

Continuity of the maximal negative graph spaces

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Abstract—In the framework of LPV and qLPV controller design one often encounter the problem of finding a maximal negative graph subspace of a parameter dependent non-singular indefinite matrix. For practical reason it is necessary that these subspaces depend continuously on the parameters. The paper provides an exhaustive condition for the existence of such a continuous solution.

I. INTRODUCTION AND MOTIVATION

Quasi linear parameter varying (qLPV) description, which is based on the possibility of rewriting the plant in a form in which nonlinear terms can be hidden by using suitably defined scheduling variables by maintaining the linear structure of the model, is frequently used in modern control design, [1]. An advantage is that in the entire operational interval nonlinear systems can be defined and a well-developed linear system theory to analyze and design nonlinear control system can be used. Modern control design approaches use a combined IQC and LMI technique to solve such problems, see [3], [8], [13], [15], [16], [17].

During the solution of control design task one often encounter the problem of finding a maximal negative space of a parameter dependent non-singular indefinite matrix, i.e., to solve for $Z(\rho)$ the inequality

$$\begin{pmatrix} I_m \\ Z(\rho) \end{pmatrix}^T P(\rho) \begin{pmatrix} I_m \\ Z(\rho) \end{pmatrix} < 0, \quad (1)$$

where ρ is the parameter, $P(\rho) \in \mathbb{R}^{(m+n) \times (m+n)}$ with inertia $in(P) = (m, 0, n)$.

Solving equation (1) involves the theory of indefinite finite dimensional inner product spaces, see [4], and in the general case the problem is not trivial. In this subject the Möbius transformation of matrices is elementary. A numerically reliable algorithm and a parametrization of the solutions for obtaining the solution for fixed ρ was given by the authors in a recent paper, see [14].

As a motivation example try to find block diagonal solu-

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tions $\Delta = \text{diag}\{\Delta_p, \Delta_c\}$ for the inequality

$$\begin{pmatrix} I & 0 \\ 0 & I \\ \Delta_p & 0 \\ 0 & \Delta_c \end{pmatrix}^T P_e \begin{pmatrix} I & 0 \\ 0 & I \\ \Delta_p & 0 \\ 0 & \Delta_c \end{pmatrix} < 0, \quad (2)$$

where P_e is non-singular and the graph subspace is maximal. Let us suppose that Δ_p is fixed and the question is whether exists a Δ_c that fulfills the inequality.

The solvability condition of the problem can be obtained by applying by the Elimination lemma, see the Appendix, with

$$C = \begin{pmatrix} \Delta_p & 0 \\ 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 \\ I \end{pmatrix}, \quad B = \begin{pmatrix} I \\ 0 \end{pmatrix}$$

and considering as variable $X = \Delta_c$.

By applying the conditions of the lemma a short computation reveals that the equation is solvable if and only if

$$\begin{pmatrix} I \\ \Delta_p \end{pmatrix}^T P_{11} \begin{pmatrix} I \\ \Delta_p \end{pmatrix} < 0, \quad \text{and} \quad \begin{pmatrix} -\Delta_p^T \\ I \end{pmatrix}^T \tilde{P}_{11} \begin{pmatrix} -\Delta_p^T \\ I \end{pmatrix} > 0,$$

where P_{11} and \tilde{P}_{11} are the corresponding upper block of P_e and P_e^{-1} , respectively. If these conditions holds then Δ_c can be obtained as a solution of a problem of type (1).

A similar setting appears in the context of the construction of the scheduling block for the controller in an LPV design, see [11].

For practical reasons we would like to obtain continuous solutions $\Delta_c(\Delta)$. As opposed to the LMI case, where the continuity of the defining matrices and the existence of a feasible solution guarantees existence of a continuous solution, the bicuadratic inequality (2) does not always have continuous solutions at all.

If some additional conditions are fulfilled, for the particular motivation problem one can construct such a solution: if P_{11} and \tilde{P}_{11} has a certain block diagonal sign constraint, see [10], or if for either P_{11} or \tilde{P}_{11} the corresponding graph subspace is maximal, see [11].

The general, however, the problem is highly non trivial and the aim of the present paper is to provide a condition that ensures the continuity of the solutions in the general case.

Example 1: In order to provide an example for which there is no continuous solution let us consider

$$P(t) = M^T(t) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} M(t)$$

with

$$M(t) := \begin{pmatrix} 0 & 1 \\ 1 & 1 - 2t \end{pmatrix}$$

and $t \in \Delta = [0, 1]$.

Suppose that there were such a function $Z(t)$. Then

$$Z(t) = \frac{1}{-1 + 2t + K(t)},$$

where $K(t)$ should be a continuous function from $[0, 1]$ into $(-1, 1)$.

It is obvious then that $Z(t)$ has a fixed sign. Suppose that $Z(t) < 0$. It means that $K(t) < 1 - 2t$ for all $t \in [0, 1]$. But if we set $t = 1$, then we have $K(t) < -1$, which is a contradiction.

Section II provides some basic facts about the matrix Möbius transformation and highlights its role in the parametrization of the maximal negative subspaces of a nonsingular matrix. The main result of the paper is formulated in Section III which gives a necessary and sufficient condition for the existence of a maximal negative subspace that continuously depends on the parameter.

II. MÖBIUS TRANSFORMATION AND INDEFINITE MAXIMAL SUBSPACES

For a nonsingular matrix M partitioned as

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (3)$$

the Möbius transformation T_M is defined by the equation

$$T_M(L) = (C + DL)(A + BL)^{-1} \quad (4)$$

for $L \in \text{dom}(T_M) = \{L : \exists(A + BL)^{-1}\}$.

The dual Möbius transformation is defined by

$$T_M^d(Z) = (ZB + D)^{-1}(ZA + C), \quad (5)$$

and

$$\text{dom}(T_M^d) = \{Z \in \mathbb{F}^{n \times m} : \exists(ZB + D)^{-1}\}.$$

Theorem 1: Let $M \in \mathbb{F}^{(m+n) \times (m+n)}$. Then

$$X \in \text{dom}(T_M^d) \Leftrightarrow X^T \in \text{dom}(T_{L^T M^T L}).$$

Moreover

$$T_M^d(X) = T_{L^T M^T L}^T(X^T), \quad (6)$$

where

$$L = \begin{pmatrix} 0 & I_m \\ I_n & 0 \end{pmatrix}. \quad (7)$$

If $M \in \mathbb{F}^{(m+n) \times (m+n)}$ is a nonsingular matrix, then

$$T_M(X) = -T_{M^{-1}}^d(-X). \quad (8)$$

The following result describes the maximal negative graph subspaces of a symmetric matrix P , i.e., all the matrices Z such that inequality (1) holds.

Theorem 2: Let P be a symmetric matrix such that there is a nonsingular matrix M for which $P = M^{-T} J M^{-1}$, where $J = \text{diag}(-I_m, I_n)$. Then all solutions of (1) are given by

$$Z = T_M(K) \quad (9)$$

for K is an arbitrary contraction ($\|K\| < 1$) in $\text{dom}(T_M)$.

Thus, the parametrization of the solution set relies on describing $\text{dom}(T_M)$.

Lemma 1: Let $T_M : \mathbb{F}^{n \times m} \rightarrow \mathbb{F}^{n \times m}$ be an arbitrary Möbius transformation, with nonsingular coefficient $M \in \mathbb{F}^{(m+n) \times (m+n)}$. Then $\text{dom}(T_M)$ is an open set in $\mathbb{F}^{n \times m}$ and the Lebesgue measure of the set of singular points $\text{dom}(T_M)^c := \mathbb{F}^{n \times m} \setminus \text{dom}(T_M)$ is zero, namely $\mu_L(\text{dom}(T_M)^c) = 0$.

Proof: $X \notin \text{dom}(T_M)$ if and only if $\det(A + BX) = 0$. The determinant function is an analytic function and obviously

$$\text{dom}(T_M)^c = \det^{-1}(0).$$

From this we have the statements of the Lemma. (If the set of zeros of an analytic function f has an accumulation point inside its domain, then f is zero everywhere on the connected component containing the accumulation point. In other words, if (r_n) is a sequence of distinct numbers such that $f(r_n) = 0$ for all n and this sequence converges to a point r in the domain of D , then f is identically zero on the connected component of D containing r .) So if we have an open set $U \in \text{dom}(T_M)^c$ then $\det(A + BX) \equiv 0$, which means that $A = 0$ and $B = 0$. ■

An exhaustive description of the set

$$\mathcal{X}_{A,B} = \{X \mid A + BX \text{ nonsingular}\}$$

can be done by using the generalized singular value decomposition (GSVD), however for our purposes it is sufficient the following result based on the more familiar singular value decomposition (SVD).

Consider the SVD of A as $A = U_A \Sigma_A V_A^T$ with

$$U_A = (U_a \quad U_{as}), \Sigma_A = \begin{pmatrix} \Sigma_a & 0 \\ 0 & 0_{as} \end{pmatrix}, V_A = (V_a \quad V_{as}). \quad (10)$$

and that of $B = U_B \Sigma_B V_B^T$ with

$$U_B = (U_b \quad U_{bs}), \Sigma_B = \begin{pmatrix} \Sigma_b & 0 \\ 0 & 0_{bs} \end{pmatrix}, V_B = (V_a \quad V_{bs}). \quad (11)$$

With these notations one has:

Lemma 2: The matrices

$$X_0(\gamma) = V_B \begin{pmatrix} 0 & \gamma \Sigma_b U_b^T U_{as} \\ 0 & 0 \end{pmatrix} V_A^T. \quad (12)$$

make $A + BX_0(\gamma)$ nonsingular for every $\gamma \neq 0$.

Moreover, for $\gamma^T \leq \frac{1}{\|B\|}$ the matrix $X_0(\gamma)$ is contraction for all $|\gamma| < \gamma^T$.

More details on the construction and the proofs can be found in [14]. A general overview on indefinite matrix analysis can be found in [4].

Let us denote by $\mathcal{B}_{n \times m}(0, r)$ the following set

$$\mathcal{B}_{n \times m}(0, r) := \{K \in \mathbb{R}^{n \times m} : \|K\| < r\}.$$

If the context makes it unambiguous, then we use the notation $\mathcal{B}(0, r)$. We call the set $\mathcal{B}_{n \times m}(0, 1)$ the open unit

ball in $\mathbb{R}^{n \times m}$. For the sake of simplicity the unit ball will be denoted by \mathcal{B}_1 .

Lemma 3: Let T_M be a Möbius transformation. Then we have two cases concerning the domain of T_M . Namely

- 1) $\text{dom}(T_M) = \mathbb{R}^{n \times m}$ if and only if $B = 0$
- 2) if $B \neq 0$, then $\text{dom}(T_M) = U_1 \cup U_2$, where $U_1, U_2 \subset \mathbb{R}^{n \times m}$ are open connected sets and $U_1 \cap U_2 = \emptyset$.

Lemma 4: T_M is a bounded Möbius transformation, namely there exists $\kappa > 0$ such that

$$\|T_M(Z)\| < \kappa(1 + \|Z\|)$$

for all $Z \in \text{dom}(T_M)$, if and only if $\text{dom}(T_M) = \mathbb{R}^{n \times m}$.

Proof: Suppose that $\text{dom}(T_M) = \mathbb{R}^{n \times m}$. Then we have that A is non-singular and $B = 0$. In this case

$$\|T_M(Z)\| = \|C + DZA^{-1}\| \leq \|C\| + \|D\| \|A^{-1}\| \|Z\|.$$

Conversely suppose that there exists $Z_0 \notin \text{dom}(T_M)$. Then there exists a sequence $Z_i \rightarrow Z_0$ such that $Z_i \in \text{dom}(T_M)$. But $\|T_M(Z_i)\| \leq \kappa \|Z_i\|$, we infer that $Z_0 \in \text{dom}(T_M)$. ■

Let us now consider the Möbius transformation on the open unit ball.

Lemma 5: Let T_M be a Möbius transformation on the unit ball $\mathcal{B}_{n \times m}(0, 1)$. Then we have two cases.

- 1) $\text{dom}(T_M) \cap \mathcal{B}_1 = \mathcal{B}_1$
- 2) $\text{dom}(T_M) \cap \mathcal{B}_1 = U_1 \cup U_2$, where U_1, U_2 are open connected sets with $U_1 \cap U_2 = \emptyset$.

In the latter case the points of the unit ball, which are not in the domain of T_M , form a hyper-surface, which cuts the unit ball into two parts (U_1, U_2) .

III. CONTINUITY

Definition 1: We say that the function $\Delta \rightarrow T_{M(\Delta)}$ is continuous if the function $\Delta \rightarrow T_{M(\Delta)}(Z(\Delta))$ is continuous for all $\Delta \rightarrow Z(\Delta)$, such that $Z(\Delta) \in \text{dom}(T_{M(\Delta)})$.

Lemma 6: The map $\Delta \rightarrow T_{M(\Delta)}(Z(\Delta))$ is continuous for all $\Delta \rightarrow Z(\Delta)$, such that $Z(\Delta) \in \text{dom}(T_{M(\Delta)})$ if the map $\Delta \rightarrow M(\Delta)$ is a continuous function.

Lemma 7: Suppose that $M(\Delta)$ and $M^{-1}(\Delta)$ is continuous. Then $Z(\Delta)$ is continuous if and only if

$$K(\Delta) = T_{M^{-1}(\Delta)}(Z(\Delta))$$

is continuous.

Lemma 8: Let Δ be a compact connected set and the map $\Delta \rightarrow M(\Delta)$, $\Delta \in \Delta$ be a continuous function, where $M(\Delta)$ is nonsingular matrix for all $\Delta \in \Delta$. Then the function $\Delta \rightarrow M^{-1}(\Delta)$ is also continuous on $\Delta \in \Delta$.

Proof: Because of the compactness of Δ we have that

$$\inf_{\Delta \in \Delta} \{\|M(\Delta)\|\} > 0,$$

which means that $M(\Delta) \rightarrow M^{-1}(\Delta)$ is continuous and so is the function $\Delta \rightarrow M^{-1}(\Delta)$. ■

Suppose that Δ is a compact connected set and the map $\Delta \rightarrow T_{M(\Delta)}$ is a continuous function from Δ into the set of fixed-size Möbius transformations. Moreover

let $\Delta \rightarrow Z(\Delta)$, $\Delta \in \Delta$ be a continuous function with $\|T_{M(\Delta)}(Z(\Delta))\| < 1$. We will use the following notations

$$M(\Delta) = \begin{pmatrix} A(\Delta) & B(\Delta) \\ C(\Delta) & D(\Delta) \end{pmatrix},$$

$$R_\Delta := \mathcal{B}_1 \cap X(\Delta),$$

where

$$X(\Delta) = \{X : \det(A(\Delta) + B(\Delta)X) = 0\}.$$

Let us denote by R the following set

$$R := \bigcup_{\Delta \in \Delta} R_\Delta.$$

Then, the main result of the paper on the existence of continuous graph subspaces can be formulated as follows:

Theorem 3: If

$$R \supset \mathcal{B}_1,$$

then does not exist a continuous function $Z(\Delta)$ with

$$\|T_{M(\Delta)}(Z(\Delta))\| < 1.$$

If

$$R \subset \mathcal{B}_1,$$

then there exists a fixed contraction $\|K_0\| < 1$ such that

$$Z(\Delta) = T_{M^{-1}(\Delta)}(K_0)$$

is continuous.

Proof: First we show that $R \cap \overline{\mathcal{B}_1}$ is a compact set. Indeed, suppose that there exists a sequence $X_n \in R \cap \overline{\mathcal{B}_1}$ with $\lim X_n = X_0$ and $X_0 \notin R \cap \overline{\mathcal{B}_1}$.

Then, for all X_n there exists a $\Delta_n \in \Delta$ such that

$$\det(A(\Delta_n) + B(\Delta_n)X_n) = 0.$$

Because of the compactness of Δ there exists a subsequence Δ_{n_k} such that $\lim_{k \rightarrow \infty} \Delta_{n_k} = \Delta_0 \in \Delta$. But for all Δ_{n_k} we have $\det(A(\Delta_{n_k}) + B(\Delta_{n_k})X_{n_k}) = 0$, which implies by continuity that $\det(A(\Delta_0) + B(\Delta_0)X_0) = 0$, which means that $X_0 \in R \cap \overline{\mathcal{B}_1}$, i.e., $R \cap \overline{\mathcal{B}_1}$ is a closed and bounded set.

Let us suppose now that $R \supset \mathcal{B}_1$, and that $Z(\Delta)$ is a continuous function with $\|T_{M(\Delta)}(Z(\Delta))\| < 1$. Then $K(\Delta) = T_{M(\Delta)}(Z(\Delta))$ is also a continuous function from Δ into \mathcal{B}_1 , which means that the set

$$K = \{K(\Delta) : \Delta \in \Delta\}$$

is a compact connected subset of \mathcal{B}_1 . As $R \supset \mathcal{B}_1$, there exists a Δ_0 such that $R_{\Delta_0} \cap K \neq \emptyset$ and has at least two points. In this case R_{Δ_0} splits K into two parts K_1 and K_2 , which is a contradiction.

Suppose now, that $R \subset \mathcal{B}_1$. From this it follows that there exists a contraction K_0 such that $K_0 \in \text{dom}(T_{M^{-1}(\Delta)})$ for all $\Delta \in \Delta$. Define the function $Z(\Delta) = T_{M^{-1}(\Delta)}(K_0)$. This function Z is a continuous function on Δ . ■

Example 2: Applying the theorem for Example 1 we have that $R = [-1, 1] \supset \mathcal{B}_1 = (-1, 1)$. Thus we cannot construct a continuous function $Z(t), t \in [0, 1]$ such that

$$|T_{M(t)}(Z(t))| = \left| \frac{1 + (1 - 2t)Z(t)}{Z(t)} \right| < 1.$$

In the two dimensional case this argument can be used in a more general context. Let

$$\begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$$

be a non-singular matrix function on a compact interval $[e, f]$.

Suppose that $R \subset \mathcal{B}_1$. Then

$$K(t) = (c(t) + d(t)Z(t))(a(t) + b(t)Z(t))^{-1} \in (-1, 1)$$

for all $t \in [e, f]$. Let the set of "wrong" points be the range of the function $\tilde{K}(t)$, (in the two dimensional case it is a continuous function) on $[e, f]$.

Then obviously the equation $K(t) = \tilde{K}(t)$ has no solution, which means that $K(t) < \tilde{K}(t)$ or $K(t) > \tilde{K}(t)$ on $[e, f]$. Suppose that $K(t) < \tilde{K}(t)$ for all $t \in [e, f]$. As

$$\left\{ \tilde{K}(t) : t \in [e, f] \right\} \subset \mathcal{B}_1$$

and it is a compact interval, there exists a $t_0 \in [e, f]$ such that $\tilde{K}(t_0) = -1$. But $K(t_0) < \tilde{K}(t_0) = -1$, which is a contradiction.

Remark 1: If we have

$$P(\rho) = \begin{pmatrix} Q(\rho) & S(\rho) \\ S(\rho)^T & R(\rho) \end{pmatrix}$$

with $Q(\rho) < 0$ and $R(\rho) > 0$ then the solution set is an ellipsoid, i.e.,

$$Z(\rho) = Z_0(\rho) + Z_{12}(\rho)KZ_{21}(\rho)$$

with $Z_0(\rho) = R^{-1}(\rho)S(\rho)^T$.

In this case the choice $K_0 = 0$ always gives the continuous solution $Z_0(\rho)$.

This is the case for the problem presented in the introduction: see, e.g., [10]. By permuting the blocks of P_e one has the partitioning

$$\begin{pmatrix} Q_e & S_e \\ S_e^T & R_e \end{pmatrix}$$

with $Q_e < 0$ and $R_e > 0$. The scheduling block Δ_c of the controller can be obtained from the condition

$$\left(\begin{array}{cc|cc} U_{11} & U_{12} & (W_{11} + \tilde{\Delta})^T & W_{21}^T \\ U_{21} & U_{22} & W_{12}^T & (W_{22} + \Delta_c)^T \\ \hline W_{11} + \tilde{\Delta} & W_{12} & V_{11} & V_{12} \\ W_{21} & W_{22} + \Delta_c & V_{21} & V_{22} \end{array} \right) > 0,$$

where $U = R_e - S_e^T Q_e^{-1} S_e$, $V = -Q_e^{-1}$, $W = Q_e^{-1} S_e$.

All scheduling variables Δ_c that can be expressed with the scheduling variables of the plant are given as

$$\Delta_c = -W_{22} + (W_{21} \quad V_{12}) \Phi_{\tilde{\Delta}} \begin{pmatrix} U_{12} \\ W_{12} \end{pmatrix} + \Gamma_{\tilde{\Delta}} L \Lambda_{\tilde{\Delta}}$$

where L ($\|L\| < 1$) is an arbitrary contraction and

$$\Gamma_{\tilde{\Delta}} = \left[U_{22} - (U_{21} \quad W_{12}^T) \Phi_{\tilde{\Delta}} \begin{pmatrix} U_{12} \\ W_{12} \end{pmatrix} \right]^{1/2},$$

$$\Lambda_{\tilde{\Delta}} = \left[V_{22} - (W_{21} \quad V_{21}) \Phi_{\tilde{\Delta}} \begin{pmatrix} W_{21}^T \\ V_{12} \end{pmatrix} \right]^{1/2}.$$

The most simple choice is $L = 0$, i.e.,

$$\Delta_c = -W_{22} + (W_{21} \quad V_{12}) \begin{pmatrix} U_{11} & W_{11}^T + \tilde{\Delta}^T \\ W_{11} + \tilde{\Delta} & V_{11} \end{pmatrix}^{-1} \begin{pmatrix} U_{12} \\ W_{12} \end{pmatrix},$$

however, with other choices one can modify, e.g., the constant term, that influences the closed loop matrix, hence, stability, of the nominal system.

Remark 2: For the problem formulated in the introduction if one of the graph subspaces in the condition

$$\begin{pmatrix} I \\ \Delta_p \end{pmatrix}^T P_{11} \begin{pmatrix} I \\ \Delta_p \end{pmatrix} < 0, \text{ and } \begin{pmatrix} -\Delta_p^T \\ I \end{pmatrix}^T \tilde{P}_{11} \begin{pmatrix} -\Delta_p^T \\ I \end{pmatrix} > 0,$$

turns out to be maximal, where P_{11} and \tilde{P}_{11} are the corresponding upper blocks of P_e and P_e^{-1} , respectively, then one can construct a continuous solution by slightly modifying the algorithm provided in [11].

Maximality imposes an inertia condition for P_{11} or \tilde{P}_{11} .

Only the special cases shown in Remark 1 and Remark 2 provides an explicit continuous solution. In the general case one can use the result of Theorem 3 to provide such a solution: factorize $P(\Delta)$ and find a contraction K_0 in the intersection of the domains of the corresponding Möbius transforms. Then a continuous solution is provided by $Z(\Delta) = T_{M^{-1}(\Delta)}(K_0)$.

IV. CONCLUSION

The paper provides an exhaustive condition for the existence of a continuous maximal negative graph subspace for a parameter dependent non-singular indefinite matrix.

The existence condition can be formulated as a non emptiness of the intersection of the domains of certain parameter dependent Möbius transform. Unfortunately this condition is formulated by using a factorization of the original symmetric matrix. Further research has to be done in order to obtain a general condition that is formulated in terms of the original data.

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APPENDIX

A fundamental result of the LMI framework in the derivation of the design equations is the Elimination lemma. The conditions of the lemma leads directly to the analysis equations that are the starting point of any controller design.

Lemma 9 (Elimination Lemma): Let $Q = Q^T$ be a nonsingular matrix with inertia $in(Q) = (m, 0, n)$ and let us consider the quadratic matrix inequality

$$\begin{pmatrix} I \\ C + AXB \end{pmatrix}^T Q \begin{pmatrix} I \\ C + AXB \end{pmatrix} < 0. \quad (13)$$

This inequality has a solution if and only if

$$B_{\perp}^T \begin{pmatrix} I \\ C \end{pmatrix}^T Q \begin{pmatrix} I \\ C \end{pmatrix} B_{\perp} < 0 \quad (14)$$

and

$$A_{\perp} \begin{pmatrix} -C^T \\ I \end{pmatrix}^T Q^{-1} \begin{pmatrix} -C^T \\ I \end{pmatrix} A_{\perp}^T > 0. \quad (15)$$

Here A_{\perp} denotes a matrix with $A_{\perp}A = 0$ and $A_{\perp}A_{\perp}^T > 0$ while B_{\perp} denotes an arbitrary basis matrix such that $BB_{\perp} = 0$ and that $B_{\perp}^TB_{\perp} > 0$. For a classical proof see, e.g., [5], [11]. For the sake of completeness we provide here a proof based entirely on the properties of the Möbius transform.

Proof: Let us write Q in the form $Q = M^{-T}HM^{-1}$, where $H = \text{diag}(-I_m, I_n)$, and M and M^{-1} is written in the form

$$M = \begin{pmatrix} E & F \\ G & H \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{G} & \tilde{H} \end{pmatrix}.$$

We have that

$$\begin{pmatrix} I \\ AXB \end{pmatrix}^T \begin{pmatrix} I & 0 \\ C & I \end{pmatrix}^T M^{-T}HM^{-1} \begin{pmatrix} I & 0 \\ C & I \end{pmatrix} \begin{pmatrix} I \\ AXB \end{pmatrix} < 0.$$

From this inequality we obtain that

$$AXB = T_{\tilde{C}M}(K), \quad \|K\| < 1, \quad \tilde{C} = \begin{pmatrix} I & 0 \\ -C & I \end{pmatrix},$$

from which we get the conditions

$$\begin{aligned} A_{\perp}T_{\tilde{C}M}(K) &= 0 \\ T_{\tilde{C}M}(K)B_{\perp} &= 0 \Leftrightarrow -T_{M^{-1}\tilde{C}^{-1}}^d(-K) = 0, \end{aligned}$$

which leads to the equations

$$\begin{aligned} A_{\perp} \begin{pmatrix} -CE + G & -CF + H \end{pmatrix} \begin{pmatrix} I \\ K \end{pmatrix} &= 0, \\ (-K \quad I) \begin{pmatrix} \tilde{E} + \tilde{F}C \\ \tilde{G} + \tilde{H}C \end{pmatrix} B_{\perp} &= 0. \end{aligned}$$

The first equation after transposition can be written in the form

$$\begin{pmatrix} -E^TC^T + G^T \\ -F^TC^T + H^T \end{pmatrix} A_{\perp}^T = \begin{pmatrix} -K^T \\ I \end{pmatrix} S_1,$$

with some nonsingular matrix S_1 .

This equation can be also written in the form

$$M^T \begin{pmatrix} -C^T \\ I \end{pmatrix} A_{\perp}^T = \begin{pmatrix} -K^T \\ I \end{pmatrix} S_1$$

which means that

$$A_{\perp} \begin{pmatrix} -C^T \\ I \end{pmatrix}^T P^{-1} \begin{pmatrix} -C^T \\ I \end{pmatrix} A_{\perp}^T > 0.$$

Similarly

$$\begin{pmatrix} \tilde{E} + \tilde{F}C \\ \tilde{G} + \tilde{H}C \end{pmatrix} B_{\perp} = \begin{pmatrix} I \\ K \end{pmatrix} S_2,$$

with nonsingular matrix S_2 .

This equation is equivalent to

$$M^{-1} \begin{pmatrix} I \\ C \end{pmatrix} B_{\perp} = \begin{pmatrix} I \\ K \end{pmatrix} S_2,$$

which means that

$$B_{\perp}^T \begin{pmatrix} I \\ C \end{pmatrix}^T P \begin{pmatrix} I \\ C \end{pmatrix} B_{\perp} < 0.$$

The proof is complete. ■