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Multivariate S-procedure

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Abstract— **Different variants of the S-procedure provides a very important tool in robust stability and robust performance analysis. Concerning performance assessment this paper shows that the design framework based on the full block S-procedure (extended KYP lemma) contains an inherent conservativeness. The main result of the paper is a multivariate version of the classical S-procedure, stated for negative graph subspaces.**

A new solvability condition for the Elimination lemma is also provided.

I. Introduction and motivation

Robust stability and robust performance analysis and synthesis of control systems with parameter uncertainties and parameter variations is one of the fundamental issues in system theory.

In the most common framework models are augmented with performance specifications and uncertainties. Weighting functions are applied to the performance signals to meet performance specifications and guarantee a tradeoff between performances. The uncertainties are modeled by both unmodeled dynamics and parametric uncertainties. As a result of this construction a linear fractional transformation (LFT) interconnection structure, which is the basis of control design, is achieved.

As a common structure, these algorithms have an analysis phase and a synthesis phase. The analysis phase consists of solving a set of linear matrix inequalities (LMIs) that are obtained by using some variant of the S-procedure and usually involves a relaxation of an infinite number of conditions to a set of finite number of constraints. The synthesis phase consists of obtaining the LTI part of the controller and possibly the scheduling variables of the controller for qLPV design, see, e.g., [4], [21], [7], [15], [23].

The main theoretical tools in this respect are the full-block S-procedure, a variant of the Elimination lemma and some variant of the classical S-procedure. KYP lemma and the full block S-procedure (extended KYP lemma), [11], [15], performs a separation step in the analysis: by introducing a multiplier for the uncertainties (and scheduling variables in LPV design) the task of finding the controller is formulated as an inequality containing fix matrices. This inequality will be the source of the analysis conditions by using the Elimination lemma, [6], [15]. The classical S-procedure, [24], [3], [16], [10] is a relaxation method: it tries to solve a system of quadratic inequalities via a LMI relaxation.

It is very important both from theoretical and practical point of view in what extent these methods introduce conservativeness in the design. A lot of research was done in this respect. The starting point of this paper is different. We approach the full block S-procedure from a pure analysis, i.e., performance assessment point of view. From this perspective it is shown that there is always a certain amount of conservativeness in the classical design chain. When it comes to the solution set of the inequalities, it will be pointed out the intimate relationship between the full block S-procedure and the Elimination lemma. As a result a new solvability condition for the Elimination lemma is given.

The main result of the paper is a multivariate version of the classical S-procedure, see the Appendix. A similar result of this type was provided in a very special case in [9].

Section II gives an other perspective of the extended KYP lemma and some implications concerning the Elimination lemma. The main result of the paper, i.e., the multivariate S-procedure, is formulated in Section IV while the intermediate technical results concerning inclusions of maximal negative subspace sets are listed in Section III.

II. Extended KYP lemma and performance assessments

The following result is a consequence of the full block S-procedure. This variant is the most often version used in practice.

Lemma 1 (Extended KYP lemma): For a given compact set $\mathbf{\Delta}$ we have

$$\begin{pmatrix} I \\ F(\delta) \end{pmatrix}^* P_p \begin{pmatrix} I \\ F(\delta) \end{pmatrix} < 0, \quad \forall \delta \in \mathbf{\Delta} \tag{1}$$

where $F(\delta) = D + C\delta(I - A\delta)^{-1}B$, if and only if there exists a symmetric (Hermitian) multiplier P which satisfies

M-1:

$$\begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^* P \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^* P_p \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} < 0,$$

M-2:

$$\begin{pmatrix} \delta \\ I \end{pmatrix}^* P \begin{pmatrix} \delta \\ I \end{pmatrix} > 0, \quad \forall \delta \in \mathbf{\Delta}.$$

This result is a generalization of the Finsler's lemma (see, e.g., [2]) and for its proof see [7]. The result is also a consequence of the full block S-procedure, see [12]. For the applicability of this result for unbounded domains and the motivation of labeling this statement as an extended KYP lemma see [19].

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It is not a condition for the lemma, but in the practical context often the performance multiplier P_p is nonsingular and the corresponding graph subspace is a maximal negative subspace.

These inequalities play a crucial role in the design of robust controllers. Much effort has been done in the lossless parametrization of the multiplier P for a given uncertainty set. This topic is not our concern here.

In the context of this paper we suppose that we already performed the design step and we also have the multipliers. Then $\mathbf{M-2}$ provides the a posteriori performance assessment, i.e., the largest set $\mathbf{\Delta}_a$ for which (1) is guaranteed by design. Observe that the lemma does not require and does not claim that $\mathbf{\Delta}_a = \mathbf{\Delta}$. In general we have $\mathbf{\Delta} \subset \mathbf{\Delta}_a$.

It turns out, however, that the set $\mathbf{\Delta}_p$ for which the actual performance, i.e., inequality (1), holds is always larger than this: $\mathbf{\Delta}_a \subset \mathbf{\Delta}_p$. This means that every design that is based on Lemma 1 is necessarily conservative regardless, whether the relaxation method used for the multiplier search was lossless or not.

To see this, consider first the following special setting: $D=0,\ A=0$, $B=I$ and $C=I$. Then (1) reads as

$$\begin{pmatrix} I \\ \delta \end{pmatrix}^* P_p \begin{pmatrix} I \\ \delta \end{pmatrix} < 0,$$

while from $\mathbf{M-1},\ \mathbf{M-2}$ we have

$$P_p - \bar{P} < 0, \quad \begin{pmatrix} I \\ \delta \end{pmatrix}^* \bar{P} \begin{pmatrix} I \\ \delta \end{pmatrix} < 0,$$

with

$$\bar{P} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}^* (-P) \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

It is not entirely trivial, but if the two negativity sets are equal, then $\bar{P} = \alpha P_p$ with some $\alpha > 0$. But P_p is indefinite, thus only inclusion is possible.

For the general case rewrite (1) as

$$\begin{pmatrix} I \\ D+CXB \end{pmatrix}^T P_p \begin{pmatrix} I \\ D+CXB \end{pmatrix} < 0$$

where $X = \delta(I - A\delta)$. It is not hard to recognize the problem of the Elimination lemma, see the Appendix. Details of the solution of this inequality and the solution set is described in [18]. It turns out that in a suitable basis $\tilde{X}$ depends on the solution of a smaller inequality

$$\begin{pmatrix} I \\ Z \end{pmatrix}^T P_s \begin{pmatrix} I \\ Z \end{pmatrix} < 0,$$

i.e., $\tilde{X} = \begin{pmatrix} Z & \tilde{X}_{12} \\ \tilde{X}_{21} & \tilde{X}_{22} \end{pmatrix}$, with $\tilde{X}_{ij}$ arbitrary. However, the multiplier inequality reads as

$$\begin{pmatrix} \begin{pmatrix} \bar{Z} & \bar{X}_{12} \\ \bar{X}_{21} & \bar{X}_{22} \end{pmatrix} \\ I \end{pmatrix}^T \bar{P} \begin{pmatrix} \begin{pmatrix} \bar{Z} & \bar{X}_{12} \\ \bar{X}_{21} & \bar{X}_{22} \end{pmatrix} \\ I \end{pmatrix} > 0,$$

which always determine a smaller set, since $\bar{Z},\ \bar{X}_{ij}$ are constrained by the inequality.

At this point we have the opportunity to present a new solvability condition for the Elimination lemma. It is known that the solution set of the inequality

$$\begin{pmatrix} I \\ C+AXB \end{pmatrix}^T Q \begin{pmatrix} I \\ C+AXB \end{pmatrix} < 0$$

is either empty or it is a set obtained as an image of the contractive ball through a Möbius transform, see [18]. Thus, if the problem is solvable, there always exists a compact set formed entirely by solutions of the inequality. It follows that Lemma 1 is applicable and we can formulate the following result:

Lemma 2: Let $Q = Q^T$ be a non-singular matrix with inertia $in(Q) = (m, 0, n)$ and let us consider the quadratic matrix inequality

$$\begin{pmatrix} I_m \\ C+AXB \end{pmatrix}^T Q \begin{pmatrix} I_m \\ C+AXB \end{pmatrix} < 0. \tag{2}$$

This inequality has a solution if and only if there exist a symmetric matrix P with $\text{in}_-(P) \geq m$ such that

$$\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}^* P \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} + \begin{pmatrix} 0 & I \\ A & C \end{pmatrix}^* Q \begin{pmatrix} 0 & I \\ A & C \end{pmatrix} < 0.$$

This result replaces the nonlinear condition on the inverse of Q with another nonlinear condition for the inertia but for a multiplier P. Potentially this relaxation can be exploited for searching structured solutions.

We conclude this section with reformulating Lemma 1 as an inclusion property. The lemma states that inequality (1) holds if and only if there is a multiplier P for which $\mathbf{\Delta}_a \subset \mathbf{\Delta}_p$ and $\mathbf{M-1}$ holds. An important application of the multivariate S-procedure provided by this paper is that in the general setting of Lemma 1, i.e., from knowing that for a multiplier P we have $\mathbf{\Delta}_a \subset \mathbf{\Delta}_p$ we have $\mathbf{M-1}$ with αP and some $\alpha > 0$.

III. Inclusion theorems

In what follows we deal with possible non maximal negative graph subspaces. We consider only the real valued case but the assertions remain valid for the complex case, too. Let us introduce the following notations: let $P \in \mathbb{R}^{(m+z+n)\times(m+z+n)}$ be symmetric matrix with inertia (m, z, n), i.e., having exactly m negative, n positive and z zero eigenvalues, and for $d \leq m$ consider the set

$$\mathcal{G}_{<_d}(P) = \left\{ Z \in \mathbb{R}^{(n+z+m-d)\times d} : \begin{pmatrix} I_d \\ Z \end{pmatrix}^* P \begin{pmatrix} I_d \\ Z \end{pmatrix} < 0 \right\}.$$

For the maximal subspace, i.e., if $d = m$, we also use the notation $\mathcal{G}_{<_m}(P) = \mathcal{G}_<(P)$. Obviously, if $d > m$, then $\mathcal{G}_{<_d}(P) = \emptyset$.

To ease the notation for two symmetric matrices consider $m = \min\{m_1, m_2\}$ and $\mathcal{G}_<(P)$ corresponds to this m. Moreover, if otherwise not stated, it is supposed that $d \leq m$. In what follows it is supposed that $n_i \neq 0$, i.e., the matrices are indefinite.

Lemma 3: Let us consider a symmetric matrix P with inertia $\mathrm{in}(P) = (m, z, n)$. Then for any $1 \leq d_1 < d_2 \leq m$ a negative graph subspace of dimension d_1 can be complemented to a negative graph subspace of dimension d_2.

Proof: Let us consider a fixed negative graph subspace of dimension d_1:

$$\mathcal{Z} = \begin{pmatrix} I_{d_1} \\ Z_0 \\ Z_1 \\ Z_2 \end{pmatrix}$$

partitioned according to the dimensions $(d_1, d_2 - d_1, m - d_2, n + z)$, and define the transformation matrix

$$T = \begin{pmatrix} I_{d_1} & 0 & 0 & 0 \\ Z_0 & I_{d_2-d_1} & 0 & 0 \\ Z_1 & 0 & I_{m-d_2} & 0 \\ Z_2 & 0 & 0 & I_{n+z} \end{pmatrix}.$$

Then we have

$$\mathcal{Z}^T P \mathcal{Z} = \mathcal{Z}^T T^{-T} T^T P T T^{-1} \mathcal{Z}, \quad \tilde{\mathcal{Z}} = T^{-1}\mathcal{Z} = \begin{pmatrix} I_{d_1} \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The map T defines a one to one correspondence between the d_1 dimensional negative graph subspaces of P and the d_1 dimensional negative graph subspaces of $\tilde{P} = T^T P T$.

Observe that

$$\tilde{P} = \begin{pmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{pmatrix}$$

with $\tilde{Q} < 0$ a d_1 dimensional symmetric matrix.

The subspace

$$\mathcal{W} = \begin{pmatrix} 0 \\ I_{d_2-d_1} \\ W_1 \\ W_2 \end{pmatrix}$$

completes $\tilde{\mathcal{Z}}$ to a negative graph subspace of dimension d_2 if and only if $\begin{pmatrix} \tilde{\mathcal{Z}} & \mathcal{W} \end{pmatrix}^T \tilde{P} \begin{pmatrix} \tilde{\mathcal{Z}} & \mathcal{W} \end{pmatrix} =$

$$= \begin{pmatrix} \tilde{Q} & \tilde{S} \begin{pmatrix} I_{d_2-d_1} \\ W_1 \\ W_2 \end{pmatrix} \\ \begin{pmatrix} I_{d_2-d_1} \\ W_1 \\ W_2 \end{pmatrix}^T \tilde{S}^T & \begin{pmatrix} I_{d_2-d_1} \\ W_1 \\ W_2 \end{pmatrix}^T \tilde{R} \begin{pmatrix} I_{d_2-d_1} \\ W_1 \\ W_2 \end{pmatrix} \end{pmatrix} < 0.$$

The last inequality is equivalent to $\tilde{Q} < 0$ and

$$\begin{pmatrix} I_{d_2-d_1} \\ W_1 \\ W_2 \end{pmatrix}^T (\tilde{R} - \tilde{S}^T \tilde{Q}^{-1} \tilde{S}) \begin{pmatrix} I_{d_2-d_1} \\ W_1 \\ W_2 \end{pmatrix} < 0.$$

Since

$$\mathrm{in}(P) = \mathrm{in}(\tilde{P}) = \mathrm{in}(\tilde{Q}) + \mathrm{in}(\tilde{R} - \tilde{S}^T \tilde{Q}^{-1} \tilde{S})$$

we have $\mathrm{in}(\tilde{R} - \tilde{S}^T \tilde{Q}^{-1} \tilde{S}) = (m - d_1, n, z)$, i.e., there exists such a $W_1,\ W_2$.

Since $T\mathcal{W} = \mathcal{W}$, we have that $\begin{pmatrix} \mathcal{Z} & \mathcal{W} \end{pmatrix}$ is a negative subspace of dimension d_2 for P, as desired. ■

Lemma 4: Inclusion $\mathcal{G}_{<d_2}(P_2) \subset \mathcal{G}_{<d_2}(P_1)$ implies $\mathcal{G}_{<d_1}(P_2) \subset \mathcal{G}_{<d_1}(P_1)$ for all $1 \leq d_1 \leq d_2 \leq m$.

Proof: As a consequence of Lemma 3 every d_1 dimensional negative graph subspace $\mathcal{Z}$ can be obtained as

$$\mathcal{Z} = \mathcal{X} V$$

where $\mathcal{X}$ is a suitable d_2 dimensional negative graph subspace and V is a full column rank d_1 dimensional vector.

Since

$$\mathcal{X}^T P \mathcal{X} < 0 \quad \Rightarrow \quad V^T \mathcal{X}^T P \mathcal{X} V < 0$$

for every d_1 dimensional full column rank vector V, we have $\mathcal{Z}^T P \mathcal{Z} < 0$ for all d_1 dimensional negative graph subspaces $\mathcal{Z}$.

Thus we have the desired result. ■

Remark 1: The assertion remain valid also in the case when the strict inequalities are changed to non-strict inequalities.

In order to prove the converse statement of Lemma 4 we need some additional auxiliary results.

Lemma 5: If

$$\begin{pmatrix} I_d \\ V \end{pmatrix}^T Q \begin{pmatrix} I_d \\ V \end{pmatrix} < 0 \tag{3}$$

for every $V \in \mathbb{R}^{(m-d)\times d}$ then $Q \leq 0$.

Proof: Let us consider a partitioning

$$Q = \begin{pmatrix} E & F \\ F^T & G \end{pmatrix}.$$

Then, from $V = 0$ one has $E < 0$. Thus Q can be transformed in the form

$$Q = \mathrm{diag}(|E|^{\frac{1}{1}}, M_G)^T \ \tilde{Q} \ \mathrm{diag}(|E|^{\frac{1}{1}}, M_G),$$

where

$$\tilde{Q} = \begin{pmatrix} -I_d & \tilde{F} \\ \tilde{F}^T & \Gamma \end{pmatrix},$$

with Γ diagonal.

With the choice $\tilde{V} = \lambda E_{ij}$ and conjugating with e_j, where $\lambda \in \mathbb{R}$, E_{ij} has its only non-vanishing element on the position (i, j) and e_j is the jth canonical unit vector, one has

$$-1 + 2\lambda \tilde{F}_{ji} + \lambda^2 \gamma_i < 0.$$

It follows that either $\gamma_i = 0, \tilde{F}_{ji} = 0$ or

$$\begin{pmatrix} -1 & \tilde{F}_{ji} \\ \tilde{F}_{ji} & \gamma_i \end{pmatrix} < 0.$$

It follows that $\tilde{Q}$ can be transformed into

$$\bar{Q} = \begin{pmatrix} -I_d & \bar{F} & 0 \\ \bar{F}^T & \bar{\Gamma} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with $\bar{\Gamma} < 0$. Taking a Schur complement we have condition

$$\begin{pmatrix} I_d \\ \bar{V}_1 + \bar{\Gamma}^{-1}\bar{F}^T \end{pmatrix}^T \begin{pmatrix} -I_d - \bar{F}^T\bar{\Gamma}^{-1}\bar{F} & 0 \\ 0 & \bar{\Gamma} \end{pmatrix} \begin{pmatrix} I_d \\ \bar{V}_1 + \bar{\Gamma}^{-1}\bar{F}^T \end{pmatrix} < 0,$$

i.e., $-I_d - \bar{F}^T\bar{\Gamma}^{-1}\bar{F} < 0$. Thus $\bar{Q} \leq 0$, hence $Q \leq 0$. ■

Remark 2: The assertion of the lemma remains valid for the non-strict inequality, too.

Lemma 6: Let U_1 and U_2 be two opens sets of a topologial space $\mathcal{X}$ with $U_2 \subset \overline{U}_1$. Then $U_2 \subset U_1$.

Proof: $U_2 \setminus U_1 = U_2 \cap U_1^c = U_2 \cap (X \setminus \overline{U}_1) = \emptyset$ ■

Now we are in a position to formulate the main result of this section:

Theorem 1: Let $P_1, P_2 \in \mathbb{R}^{(m+n)\times(m+n)}$ be symmetric matrices with inertia $\mathrm{in}(P_i) = (m_i, z_i, n_i)$ and consider that $d < m = \min\{m_1, m_2\}$.

Then the following are equivalent

a.) $\mathcal{G}_{<_d}(P_2) \subset \mathcal{G}_{<_d}(P_1)$,
b.) $\mathcal{G}_{<}(P_2) \subset \mathcal{G}_{<}(P_1)$.

Proof: Assertion $(b) \Rightarrow (a)$ is Lemma 4.

To prove $(a) \Rightarrow (b)$ suppose that $\mathcal{G}_{<_d}(P_2) \subset \mathcal{G}_{<_d}(P_1)$, i.e., there exists X_0 such that $X_0 \in \mathcal{G}_{<}(P_2)$ but $X_0 \notin \mathcal{G}_{<}(P_1)$. Let

$$Q = \begin{pmatrix} I_m \\ X_0 \end{pmatrix}^* P_1 \begin{pmatrix} I_m \\ X_0 \end{pmatrix},$$

then Q is non negative definite on the subspaces

$$\begin{pmatrix} I_d \\ V \end{pmatrix}$$

for every $V \in \mathbb{R}^{(m-d)\times d}$. From Lemma 5 we have that $Q \leq 0$, i.e.,

$$\begin{pmatrix} I_m \\ X_0 \end{pmatrix}^* P_1 \begin{pmatrix} I_m \\ X_0 \end{pmatrix} \leq 0.$$

Thus $\mathcal{G}_{<}(P_2) \subset \overline{\mathcal{G}_{<}(P_1)}$. Then, according to Lemma 6 we have $\mathcal{G}_{<}(P_2) \subset \mathcal{G}_{<}(P_1)$. ■

Remark 3: The assertion of the theorem remains valid for the non-strict inequality, i.e., if we consider non-positive graph subspaces instead of negative subspaces.

IV. MULTIVARIATE S-PROCEDURE

The following theorem holds.

Theorem 2: Let $P_1, P_2 \in \mathbb{R}^{(m+n)\times(m+n)}$ be symmetric matrices with inertia $\mathrm{in}(P_i) = (m_i, z_i, n_i)$ and consider $d < m = \min\{m_1, m_2\}$.

Then the following are equivalent

a.) $\mathcal{G}_{<_d}(P_2) \subset \mathcal{G}_{<_d}(P_1)$ for some d such that $1 \leq d \leq m$,
b.) there exists $\alpha > 0$ such that

$$P_1 - \alpha P_2 \leq 0.$$

Proof: The assertion $(b) \Rightarrow (a)$ is trivial.

To prove $(a) \Rightarrow (b)$ consider Theorem 1. It follows that if (a) is valid for a $d \leq m$, then it is also valid for $d = 1$. Moreover, taking the closure of the open sets, one has the implication for non-positive graph subspaces, too. Then we can apply the statement of the classical S-procedure, to obtain the existence of $\alpha \geq 0$ such that $P_1 - \alpha P_2 \leq 0$. Since we have indefinite matrices, it follows that $\alpha > 0$. ■

Corollary 1: If $\mathcal{G}_{<_d}(P_2) \subset \mathcal{G}_{<_d}(P_1)$ for some d such that $1 \leq d \leq m$ then $m_2 \leq m_1$ and $n_2 \geq n_1$.

Proof: According to Theorem 2 we have $P_1 \leq \alpha P_2$. If $m_2 > m_1$, then it were a v such that $v^T P_2 v < 0$ but $v^T P_1 v \geq 0$, a contradiction.

The inequality $n_2 \geq n_1$ follows analogously. ■

This shows that if inclusion of the negative graph subspaces holds, then we have

$$J_1 \leq J_2$$

where $J_i = \mathrm{diag}(-I_{m_i}, 0_{z_i}, I_{n_i})$ and $P_i = M_i^T J_i M_i$ with M_i nonsingular.

Corollary 2: Let us consider the set $\mathbf{\Delta}_a$ defined by the inequality

$$\begin{pmatrix} \delta \\ I \end{pmatrix}^* P \begin{pmatrix} \delta \\ I \end{pmatrix} > 0, \quad \forall \delta \in \mathbf{\Delta}_a.$$

Then

$$\begin{pmatrix} I \\ F(\delta) \end{pmatrix}^* P_p \begin{pmatrix} I \\ F(\delta) \end{pmatrix} < 0, \quad \forall \delta \in \mathbf{\Delta}_a$$

where $F(\delta) = D + C\delta(I - A\delta)^{-1}B$, implies that

$$\begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^* \alpha P \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^* P_p \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} < 0,$$

for some $\alpha > 0$.

Proof: According to Lemma 1 we have P_1 such that

$$\begin{pmatrix} I \\ F(\delta) \end{pmatrix}^* P_p \begin{pmatrix} I \\ F(\delta) \end{pmatrix} < 0$$

holds for all $\delta \in \mathbf{\Delta}_{P_1}$, moreover, $\mathbf{\Delta}_a \subset \mathbf{\Delta}_{P_1}$ and

$$\begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^* P_1 \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^* P_p \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} < 0.$$

According to Theorem 2 we have $\bar{P}_1 \leq \alpha \bar{P}$ for some $\alpha > 0$, where

$$\bar{P} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}^* (-P) \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

and

$$\bar{P}_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}^* (-P_1) \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

i.e., $P_1 \geq \alpha P$. It follows that

$$\begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^* \alpha P \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^* P_p \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} < 0.$$

■

V. CONCLUSION

The paper presents the full block S-procedure (extended KYP lemma) from a performance assessment perspective and it shows that the design framework based on the extended KYP lemma contains an inherent conservativeness. A new solvability condition for the Elimination lemma is also provided.

The main result of the paper is a multivariate version of the classical S-procedure, stated for negative graph subspaces.

REFERENCES

[1] A. Packard and G. Balas, "Theory and application of linear parameter varying control techniques," *American Control Conference, Workshop I, Albuquerque, New Mexico*, 1997.

[2] S.P. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishan. *Linear matrix inequalities in system and control theory*. SIAM Studies in Applied Mathematics 15. SIAM, Philadelphia, 1994.

[3] K. Derinkuyu and M. Pinar, "On the S-procedure and Some Variants," *Mathematical Methods of Operations Research*, vol. 64, pp. 55–77, 2006.

[4] P. Gahinet, P. Apkarian, and M. Chilali, "Affine parameter dependent Lyapunov functions and real parameter uncertainity," *IEEE Transactions on Automatic Control*, vol. 41, pp. 436–442, 1996.

[5] I. Gohberg, P. Lancaster, and L. Rodman. *Indefinite linear algebra and applications*. Birkhauser, 2005.

[6] A. Helmersson. IQC synthesis based on inertia constraints. *Proceedings of the 14th IFAC World Congress, Bejing, China* pages 163–168, 1998.

[7] T. Iwasaki and S. Hara. Well-posedness of feedback systems: Insights into exact robustness analysis and approximate computations. *IEEE Trans. Automat Contr*, 43619–630, 1998.

[8] T. Iwasaki and G. Shibata, "LPV system analysis via quadratic separator for uncertain implicit systems," *IEEE Transactions on Automatic Control*, vol. 46, pp. 1195–1208, 2001.

[9] Z. Luo, J. Sturm and S. Zhang, "Multivariate Nonnegative Quadratic Mappings," *SIAM Journal on Optimization*, vol. 14, pp. 1140–1162, 2004.

[10] I. Polik and T. Terlaky, "A Survey of the S-Lemma," *SIAM Review*, vol. 49, pp. 371–418, 2007.

[11] A. Rantzer, "On the Kalman-Yakubovich-Popov lemma, " *Systems & Control Letters*, vol. 28, pp. 7–10, 1996.

[12] C. Scherer, "A full block S-procedure with applications," *Proc. Conference on Decision and Control*, pp. 2602–2607, 1997.

[13] M. Dettori and C. Scherer, "Robust stability analysis for parameter dependent systems using full block S-procedure," *Proc. Conference on Decision and Control*, pp. 2798–2799, 1998.

[14] C. Scherer, "Robust Mixed Control and Linear Parameter-Varying Control with Full Block Scalings," in *Advances in Linear Matrix Inequality Methods in Control*, SIAM, 2000.

[15] C. Scherer, "LPV control and full block multipliers," *Automatica*, vol. 37, pp. 361–375, 2001.

[16] C. Scherer, "LMI relaxations in robust control," *European Journal of Control*, vol. 12, p. 3–29, 2006.

[17] Z. Szabó, P. Gáspár and J. Bokor, "A novel control-oriented multi-affine qLPV modeling framework," *Proc. of the 18th Mediterranean conference on control and automation, Marrakech, Maroc*, pp. 1019–1024, 2010.

[18] Z. Szabó, Zs. Biró and J. Bokor, "All controllers for an LPV robust control problem," *Proc. of the 7th IFAC Symposium on Robust Control Design, Aalborg, Denmark*, 2012.

[19] Z. Szabó, Zs. Biró and J. Bokor, "Möbius transform and efficient LPV synthesis," *Proc. of the 51th IEEE Conference on Decision and Control, Maui, Hawaii*, 2012.

[20] C.P. Wei and L. Lee, "On the Equivalent Relationship Between Generalized Performance, Robust Stability, and Quadratic Stability," *IEEE Transactions on Automatic Control*, vol. 55, pp. 2811–2816, 2010.

[21] F. Wu, X. H. Yang, A. Packard, and G. Becker, "Induced $\mathcal{L}^2$ norm control for LPV systems with bounded parameter variation rates," *International Journal of Nonlinear and Robust Control*, vol. 6, pp. 983–998, 1996.

[22] F. Wu, "A generalized LPV system analysis and control synthesis framework," *International Journal of Control*, vol. 74, pp. 745–759, 2001.

[23] F. Wu and K. Dong, "Gain scheduling control of LFT systems using parameter dependent Lyapunov functions," *Automatica*, vol. 42, pp. 39–50, 2006.

[24] V. A. Yakubovich, "S-procedure in nonlinear control theory," *Automatica*, vol. 4, pp. 73–93, 1977.

APPENDIX

Lemma 7 (S-procedure): The following statements are equivalent:

i.) there is an x_0 such that

$$\begin{pmatrix}1\\x_0\end{pmatrix}^T P_2 \begin{pmatrix}1\\x_0\end{pmatrix} < 0$$

and for all $x \in \mathbb{R}^n$

$$\begin{pmatrix}1\\x\end{pmatrix}^T P_2 \begin{pmatrix}1\\x\end{pmatrix} \leq 0 \quad \Rightarrow \quad \begin{pmatrix}1\\x\end{pmatrix}^T P_1 \begin{pmatrix}1\\x\end{pmatrix} \leq 0$$

ii.) there exists $\alpha \geq 0$ such that

$$P_1 - \alpha P_2 \leq 0.$$

A fundamental result of the LMI framework in the derivation of the design equations is the Elimination lemma. The conditions of the lemma leads directly to the analysis equations that are the starting point of any controller design.

Lemma 8 (Elimination Lemma): Let $Q = Q^T$ be a nonsingular matrix with inertia $in(Q) = (m, 0, n)$ and let us consider the quadratic matrix inequality

$$\left(\begin{array}{c} I \\ C + AXB \end{array} \right)^T Q \left(\begin{array}{c} I \\ C + AXB \end{array} \right) < 0. \tag{4}$$

This inequality has a solution if and only if

$$B_\dashv^* \left(\begin{array}{c} I \\ C \end{array} \right)^T Q \left(\begin{array}{c} I \\ C \end{array} \right) B_\dashv < 0 \tag{5}$$

and

$$A_\perp \left(\begin{array}{c} -C^T \\ I \end{array} \right)^T Q^{-1} \left(\begin{array}{c} -C^T \\ I \end{array} \right) A_\perp^* > 0. \tag{6}$$

Here $A_\perp$ denotes a matrix with $A_\perp A = 0$ and $A_\perp A_\perp^* > 0$ while $B_\dashv$ denotes an arbitrary basis matrix such that $BB_\dashv = 0$ and that $B_\dashv^* B_\dashv > 0$. For a proof see, e.g., [6], [15].

The following result describes the maximal negative graph subspaces of a symmetric matrix P, i.e., all the matrices Z such that

$$\begin{pmatrix}I_q\\Z\end{pmatrix}^* P \begin{pmatrix}I_q\\Z\end{pmatrix} < 0, \tag{7}$$

where $P \in \mathbb{F}^{(q+p)\times(q+p)}$ with inertia $in(P_s) = (q, 0, p)$.

Theorem 3: Let $\mathcal{M}$ be a symmetric matrix such that there is a nonsingular matrix M for which $P = M^{-T} J M^{-1}$, where $J = \operatorname{diag}(-I_m, I_n)$. Then all solutions of (7) are given by

$$Z = T_M(K) \tag{8}$$

for K is an arbitrary contraction ($\|K\| < 1$) in $\operatorname{dom}(T_M)$.

For a matrix M partitioned as

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \tag{9}$$

the Möbius transformation T_M is defined by the equation

$$T_M(L) = (M_{21} + M_{22}L)(M_{11} + M_{12}L)^{-1} \tag{10}$$

for $L \in \operatorname{dom}(T_M) = \big\{ L : \exists (M_{11} + M_{12}L)^{-1} \big\}$.

Thus, the parametrization relies on describing $\mathrm{dom}(T_M)$.

An exhaustive description of the set

$$\mathcal{X}_{M_{11},M_{12}} = \{X \mid M_{11} + M_{12}X \text{ nonsingular } \}$$

can be done by using the generalized singular value decomposition (GSVD), however for our purposes it is sufficient the following result based on the more familiar singular value decomposition (SVD).

Consider the SVD of A as $A = U_A \Sigma_A V_A^*$ with

$$U_A = \begin{pmatrix} U_a & U_{as} \end{pmatrix}, \, \Sigma_A = \begin{pmatrix} \Sigma_a & 0 \\ 0 & 0_{as} \end{pmatrix}, \, V_A = \begin{pmatrix} V_a & V_{as} \end{pmatrix}. \tag{11}$$

and that of $B = U_B \Sigma_B V_B^*$ with

$$U_B = \begin{pmatrix} U_b & U_{bs} \end{pmatrix}, \, \Sigma_B = \begin{pmatrix} \Sigma_b & 0 \\ 0 & 0_{bs} \end{pmatrix}, \, V_B = \begin{pmatrix} V_a & V_{bs} \end{pmatrix}. \tag{12}$$

With these notations one has:

Lemma 9: The matrices

$$X_0(\gamma) = V_B \begin{pmatrix} 0 & \gamma \Sigma_b U_b^* U_{as} \\ 0 & 0 \end{pmatrix} V_A^*. \tag{13}$$

make $A + BX_0(\gamma)$ nonsingular for every $\gamma \neq 0$.

Moreover, for $\gamma^* \leq \frac{1}{||B||}$ the matrix $X_0(\gamma)$ is contraction for all $|\gamma| < \gamma^*$.

More details on the construction and the proofs can be found in [18]. A general overview on indefinite matrix analysis can be found in [5].