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Generic fractal structure of the optimal synthesis in problems with affine multi-dimensional control

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Abstract— **We consider a linear-quadratic deterministic optimal control problem where the control takes values in a two-dimensional simplex. The phase portrait of the optimal synthesis contains second-order singular extremals and exhibits modes of infinite accumulations of switchings in finite time, so-called chattering. We prove the presence of an entirely new phenomenon, namely a chaotic behaviour of the set of optimal trajectories. The set of optimal non-wandering trajectories has the structure of a Cantor set, and the dynamics of the system is described by a topological Markov chain. We compute the entropy and the Hausdorff dimension of the non-wandering set. This behaviour is generic for piece-wise smooth Hamiltonian systems in the vicinity of a junction of three discontinuity hypersurface strata.**

I. INTRODUCTION

The main contribution of this paper is to present a class of deterministic optimal control problems whose solution exhibits a chaotic behaviour, and to show that this behaviour is generic if a certain type of singularity is present. Previously, chaotic behaviour has been observed in dynamical systems in general, but not in systems arising from control problems.

The main tool for solving deterministic optimal control problems is the Pontryagin Maximum Principle [11]. It allows to reduce the control problem to a two-point boundary value problem in a Hamiltonian dynamical system. Suppose the control variable u is allowed to take values in some set U. Then the Hamiltonian function H defining the dynamics is given as the maximum

$$H(t,x,p) = \max_{u \in U} \mathcal{H}(t,x,p,u)$$

over the control u of the *Pontryagin function* $\mathcal{H}$, and the optimal control $\hat{u}(t,x,p)$, if it exists, is among the maximizers.

Usually the maximizer is unique in an open dense subset of the space of variables t,x,p and depends smoothly on these variables. On this set, the Hamiltonian H will then be smooth, whereas on its boundary the derivatives of H will experience discontinuities. The Hamiltonian system as a whole will then be piece-wise smooth, with an open dense subset of the extended phase space divided into non-intersecting domains $\Omega_1,\dots,\Omega_k$, on which the Hamiltonian is given by smooth functions $H_1,\dots,H_k$, respectively. The dynamics is described by an ODE with discontinuous right-hand side. In a typical situation, the set U is a convex polyhedron, and the domains Ω_i are those regions where the optimal control resides in a particular vertex v_i of the polyhedron. The set of points where the derivatives of H are discontinuous is a stratified manifold, and on each stratum the optimal control is confined to a particular face of the polyhedron U.

A trajectory of the Hamiltonian system evolving inside a smoothness domain is called *regular*. If a trajectory passes from one smoothness domain Ω_i into another Ω_j, then the corresponding optimal control will experience a jump from the vertex v_i of the polyhedron U to the vertex v_j. This process is called *switching*, and the discontinuity hypersurface is called *switching surface*. Typically an optimal trajectory intersects this surface transversally, in which case it is called *bang-bang trajectory*. It may happen, however, that a trajectory approaches the switching surface and then starts to move along it, in which case one speaks of a *singular trajectory*. A singular trajectory may also become regular by leaving the switching surface. Typically uniqueness of the solution does not hold in the vicinity of a singular trajectory, and many regular trajectories can join or leave from the same singular trajectory. This is possible because the right-hand side of the underlying ODE experiences a discontinuity.

For a singular trajectory lying inside a stratum of codimension one[1] one can define a (scalar) *order*, in dependence on up to which order the Poisson brackets of the adjoining smooth pieces H_i of the Hamiltonian vanish. The order may be *local* or *intrinsic*, in dependence on whether the brackets vanish only on the trajectory itself or in a neighbourhood of it [8]. If the intrinsic order of the singular trajectory is even, then a regular trajectory cannot join it in a piece-wise smooth manner. In this case the regular trajectory spirals into the singular trajectory and intersects the switching surface in infinite number of points in finite time, such that the joining point is an accumulation point of switchings. This phenomenon is called *chattering*, and it is well-studied for the situation where two smoothness domains meet at the singular trajectory in question [10].

In this contribution we consider the more complicated situation where three smoothness domains $\Omega_1,\Omega_2,\Omega_3$ meet at a manifold S_{123} of codimension 2. This situation is locally equivalent to an optimal control problem with 2-dimensional control. The general case of singular extremals for n-dimensional control was explored in [4]. The role of the scalar order is then played by a flag of orders. In the

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[1]In a neighbourhood of the singular trajectory this is equivalent to a problem with one-dimensional control [5].

present work we consider a singular trajectory of full second order. This work can be seen as a continuation of the paper [13], where this case was first considered and the presence of the chattering phenomenon was proven. In this paper we show that an additional phenomenon occurs, namely a chaotic behaviour of the optimal control on the trajectories of the system. This phenomenon was not yet seen in optimal control problems and is hence entirely new.

Our findings are not limited to optimal control problems, but rather hold for a whole class of piece-wise smooth Hamiltonian systems with three smoothness domains joining at singular trajectories of full second order. The complete proof of the results formulated in this paper is highly technical and far too extensive to be exposed here. We shall give an outline of the proof, which can be divided in two major parts. First we consider a particular linear-quadratic optimal control problem, which has a singularity of the type occurring in the general situation, and whose solution is hence expected to be prototypical for the general case. The first step of the proof consists in finding a complete description of the optimal synthesis of this problem. This description is not by an analytic formula, however, but rather qualitatively with the possibility of a quantitative computation up to machine precision. The fractal residing in the optimal solution manifests itself at this stage, and its presence is proven by the methods presented in [9]. The coordinates of every self-similar trajectory can be computed as roots of some system of algebraic equations and hence can be determined to arbitrary precision. The second step consists in showing that in the general situation, the optimal solution is topologically equivalent to the solution of the previously considered particular optimal control problem. The proof of this result goes along the lines of [12], where an analogous result was demonstrated for the case of a one-dimensional control, but the details are more involved. A publication of a complete proof is in preparation.

The remainder of the paper is organized as follows. We give exact formulations of our results in the next section. Then we present the particular linear-quadratic optimal control problem in Section III. Finally, we state some conjectures and possible directions for further work in the last section.

II. UBIQUITY THEOREM

To formulate the main theorem on ubiquity we need a number of definitions. Consider a smooth $2n$-dimensional symplectic manifold $\mathcal{M}^{2n}$. Let $S \subset \mathcal{M}$ be a $(2n-1)$-dimensional piece-wise smooth stratified submanifold that divides $\mathcal{M}$ into a finite number of open domains $\Omega_1, ...\Omega_k$, such that $\mathcal{M} = \overline{\bigcup \Omega_i}$. Denote the stratum of maximal dimension that divides the smoothness domains Ω_i and Ω_j by S_{ij}. Consider a continuous Hamiltonian $H(q,p) : \mathcal{M} \to \mathbb{R}$ such that its restriction $H_i = H|_{\Omega_i}$ to Ω_i is a smooth function which is C^∞-extendable to a neighborhood of the closure $\overline{\Omega_i}$. We shall consider the situation when three $(2n-1)$-dimensional strata S_{ij}, S_{jk}, S_{ik} of S join in a $(2n-2)$-dimensional stratum S_{ijk} of codimension 2. Without restriction of generality, we assume that $\{i,j,k\} = \{1,2,3\}$. Therefore, at a point $x_0 \in S_{123}$ we have $H(x_0) = H_1(x_0) = H_2(x_0) = H_3(x_0)$.

The main step in the investigation of the qualitative behaviour of solutions of differential equations is to find its typical singularities. For the study of continuous differential equations this approach was already developed by Poincaré. In deterministic optimal control, the Hamiltonian systems emanating from the Pontryagin maximum principle typically have discontinuous right hand sides, and the singularities may be more complicated.

Works of Kupka [7] and Zelikin-Borisov [12] investigated solutions $x(t)$ of piecewise smooth Hamiltonian systems which join a second order singular point x_0 lying on a discontinuity hypersurface S separating two smoothness domains of the Hamiltonian. Let the time instant where $x(t)$ joins x_0 be T, $x(T) = x_0$. It has been proved that in this situation there exists a family of trajectories $x(t)$ intersecting the discontinuity hypersurface S an infinite number of times while t approaches T, and the time instants $t_1, t_2, ..., t_n, ...$ of the intersections form an asymptotically bi-constant ratio, i.e., alternating, geometric progression. In other words, $\lim_{n\to\infty} t_n = T$ and there exist positive constants a, b, satisfying $ab < 1$, such that

$$\lim_{n\to\infty} \frac{t_{2n+2} - t_{2n+1}}{t_{2n+1} - t_{2n}} = a, \ \lim_{n\to\infty} \frac{t_{2n+3} - t_{2n+2}}{t_{2n+2} - t_{2n+1}} = b.$$

Thus the second-order singular points are such that trajectories approaching these singularities experience a chattering behaviour.

In this work we investigate the more complicated situation when the second-order singular point lies on a stratum S_{ijk} of codimension 2, as described at the beginning of this section. We show that a new phenomenon occurs, which has not been observed before in Hamiltonian systems with discontinuous right-hand side and optimal synthesis theory. Besides trajectories reaching x_0 with the usual chattering control, there exists a Cantor-like set $\mathcal{X}$ of trajectories whose sequence of intersections with the strata S_{ij} is described by a topological Markov chain Σ_Γ^+ [2].

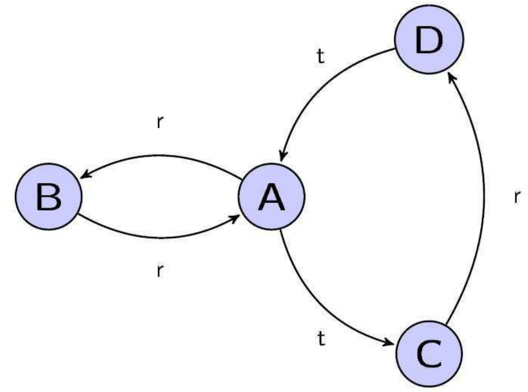

Fig. 1. Graph $\tilde{\Gamma}$ from which Γ is constructed

Here the oriented graph Γ can be constructed from the graph $\tilde{\Gamma}$ depicted in Fig. 1, as follows. The vertex set of Γ is the direct product of the vertex set $\{A, B, C, D\}$ of $\tilde{\Gamma}$ with the set of ordered pairs $(i, j) \in \{1, 2, 3\}^2$ such that $i \neq j$. Hence Γ has 24 vertices $A_{12}, A_{13}, \dots, D_{32}$. There is a directed edge in Γ from A_{ij} to $B_{i'j'}$ iff $i' = j$ and the permutation $\sigma \in S_3$ taking the pair (i, j) to the pair (i', j') has order 3 (is a **r**otation). There is a directed edge from A_{ij} to $C_{i'j'}$ iff $i' = j$ and the permutation $\sigma \in S_3$ taking the pair (i, j) to the pair (i', j') has order 2 (is a **t**ransposition). Arrows $B_{ij} \to A_{i'j'}$, $C_{ij} \to D_{i'j'}$, and $D_{ij} \to A_{i'j'}$ are constructed in a similar way, according to whether the corresponding arrow in the graph $\tilde{\Gamma}$ carries an **r** or a **t**. The other pairs of vertices in Γ are not connected by arrows.

It is easy to see that Γ is a disconnected union of two isomorphic graphs, $\Gamma = \Gamma_+ \sqcup \Gamma_-$. Moreover, each of the graphs $\Gamma_\pm$ is strongly connected and primitive. The intersection of the set $\mathcal{X}$ with a transversal hypersurface is a multi-dimensional Cantor set which is alike to a Smale horseshoe. The topological Markov chain $\Sigma^+_{\Gamma_\pm}$ is homeomorphic to the standard Smale horseshoe as a topological space but does not conjugate to it as a dynamical system.

Definition 1: A point $x_0 \in S_{123}$ will be called **strange** if the following conditions are fulfilled at x_0:

i). All commutators (Poisson brackets) of the functions H_i, $(i = 1, 2, 3)$ up to the fourth order inclusively equal zero at the point x_0. The gradients of these commutators and those of H_i has maximum rank at x_0 as far as it is allowed by anti-commutativity of the Poisson bracket and the Jacobi identities. In other words, these gradients are in general position.

ii). Set $F_0 = H_1 + H_2 + H_3$, $F_1 = H_2 - H_3$, $F_2 = H_3 - H_1$, $F_3 = H_1 - H_2$. Then the symmetric bilinear form

$$B_{ij} = \operatorname{ad} F_i (\operatorname{ad} F_0)^3 F_j|_{x_0}, \quad i, j = 1, 2, 3$$

has rank 2, more precisely is a negative multiple of the form

$$\begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix}.$$

All other fifth order commutators of the functions F_i, $i = 0, 1, 2, 3$, of the form $(\operatorname{ad} F_\alpha)(\operatorname{ad} F_\beta)(\operatorname{ad} F_\gamma)(\operatorname{ad} F_\delta) F_\epsilon$ vanish at x_0[2].

The set of trajectories reaching the point x_0 is described as follows:

Theorem 1 (on ubiquity): Consider a Hamiltonian system with piece-wise smooth Hamiltonian H as described at the beginning of this section. Let the point $x_0 \in S_{123}$ be strange in the sense of Definition 1. Then in a neighborhood of x_0 there exists a set of points $\mathcal{X}$ such that

(I) Each trajectory $X(t, y)$ beginning at a point $y \in \mathcal{X}$ reaches the point x_0 in finite time $T(y) > 0$, i.e. $X\big(T(y), y\big) = x_0$ and $X(t, y) \in \mathcal{X}$ for $t \in (0, T(y))$. Moreover, $X(t, y)$ has a countable number of successive transversal intersections with S at time instants $t_1, t_2, \dots, t_k, \dots$, and $\lim_{k\to\infty} t_k = T(y)$.

(II) Denote by $f : \mathcal{X} \cap S \to \mathcal{X} \cap S$ the map which transfers a point $y \in \mathcal{X} \cap S$ into the point of the next intersection of $X(t, y)$ with S, i.e. $f(y) = X(t_1, y)$. Then the one-way topological Markov chain Σ^+_Γ of right infinite paths on the graph Γ defined in the caption of Fig. 1, which does not depend on x_0 and H, is the topological factor of the dynamical system f, as in the following diagram:

$$\begin{array}{ccc} \mathcal{X} \cap S & \xrightarrow{f} & \mathcal{X} \cap S \\ \downarrow \Phi_\Gamma & & \downarrow \Phi_\Gamma \\ \Sigma^+_\Gamma & \xrightarrow{l} & \Sigma^+_\Gamma \end{array}$$

Here l is the Bernoulli shift to the left and Φ_Γ is a continuous surjective map. The inverse image $\Phi_\Gamma^{-1}(\sigma)$ of each point $\sigma \in \Sigma^+_\Gamma$ is homeomorphic to an open disc D^2, and the diameter of $f^k\big(\Phi_\Gamma^{-1}(\sigma)\big)$ tends to 0 as $k \to +\infty$.

(III) The Hausdorff and the Kolmogorov dimensions of $\mathcal{X}$ do not depend on x_0 and H, as long as x_0 meets the conditions of Definition 1. The following estimates are valid

$$3.204762 \leq \dim_H \mathcal{X} \leq \overline{\dim_K} \mathcal{X} \leq 3.407495 \qquad (1)$$

(IV) The topological entropy [1] of the system l equals[3]

$$h_{\mathrm{top}}(l) = \log_2 \left(\sqrt[3]{\frac{1}{2} + \frac{\sqrt{69}}{18}} + \sqrt[3]{\frac{1}{2} - \frac{\sqrt{69}}{18}} \right) \approx 0.4057.$$

(V) Besides the Cantor-like set of trajectories $\mathcal{X}$ described in (I)–(IV), there exist two other sets of chattering trajectories reaching the point x_0 in finite time:

(a) There exist two one-parameter families of three-chain trajectories R_{123} and R_{132}, attaining the point x_0 in finite time with a countable number of successive cyclic intersections with the strata S_{12}, S_{23}, and S_{31} (for R_{123}), and with the same strata in reversed order (for R_{132}).

(b) There exist three two-parameter families of four-chain trajectories Q_1, Q_2 and Q_3. Each trajectory of Q_i moving to x_0 has a countable number of successive intersections with the strata S_{ij}, S_{ik}, S_{ik}, and S_{ij} repeating cyclically.

(VI) A similar situation with the direction of time flow inverted holds for trajectories leaving the point x_0.

Theorem 2: The set $\mathcal{S} \subseteq S_{123}$ of all strange points in S_{123} is a submanifold of codimension $\operatorname{codim} \mathcal{S} = 77$ in S_{123}. Moreover, $\bigcup_{x\in\mathcal{S}} \mathcal{X}(x)$ is a locally trivial bundle over $\mathcal{S}$ with fibre $\mathcal{X}(x)$ of fractional dimension (1).

The full proofs of the theorems are too extensive to be reproduced here, a publication is in preparation.

Remark 1: The codimension 77 of $\mathcal{S}$ is so large because the conditions of Theorem 1, in particular, the definition of strange points, are excessively strong. In fact they can easily be weakened.

[2]These conditions are very strong and can easily be modified and weakened for a number of concrete situations.

[3]Formally, $\mathcal{X}$ is non-compact. The topological entropy of f on $\mathcal{X} \cup \{x_0\}$ hence equals 0. But after a blow-up of the singularity at x_0 the topological entropy of f becomes equal to $h_{\mathrm{top}}(l)$.

III. MODEL CONTROL PROBLEM

The proof of Theorem 1 proceeds by considering the following particular model problem

$$\min \int_0^\infty |x(t)|^2 dt, \ddot{x} = u, u \in U. \tag{2}$$

Here $x \in \mathbb{R}^2$ is the phase vector, and u is the control taking values in the set U, which is an equilateral triangle with centre at the origin.

Optimal problem (2) describes the control of the following hypothetical aircraft. Imagine an apparatus shaped like a flat disc which can move in a fixed horizontal plane (for instance, on an air cushion, or with the help of an auxiliary engine that keeps the disc on a constant altitude). The disc is equipped with three jet propulsions situated at the vertices of an equilateral triangle inscribed into the disc and directed outward along the radii. Suppose the total rate of fuel expenditure for all the three jets is bounded, but the distribution of the fuel among the three engines can be set arbitrarily. Then the resultant force acting on the disc will be any convex combination of the three maximal forces resulting from directing the maximum amount of fuel to one engine alone. It takes values lying at any point of an equilateral triangle.

It will be convenient to use barycentric coordinates $x = (x_1, x_2, x_3)$; $y = \dot{x} = (y_1, y_2, y_3)$, subject to the conditions

$$x_1 + x_2 + x_3 = 0;\ y_1 + y_2 + y_3 = 0.$$

These equations define the four-dimensional phase space $M = \mathbb{R}^4$. The control $u = \dot{y}$ belongs to the triangle

$$U = \{u \in \mathbb{R}^3 \mid u_1 + u_2 + u_3 = 0,\ u_i \le 1\ \forall i = 1,2,3\}.$$

The vertices of the triangle U are located at the points $a_1 = (-2,1,1), a_2 = (1,-2,1), a_3 = (1,1,-2)$.

Let $(x(t), y(t))$ be the trajectory that corresponds to the optimal control $u(t)$. In accordance with the Pontryagin Maximum Principle there exist absolutely continuous functions $(\phi(t), \psi(t))$ (where ϕ is the vector conjugated with x, and ψ is conjugated with y) and a real number $\lambda_0 \ge 0$ such that λ_0, ϕ, ψ are not simultaneously zero and together with (x, y) are solutions of the Hamiltonian system

$$\begin{gathered} \dot{x} = H_\phi,\ \dot{y} = H_\psi,\ \dot{\phi} = -H_x,\ \dot{\psi} = -H_y, \\ u \in \operatorname*{Argmax}_u \mathcal{H},\ x(0) = a, y(0) = b, \end{gathered} \tag{3}$$

where the Hamiltonian H is the maximum over the control u of the Pontryagin function $\mathcal{H}$ given by

$$\mathcal{H} = \sum_{i=1}^{3} (-\frac{\lambda_0}{2} x_i^2 + \phi_i y_i + \psi_i u_i).$$

It is easy to see that $\lambda_0 \neq 0$, otherwise the integral that defines the value of the functional in (2) diverges. Hence one can normalize to $\lambda_0 = 1$. Since ϕ and ψ belong to T^*M, we have the restrictions $\sum \phi_i = 0,\ \sum \psi_i = 0$. We shall call the space of variables (x, y, ϕ, ψ) the *extended phase space* $\mathcal{M}$.

Since the control system is linear and since the set U and the functional are convex, we have by the Kuhn-Tucker theorem [6] that the solution to problem (2) exists and is unique. Any trajectory satisfying the Pontryagin Maximum Principle is optimal.

The calculations needed to design the optimal synthesis of problem (2) are rather laborious. We provide here only the main ideas and describe the final results.

First we recall some results from [13]. Consider the symmetry groups of the optimal synthesis. There exists a 1-parametric scaling group g of the Hamiltonian System (3) which acts as

$$g_\kappa(y, x, \phi, \psi) = (\kappa y, \kappa^2 x, \kappa^3 \phi, \kappa^4 \psi).$$

The velocity along the trajectories thus scales as $g_\kappa(t) = t/\kappa$.

The symmetry group S_3 of the triangle U is the group of permutations of the indices $1, 2, 3$ and acts simultaneously on the coordinates of the vectors x, y, ϕ, ψ.

Families of trajectories of Hamiltonian systems (including the case of discontinuous right-hand side) have a saddle-type structure due to the Liouville theorem. There exists an isomorphism between the set of trajectories reaching the origin and of those leaving the origin. Indeed, this isomorphism is generated by the action of the scale group g_κ for negative values of κ that inverts the direction of time flow. Hence $\mathcal{M}$ is the product of two isomorphic manifolds $\mathcal{M}_+$ and $\mathcal{M}_-$ with $\dim \mathcal{M}_\pm = 4$ which is equal to $\dim M$. It is easy to see that any optimal trajectory of problem (2) reaches the origin in finite time and so belongs to $\mathcal{M}_+$.

The quotient manifold M/g of the space M with respect to the action of the group g (the space of orbits of the group g) is a three-dimensional sphere $\mathbb{S}^3$. The subsequent factorization relative to the action of the group S_3 leads to a manifold N having the type of a lens space.

To describe the optimal synthesis we explore the Poincaré mapping F of the control switching surface $Q \subset N$ onto itself. The manifold Q is homeomorphic to a two-dimensional disc with some special gluing of its boundary. The periodic points of F relate to cycles in the quotient space M/g.

The least energetic state from the point of view of the problem of minimization of the functional (2) is the origin. After reaching the origin the optimal value of the control u is zero and the phase point stops to move.

The control $u(t)$ is called *regular* on an interval (t_0, t_1) if on this interval one of the components ψ_i is strictly less than the two other components. In that case $u(t) \equiv a_i$.

The control $u(t)$ is called (i,j)-*singular* on an interval (t_0, t_1) if on this interval $\psi_i \equiv \psi_j < \psi_k,\ k \neq i, j$.

Proposition 1: [13] Any (i,j)-singular optimal trajectory belongs to the manifold $A^{ij} = \{x_i = x_j,\ y_i = y_j\}$. The control on an (i,j)-singular optimal trajectory is equal to $u_i^{ij} = u_j^{ij} = -1/2,\ u_k^{ij} = 1,\ k \neq i, j$.

The set A^{ij} is a two-dimensional plane in M. After factorization by g one obtains a cycle Z^{ij}. On one half of this cycle the control is nonsingular, $u = a_k$, $(k \neq i, j)$, and on the other it is (i, j)-singular. Hence the cycles Z^{ij} correspond to three singular periodic orbits of the Poincaré mapping F having period 2 (we will denote it by the same letters as the corresponding cycles and call it half-singular). The corresponding trajectories in M are spirals having discontinuities of their derivatives at the switching points and reaching the origin in finite time with an infinite number of switchings. The linking coefficient of any two cycles equals 1 [13].

Proposition 2: [13] There exist 2 cycles $Z_{\pm}^{ijk}$ which are generated by fixed points of the triple Poincaré mapping F^3. The control on $Z_{\pm}^{ijk}$ alternates between the vertices of the triangle U in a cyclical order, $a_i \to a_j \to a_k$ or $a_i \to a_k \to a_j$, respectively. The linking coefficient of these two cycles equals 1.

After factorization by the scaling group g the periodic cycles $Z_{\pm}^{ijk}$ can be shown to be repellers and the half-singular ones Z^{ij} to be attractors of Poincaré mapping F [13].

The structures described above do not exhaust the set of optimal trajectories. To obtain the full picture we had to perform both extensive calculations on a computer and further theoretical investigations. The projections of the Lagrangian manifold $\mathcal{M}_+$ both on the space of variables x, y and on the space of variables ϕ, ψ turn out to be locally bi-Lipschitz homeomorphisms. This fact allows to use the variables ϕ, ψ to parameterize $\mathcal{M}_+$, which is very convenient since the switching surface S has a simple description in terms of these variables, and the control u is defined explicitly by the vector ψ. Calculations have been performed in the quotient space with respect to the action of the group g. In this compact quotient space the dynamics is given by a direction field rather than a vector field and an infinite number of switchings in finite time is no more an obstacle to a quantitative analysis.

Consider the flow of optimal trajectories emanating from the two repellers $Z_{\pm}^{ijk}$ and the flow emanating in backwards in the reversed time direction from the three attractors Z^{ij}. The fronts of these two flows in the process of approaching each other become more and more fractal-like. The reason is that these two flows are bound to glue one-to-one pointwise to each other in a manner that respects the action of the group S_3, which is not possible in a regular manner. As a consequence, a fractal-like set NW of non-wandering trajectories does not reach neither the attractors nor the repellers. This set NW contains a countable number of periodic trajectories with infinitely growing periods and a Cantor-like continuum of non-wandering chaotic trajectories. These structures render the optimal synthesis of the problem fractal-like. More generically, this picture describes the typical behaviour of trajectories of piece-wise smooth Hamiltonian systems at strange points, where three smoothness domains join at a stratum of codimension 2.

Turning back from the quotient space to the Lagrangian manifold $\mathcal{M}_+$ one obtains a movement to the point x_0 along cone-like spirals with a chaotic sequence of controls.

The presence of the fractal is proven by pointing out a rectangular-shaped set Z on the switching surface Q of the system (3) such that the Poincaré map F is hyperbolic in a Lipschitzian sense on Z, and such that the iterated Poincaré maps F^2 or F^3 map several domains $Z_s \subset Z$ back on Z in way similar to the Smale horseshoe map. These iterated maps correspond to the maps $A \to B$ and $A \to C \to D \to A$ in the graph of admissible mappings defined in Fig. 1. The symbolic dynamics corresponding to the graph defines a topological Markov chain Σ_Γ^+ which is akin (but not coincides) to the Smale horseshoe. The periodic sequences of the chain Σ_Γ^+ yield the countable set of periodic orbits of the Poincaré map F. The closure of the union of the periodic orbits gives the set NW of all non-wandering points. The cardinality of the set of non-wandering points equals that of the continuum, and its Hausdorff dimension is fractional. The points of NW define the chaotic part of the dynamics of the system. After a countable number of transitions between the domains Z_s these points reach x_0 in finite time.

To prove Theorem 1 we introduce a prolongation of the Hamiltonian system $H = \{H_i\}$ to an auxiliary control system on the Poisson brackets of the functions H_i with a two-dimensional control. We call the obtained system *descending tree-like*, since each of its equations has in some sense a lower order than the preceding one. The main part of the descending system is then the Hamiltonian system emanating from the Pontryagin maximum principle for problem (2). That is why we called this system *model control problem.*

In standard ergodic theories, in stochastic dynamics, in attractor theories etc., only the "everlasting" or "eternal" problems are explored, i.e., the behaviour of the system as $t \to \infty$. In such theories the transient phenomena are considered as nonessential and only the asymptotic consequences are of importance. But in our case, when all optimal processes terminate at finite time, one has unduly lack of time for eternity. Nevertheless in our case all features of the above-mentioned theories arise too. The chattering control imitates eternity by infinitely many increasingly shorter intervals between sequential switchings. Following these switches, i.e., following the Poincaré maps of the switching surface to itself, one goes through eternity in finite time exactly as in the Zeno paradox about Achilles and the tortoise. If one measures time by using the number of switches, then the total time appears to be infinite.

IV. OUTLOOK FOR FURTHER WORK

In this work we introduced a class of optimal control problems whose solution exhibits a chaotic behaviour in combination with chattering. This means that the chaotic trajectories are went through in a finite time interval, at the end of which the solution becomes singular of second full order. The results hold in the setting of piece-wise smooth Hamiltonian systems, which not necessarily originate from an optimal control problem. The full complexity of the solution can be already observed in the linear-quadratic model problem (2) of dimension 4.

As a conclusion we formulate some conjectures that seem to us of importance.

Conjecture 1: The optimal synthesis for the analog of the problem (2), when the set U is a regular polygon with an odd number of vertices and when condition ii) in Definition 1 is properly changed, has a fractal structure.

Conjecture 2: The phase portrait for piece-wise smooth Hamiltonian systems in a neighbourhood of a junction of 4 strata of codimension 1 in a stratum of codimension 3, when the conditions of Theorem 1 are properly changed, has a fractal structure.

The optimal synthesis for the three-dimensional analog of problem (2), when the set U is a regular tetrahedron and the condition ii) in the definition 1 is properly changed, has a fractal structure.

Conjecture 3: The optimal synthesis for general control problems which are affine in the control, when the set U is a polyhedron, has a fractal structure.

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