

On Practical Fixed Order H_∞ Loop-Shaping

Serge Hirwa, Philippe Feyel, Gilles Duc, and Guillaume Sandou

Abstract — In this paper, a fixed order H_∞ synthesis is used to directly shape the open loop transfer function so that it matches as closely as possible, in the singular values sense, a desired frequency gain which captures both performance and robust stability objectives. First, a loop shaping weight is automatically computed, and then it is used in the controller synthesis by solving a well-suited four-block H_∞ control problem with a recently developed nonsmooth H_∞ synthesis routine. Two industrial applications are provided to illustrate the potential of the approach for low order robust controller design: a first one on the control of a piezo-actuated optronic mechanism, a second one dealing with inertial Line Of Sight stabilization loops.

I. INTRODUCTION

THE Glover-McFarlane H_∞ loop shaping technique [1] makes use of classical frequency domain design concepts, which enable to compute a robust feedback controller K by focusing only on the open-loop transfer function: $L = GK$, where G is a nominal LTI model of the controlled plant. The designer first uses a precompensator W_1 and/or a postcompensator W_2 to define a desired open-loop shape $G_s = W_2GW_1$, and then performs a robust coprime factors stabilization to compute K_∞ by minimizing:

$$\left\| \begin{bmatrix} I \\ K_\infty \end{bmatrix} (I + G_s K_\infty)^{-1} \begin{bmatrix} I & G_s \end{bmatrix} \right\|_\infty = \gamma \quad (1)$$

The final feedback controller can be of high order, since it is a combination of W_1 , W_2 and K_∞ :

$$K = W_1 K_\infty W_2$$

In [2], reduced order controllers are found by optimizing (1) over the set of fixed order and stabilizing K_∞ , using coprime factors and LMI optimization. However, in this approach the order of the final controller is still bounded below by the order of W_1 and W_2 .

In fact, in the full order case, (1) is equivalent to a more

S. Hirwa is with SAGEM (SAFRAN group), Optronics & Defense division, France (serge.hirwa@sagem.com) and SUPELEC Systems Sciences (E3S), Automatic Control Department, France (serge.hirwa@supelec.fr).

P. Feyel is with SAGEM (SAFRAN group), Optronics & Defense division, France (philippe.feyel@sagem.com).

G. Duc and G. Sandou are with SUPELEC Systems Sciences (E3S), Automatic Control Department, France ({gilles.duc, guillaume.sandou}@supelec.fr).

standard four-block H_∞ control objective (see [1]):

$$\gamma = \left\| \begin{bmatrix} W_2 \\ W_1^{-1}K \end{bmatrix} (I + GK)^{-1} \begin{bmatrix} W_2^{-1} & GW_1 \end{bmatrix} \right\|_\infty \quad (2)$$

Criterion (2) would then be more convenient for directly finding a final reduced order controller K , by using an efficient fixed order H_∞ synthesis technique. We partially used this idea in our recent paper [3], where W_1 and W_2 were considered as already well-tuned during a previous full order H_∞ loop shaping design.

In this paper, following the seductive idea introduced in [4] with the Matlab command LOOPSYN, we directly address performance and robust closed-loop stability via a target open loop frequency gain:

$$h(\omega) > 0, \quad \omega \in [\omega_{\min} \quad \omega_{\max}]$$

A square stable and minimum phase weight W is first computed so that the singular values of the compensated plant $G_s = GW$ fit this loop-shape $h(\omega)$. Then this weight is embedded in a four-block H_∞ synthesis of type (2), which is solved at fixed order, using the routine HINFSTRUCT [5].

The main steps of the proposed controller synthesis procedure are exposed in section II, and then two robust controller designs are addressed in section III and IV. In both cases the synthesis models are of significant order (>10) and specific loop shape gains are used to handle performance and robustness requirements.

II. CONTROLLER SYNTHESIS PROCEDURE

In the following sections, \mathfrak{RH}_∞ denotes the space of proper, real rational matrix functions with no poles in the closed right half plane $\text{Re } s \geq 0$; $G \equiv (A, B, C, D)$ is a state space realization of the nominal model of the controlled plant, with $D \in \mathfrak{R}^{p \times m}$.

A. Loopshape graphical construction

The controlled plant is assumed to be in the set $G_\Delta = \{(I + \Delta)G; \Delta \in \mathfrak{RH}_\infty\}$, with Δ being an *unstructured output direct multiplicative uncertainty*.

Using the frequency gain $h(\omega)$, the designer directly specifies the performance and robust stability objectives on the desired open-loop singular values of the compensated plant:

$$-- \underline{\sigma}(G(j\omega)K(j\omega)) \approx h(\omega) \gg 1 \text{ at low frequencies, to}$$

ensure good reference tracking and disturbance rejection.

-- $\bar{\sigma}(G(j\omega)K(j\omega)) \approx h(\omega) \ll 1$ at high frequencies to

ensure robustness against high frequency unmolded dynamics, which are roughly captured by the uncertainty Δ .

-- In the middle frequencies, near the desired bandwidth ω_B (or 0dB frequency), the slope of $h(\omega)$ must be small (-20dB/dec). Moreover this bandwidth should be compatible with the fundamental limitations induced by the right half plane (RHP) poles and zeros of plant G (see [6]).

Once the graphical design is done, the obtained frequency gain $h(\omega)$ is fitted with a proper stable minimum phase SISO model L_h by using a *log-Chebyshev magnitude approximation* algorithm (command FITMAGFRD of Matlab).

B. Weight construction

The computation of the loop-shaping weight W in the Matlab command LOOPSYN relies on *inner-outer factorization* of the plant G , which requires the following hypothesis:

- i) G is stabilizable and detectable.
- ii) G has neither poles nor zeros on the $j\omega$ -axis or at ∞ .

In fact physical systems might contain integrators or have transmission zero at ∞ (state space matrix D is null or not full rank); this 2nd assumption can be enforced by using an appropriate bilinear pole-zero shifting on the plant G .

We call a transfer matrix N *inner* if it is proper, stable and satisfies $N^T(-s)N(s) = I$.

Under assumptions (i)-(ii) and $p \geq m$ (plant G has more outputs than inputs), Theorem 1 p.113 in [7] provides an explicit realization of a right coprime factorization $G = NM^{-1}$ with an *inner* numerator N , by solving an Algebraic Riccati Equation :

$$\begin{aligned} M &\equiv (A + BF, BR^{-1/2}, F, R^{-1/2}) \in \mathfrak{RH}_\infty^{m \times m} \\ N &\equiv (A + BF, BR^{-1/2}, C + DF, DR^{-1/2}) \in \mathfrak{RH}_\infty^{p \times m} \\ R &= D^T D > 0 \\ F &= -R^{-1}(B^T X + D^T C) \\ X &= \text{Ric} \begin{bmatrix} A - BR^{-1}D^T C & -BR^{-1}B^T \\ -C^T D_\perp D_\perp^T C & -(A - BR^{-1}D^T C)^T \end{bmatrix} \end{aligned}$$

where D_\perp is an orthogonal complement of D (that is $D^T D_\perp = 0$), such that $\begin{bmatrix} DR^{-1/2} & D_\perp \end{bmatrix}$ is square and orthogonal (that is $\begin{bmatrix} DR^{-1/2} & D_\perp \end{bmatrix}^T \begin{bmatrix} DR^{-1/2} & D_\perp \end{bmatrix} = I$).

If G is stable, M is *outer* (proper stable and inverse stable); if G is unstable, a same type of right coprime factorization of $M^T = N_m M_m^{-1}$, with N_m square *inner*, would give M_m *outer*.

Then an inner-outer factorization $G = G_i G_o$ can be defined with:

-- $G_i = N$ and $G_o = M^{-1}$ if G is stable.

-- $G_i = N(N_m^{-1})^T$ and $G_o = M_m^T$ if G is unstable.

$G_i^T(-s)G_i(s) = I$ implies $\sigma(G_i L_h) = \sigma(L_h) = \sigma(GG_o^{-1}L_h)$.

This gives the desired loop shaping weight $W = G_o^{-1}L_h$ such that $\sigma(GW) = \sigma(L_h)$.

A. Fixed Order H_∞ Synthesis

In the square plant case ($p=m$), the four-block H_∞ synthesis objective (2) is constructed with $W_1 = W$ and $W_2 = I$, and solved at fixed order with HINFSTRUCT.

However, it must be noted that despite the established factorization is valid for the strictly tall plant case ($p > m$), the number of singular values that can be shaped to L_h using W (exactly m) is less than the number of available inputs (exactly p). SIMO design with this approach is not straightforward, as it will be illustrated in section IV.

III. A SISO DESIGN CASE: ROBUSTIFICATION OF AN EXISTING FIXED ORDER CONTROLLER

In this first example, a fixed order robust controller is designed for a piezo-actuated microscanning mechanism.

Microscanning is a well-known technique for enhancing the resolution of digital image sensors (IR cameras for example). A microscan unit acquires multiple images of the same scene, while sliding by a fraction of the detector pitch (pixel) either a lens or the detector (Fig 1). The different switched frames of the scene are processed in real-time by an embedded algorithm to produce a higher resolution image.

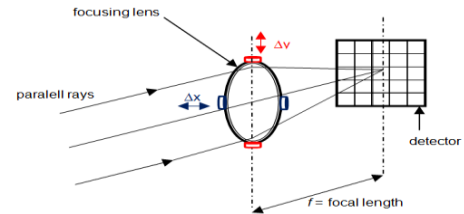


Fig. 1. Lens based microscanning

Fast microtranslations ($\Delta x, \Delta y$) in the X-Y plane of the detector are needed, so usually piezoelectric actuators are convenient because of their high dynamics. However, serious variations in the structural resonance modes (frequency and amplitude) of the whole piezo-actuated mechanism are observed for different operating conditions (temperature, mounting interactions), as illustrated in Fig 2 (Left).

The mechanism had a baseline controller K_b (the same for X and Y axis), tuned for the nominal model that has order $\mathcal{P}G = 12$ (Fig 3 Left); it gave good nominal closed-loop performance (Fig 3 Right) but couldn't guarantee closed-loop stability w.r.t the resonance mode variations (Black-

Nichols chart in Fig 2 clearly shows instability for several measured transfer functions $G_i(j\omega)$ if K_b is still used as feedback controller)

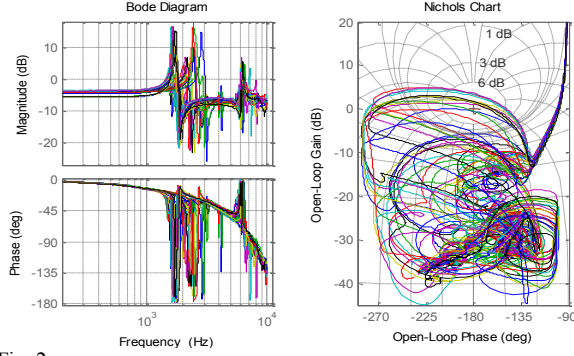


Fig. 2.

Left: Set of measured frequency responses $G_i(j\omega)$, from input voltage to lens displacement on each axis (X and Y), for different operating conditions change.
Right: Nichols charts with K_b : $L_i(j\omega) = G_i(j\omega)K_b(j\omega)$.

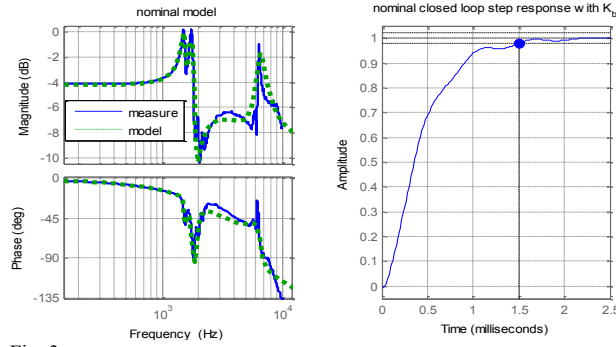


Fig. 3.

Left: Nominal model on the X-axis; the 2nd axis has similar dynamics.
Right: Nominal closed-loop step response with K .

The magnitude of the measured *output multiplicative direct uncertainty* w.r.t the nominal model G :

$$|\Delta_i(j\omega)| = \left| \frac{G_i(j\omega) - G(j\omega)}{G(j\omega)} \right|, \quad \omega \geq 0$$

is used to construct graphically a loopshape $h(\omega)$ that will keep the good nominal performance (we maintain the low frequency gain provided by the baseline controller, e.g. $h(\omega) = |G(j\omega)K_b(j\omega)|$ for $\omega \leq \omega_B$, with ω_B such that $|G(j\omega_B)K_b(j\omega_B)| = 0$) and ensure robust stability, by approximating in high frequencies ($\omega \gg \omega_B$) the robust stability condition derived from the Small Gain Theorem [8]:

$$\bar{\sigma} \left(\frac{G(j\omega)K(j\omega)}{1 + G(j\omega)K(j\omega)} \right) \approx \bar{\sigma}(G(j\omega)K(j\omega)) \approx h(\omega) < \frac{1}{\max_i |\Delta_i(j\omega)|}$$

This frequency gain is fitted by a proper stable minimum phase L_h of order $\partial^\circ L_h = 6$. Then, using the procedure described in section II, a more robust controller K of order

$\partial^\circ K = \partial^\circ K_b = 3$ is computed (Fig 4).

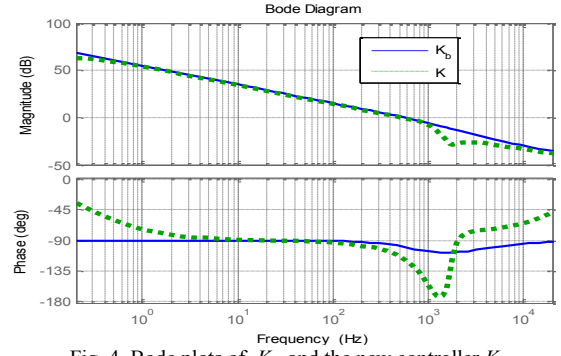


Fig. 4. Bode plots of K_b and the new controller K

The Small Gain Theorem condition is well fulfilled with this new controller (Fig. 5 Top). Robust stability is then guaranteed for the whole range of measured transfer functions whereas the step responses are very well damped with a small varying response time (Fig. 5 Bottom).

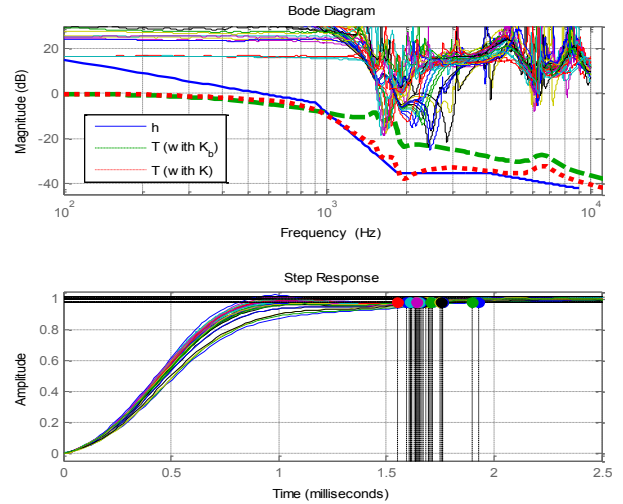


Fig. 5.

Top: Robust stability test: $|T(j\omega)|$ vs. $|\Delta_i^{-1}(j\omega)|$

Bottom: Closed-loop step responses of $T_i = G_i K (1 + G_i K)^{-1}$

IV. ADAPTATION TO A SIMO CASE

In this second case, we slightly adapted the procedure in order to design a 2 inputs 1 output controller for one axis of an Inertially Stabilized Platform (Fig. 6).

As stated in [3], to guarantee high accurate angular pointing, LOS (Line Of Sight) stabilization loops are designed to ensure good rejection of the 2 main disturbances (See Fig 7):

- Γ_f : Friction torques induced by rotational movements (pitch, yaw and roll) of the host vehicle.
- Γ_s : LOS deflections due to host vehicle vibrations.

Dispersion of manufacturing and/or change in operating conditions (temperature, mounting interactions) sometimes result in significant variations of the system (resonant modes of the mechanical platform, parameters of the motor).

Robust stability of the loop must then be ensured w.r.t to these variations.

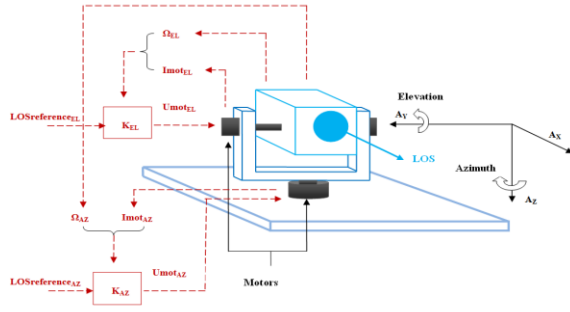


Fig. 6. A 2 Axis Inertially Stabilized Platform

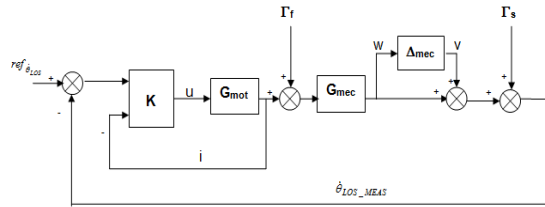


Fig. 7: The feedback loop on each axis.

The uncertainty Δ_{mec} accounts for the possible variations on the resonance modes of the gimbals assembly, u is the motor input voltage, i is the motor current.

The system is mechanically well decoupled. Usually, controller synthesis for every axis is done using a linear model:

$$\begin{bmatrix} \dot{\theta}_{LOS_MEAS} \\ i \end{bmatrix} = \begin{bmatrix} G_{mec}(s)G_{mot}(s) \\ G_{mot}(s) \end{bmatrix} u$$

The motor and gimbals assembly linear models are obtained by frequency response identification (Fig. 8).

In order to capture the performance objectives, the desired open loop gain $h(\omega)$ must have high gain at very low frequency (for friction torques rejection) but also in some precise low and middle frequency zones (for host vehicle vibrations rejection). Robust stability is addressed as in the previous SISO example, by imposing sufficiently low gain in the resonant mode frequency zone.

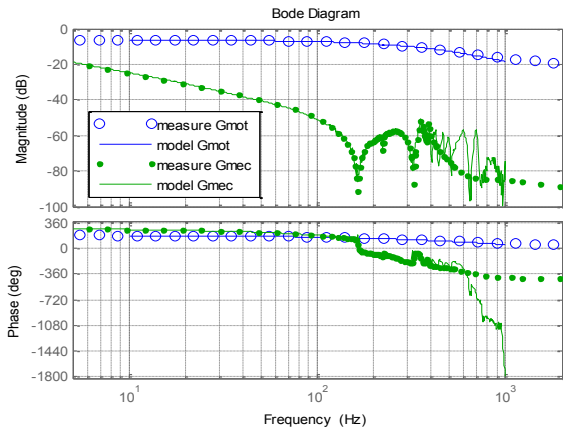


Fig. 8: Electrical and mechanical transfer function responses

Despite in this case we don't have any set of measured *output multiplicative direct uncertainty* $\Delta_{mec}(j\omega)$ as in the previous SISO example, a same type of variations in terms of magnitude can be expected in high frequencies. Then for sake of illustration, one can take as $h(\omega)$ the plot in Fig.9.

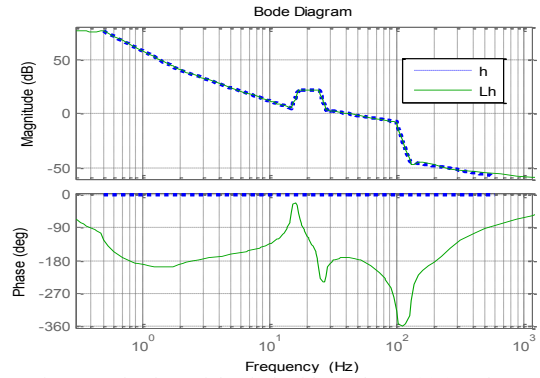


Fig. 9: Bode plots of the desired loop shape: $h(\omega)$ and L_h

This frequency gain is fitted by a proper stable minimum phase $L_h(s)$, then W such that $\sigma(G_{mec}G_{mot}W) = \sigma(L_h)$ is computed using the method of section II.B.

In our design problem, the regulated variable is the angular rate ($\dot{\theta}_{LOS_MEAS}$), the motor current i is just used as a supplementary measure. Actually there is no need of high gain motor current loop, as it is usually the case in classical cascade servo loops, where PI controllers are used. This is because on this specific system, the motor directly drives the assembly without any gear transmission. The back electromotive force disturbance is then negligible since the relative speed between motor and load is low. So the current loop is rather used here to enhance the stability margins (phase and gain margins) at input. Typically a good gain margin is needed to account for variations of the motor resistance due to high temperature changes.

Then, to perform a MISO controller synthesis, W is factorized in W_i (roll-off term) and W_o (low frequency high gain term) such that $W = W_o W_i$ (Fig. 10), w.r.t the desired bandwidth ω_B (where $h(\omega_B) = 1$):

$$W(s) = k_W \prod_{j=1}^r \frac{s - z_j}{s - p_j}$$

$$W_i(s) = \lambda \left(\prod_{|z_j| > \omega_B, |p_j| > \omega_B} \frac{s - z_j}{s - p_j} \right), \quad \lambda = \prod_{|z_j| > \omega_B, |p_j| > \omega_B} \left| \frac{p_j}{z_j} \right|$$

$$W_o(s) = \frac{k_W}{\lambda} \left(\prod_{|z_j| < \omega_B, |p_j| < \omega_B} \frac{s - z_j}{s - p_j} \right)$$

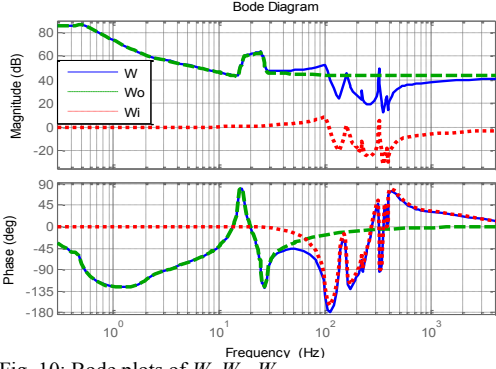


Fig. 10: Bode plots of W , W_o , W_i

A H_∞ loop shaping synthesis problem can then be established, by considering:

$$G = \begin{bmatrix} G_{mec} & G_{mot} \\ G_{mot} & \end{bmatrix}, W_1 = W_i, W_2 = \begin{bmatrix} W_o & 0 \\ 0 & \rho \end{bmatrix}$$

The constant scaling ρ is used on the motor current measure, and adjusted manually to increase the classical stability margins (phase and gain margins) at input.

The use of motor current measurement slightly modifies the target loop shape singular value in this way:

$$\begin{aligned} & \sigma \left(\begin{bmatrix} W_o & 0 \\ 0 & \rho \end{bmatrix} \begin{bmatrix} G_{mec} & G_{mot} \\ G_{mot} & \end{bmatrix} W_i \right) (j\omega) \\ &= \sqrt{\left[(W_o G_{mec} G_{mot} W_i)(-j\omega) \quad \rho(G_{mot} W_i)(-j\omega) \right] \begin{bmatrix} (W_o G_{mec} G_{mot} W_i)(j\omega) \\ \rho(G_{mot} W_i)(j\omega) \end{bmatrix}} \\ &= \sqrt{|(W_o G_{mec} G_{mot} W_i)(j\omega)|^2 + |\rho(G_{mot} W_i)(j\omega)|^2} \\ &= |(W_o G_{mec} G_{mot} W_i)(j\omega)| \sqrt{1 + \left| \frac{\rho}{(W_o G_{mec})(j\omega)} \right|^2} \\ &= |L_h(j\omega)| \sqrt{1 + \left| \frac{\rho}{(W_o G_{mec})(j\omega)} \right|^2} \end{aligned}$$

At this point, it must be highlighted that the order of W is quite significant in our example since the desired loop shape gain is complex (See Table I). Full order H_∞ loop-shaping synthesis might then be seen as awkward, since it results in an excessively high order 2 inputs 1 output controller (Table II). Despite it exhibits very nice stability margins, it cannot be implemented on the real system.

TABLE I
SYNTHESIS DATA

$\partial^\circ G$	$\partial^\circ L_h$	$\partial^\circ W$	$\partial^\circ W_o$	$\partial^\circ W_i$
15	30	45	24	21

TABLE II
FULL ORDER SYNTHESIS

γ	GM(dB)	PM($^\circ$)	$\partial^\circ K$
3	13	68	105

Then, our main goal is to find a much lower order controller that will shape the loop in the same way, by solving the four-block H_∞ control problem (2), at fixed order and fixed structure.

For fixed order H_∞ synthesis, we choose the cascade structure for the controller (Fig 11):

$$K = K_i \begin{bmatrix} K_{\theta_{LOS}} & 1 \end{bmatrix}$$

This structure enables to assign directly the roll-off constraint to K_i , and the performance objective to $K_{\theta_{LOS}}$.

Significant reduction of the order of K is then achieved without deteriorating the good stability margins (Table III, Fig 12) and the closed-loop sensitivity functions shape (Fig 13) obtained with the full order controller.

TABLE III
FIXED ORDER SYNTHESIS

γ	GM(dB)	PM($^\circ$)	$\partial^\circ K_{\theta_{LOS}}$	$\partial^\circ K_i$
3.4	18	55	6	6

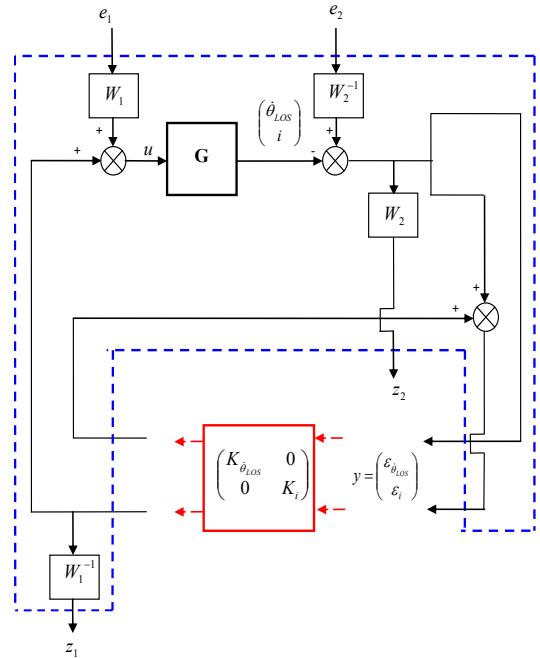


Fig. 11: Cascade Fixed Order Controller Synthesis

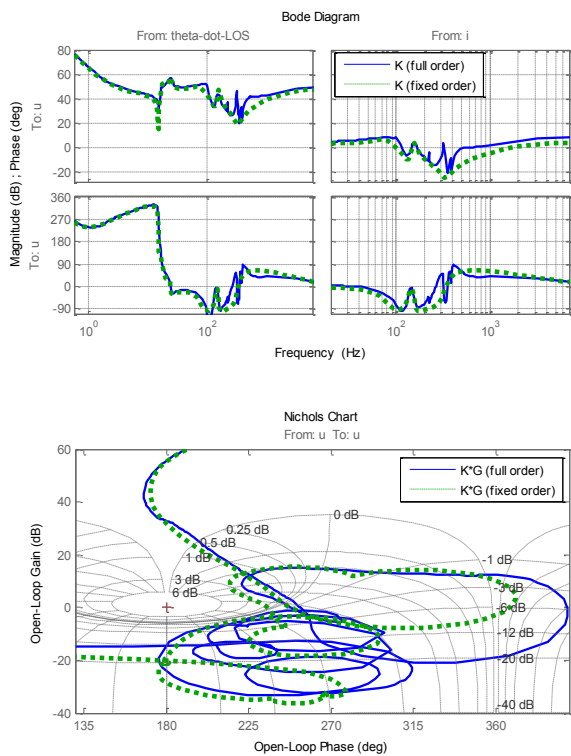


Fig. 12
 Top: Bode plots of controllers
 Bottom: Nichols chart of the open loop at input u .

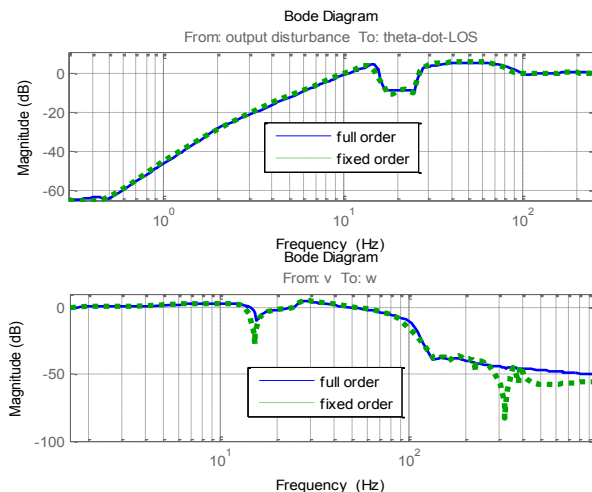


Fig. 13:
 Top: closed-loop sensitivity function (disturbance rejection).
 Bottom: closed-loop sensitivity function (robustness w.r.t Δ_{mec}).

V. CONCLUSION

In this paper we used the key ingredients of the H_∞ loop shaping technique to design reduced order and structured robust controllers, without having to manually tune the loop-shaping filters. For square plants (only a SISO case has been addressed but the extension to the square MIMO case would be straightforward), the design is quasi-automatic and only

requires defining a suitable target open loop shape, without taking much care of the order of this loop-shape and the resulting automatically computed loop-shaping filter. Moreover SIMO cases can also be well handled with a little bit more emphasis on the structure of the controller. This is made possible by the use of an efficient nonsmooth H_∞ synthesis algorithm: HINFSTRUCT.

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