section V. We obtain the zero dynamics of $\bar{G}(s)$ as

$$H_0(s) = \frac{-1.0370}{3s^{0.5}+1} \tag{34}$$

and its time domain version

$$\eta_0(t) = \frac{-1.0370}{3}\varepsilon_{k_i}\left(t, \frac{1}{3}; 0.5, 0.5\right). \tag{35}$$

Subsequently, the inversion-based feedforward signal $f(t;\tau)$ (note that it coincides with the control variable) can be numerically computed via (12):

$$\begin{aligned} f(t;\tau) &= 0.3333D^1 y(t;\tau) - 0.1111D^{0.5}y(t;\tau) + 0.0370 \\ &\quad + \int_0^t \eta_0(t-\xi)y(\xi;\tau)d\xi. \end{aligned} \tag{36}$$

Now consider the following set of constraints:

$$\begin{gathered} u_M^0 \leq 1.5,\ \ u_M^1 \leq 5 \\ y_M^1 \leq 2 \end{gathered} \tag{37}$$

The bisection algorithm has been applied resulting in the minimum value of the transition time $\tau_i^* = 0.72$ while, in order to satisfy the constraint on the output signal, $\tau_o^* = 1.12$ is obtained. The tightest constraint is the one on the derivative of the output signal, therefore $\tau^* = \tau_o^* = 1.12 > \tau_i^*$. Numerically computing $u(t;\tau)$ with the obtained transition time τ^* and simulating the system response, the result shown in Figure 2 is obtained. As expected, the system response is smooth and monotonic and the input variable is smooth and satisfies the constraints (37). Note that the system output and the transition polynomial have that same shape, except for the time shift depending on the process delay.

For the purpose of simulation, the fractional-order dynamics has been approximated in the frequency domain by using the well-known Oustaloup approximation [25]. In order to obtain a precise approximation of the real fractional system, an high number of poles and zeros has been used, namely, 20 cells in a frequency band $[0.0001\ \ 10000]$.

In order to speed up the system response the constraints on the output slew rate are then relaxed by selecting

$$\begin{gathered} u_M^0 \leq 1.5,\ \ u_M^1 \leq 5 \\ y_M^1 \leq 5 \end{gathered} \tag{38}$$

The previously computed value $\tau^* = \tau_i^* = 0.72$ can now be used as in this case the constraint on the output slew rate is satisfied. A new simulation has been performed recomputing the input signal, the results are shown in Figure 3. Now, as expected, the tightest constraint is the one imposed on the control variable slew rate.

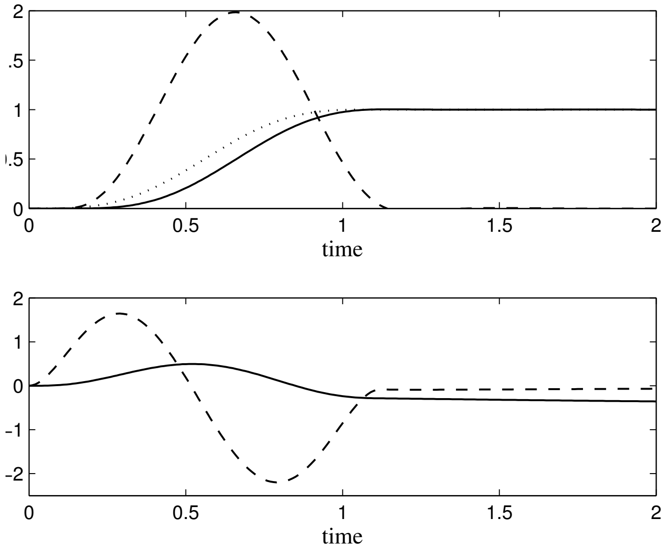

Fig. 2. Process output (top, solid line), derivative of the process output (top, dashed line), transition polynomial (top, dotted line), control variable (bottom, solid line) and control variable slew rate (bottom, dotted line) for the set of constraints (37).

VIII. CONCLUSIONS

In this paper we have proposed a methodology for the design of the feedforward action in a fractional system in order to achieve a minimum-time process variable transition from a steady-state value to another subject to constraints on the process input and output signals and their derivatives. The method is based on the selection of a transition polynomial (parameterized by the transition time) as a desired output function and then on the synthesis of the feedforward signal (to be applied either to the process input or to the closed-loop system) by applying an input-output inversion procedure.

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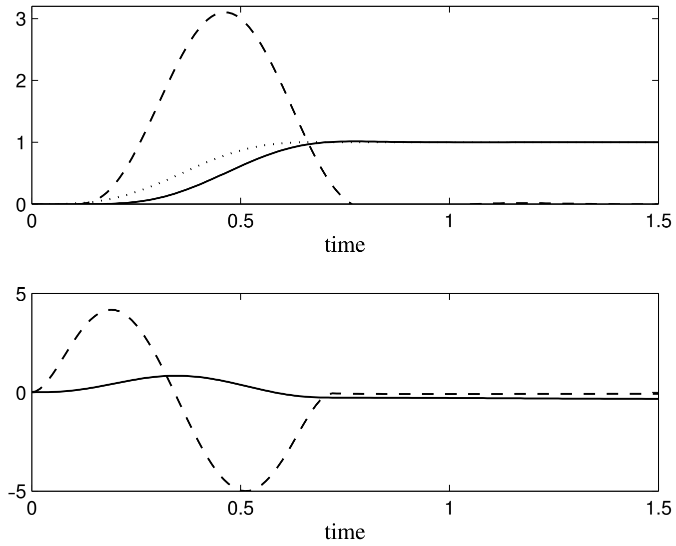

Fig. 3. Process output (top, solid line), derivative of the process output (top, dashed line), transition polynomial (top, dotted line), control variable (bottom, solid line) and control variable slew rate (bottom, dotted line) for the set of constraints (38).

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