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Discrete-time Incremental ISS: A Framework for Robust NMPC

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Abstract— **In this paper, Incremental Input-to-State Stability is studied as a system theoretic framework to address the challenges of robust nonlinear model predictive control. In the first part of the paper, a Lyapunov framework for Incremental Input-to-State Stability of nonlinear discrete-time dynamical systems is established. In the second part, Incremental Input-to-State Stability is shown to lead to an efficient MPC method for disturbed nonlinear systems. Based on the Incremental Input-to-State Stability Lyapunov function, a tightening of the constraints is proposed. Satisfaction of the tightened constraints can be guaranteed under the disturbances. By this concept, a robust nonlinear model predictive control problem is handled and the effectiveness is shown through an example from the literature.**

Index Terms— **Incremental ISS Lyapunov, Robust Optimization, Robust MPC**

I. INTRODUCTION

Throughout the last decades, model predictive control (MPC) has become one of the most widely used control techniques for industrial applications (see, e.g., [1]). This development is due to the fact that MPC is an optimal control technique being able to deal with constraints on the states and the inputs. Under given assumptions, MPC schemes can provide guarantees for stability and recursive feasibility of the closed loop (see, e.g., [2] and the references therein).

Dealing with disturbances and uncertainties is of great interest. Even though MPC is inherently robust against small disturbances in many cases, as presented, e.g., in [3], robustness cannot be guaranteed against all disturbances or for all systems. In order to provide at least bounds for the states and inputs of the disturbed system, different ideas have been brought up in the last years.

In the case of linear systems, the framework of *Tube MPC* ([4], [5]) has attracted much interest. Tubes are sets around a nominal trajectory of a system capturing the effect of disturbances and uncertainties. In terms of linear systems, tubes can be computed exactly. However, for nonlinear systems, tubes cannot be determined exactly. Therefore, other concepts have been introduced to provide bounds for non-linear systems. In [6], for example, a second MPC scheme is introduced for the error system, providing bounds for the states and inputs.

Another approach when dealing with uncertain nonlinear systems is to use Input-to-State Stability (ISS) (see, e.g., [7], [8]). When using ISS in the framework of Robust MPC, the idea is to find a bound around the origin. Instead of finding a bound around the origin as it is done in ISS, δISS aims to find a bound around a given trajectory. Thus, δISS is as a system theoretic concept more restrictive than ISS, but provides better bounds for Robust NMPC. A control system is said to be Incremental Input-to-State Stabile (δISS) with respect to external disturbances, if the distance between any two system trajectories can be bounded by the distance of the disturbance sequences and the initial states. δISS has not attracted much attention in the context of MPC. In [9], the related concept of δISOS is used to prove stability of a moving horizon estimator under vanishing disturbances. However, δISS proves to be a well suited concept providing tubes for Robust MPC. In fact, we show that the idea of δISS for MPC is a combination of the two main directions of Robust MPC, taking the bounds for the uncertain system from the definition of δISS and taking invariant sets as bounds around the nominal system, a behavior adapted from Tube MPC.

The contribution of this paper is twofold. In the first part, we show that the existence of a δISS Lyapunov function in discrete-time implies δISS by extending the results from [10] to discrete time. In the second part of the paper, a robust MPC problem is formulated, including an uncertainty affected nonlinear control system. We present classes of δISS systems for which a nominal MPC problem with tightened constraints can be formulated efficiently.

The remainder of this paper is organized as follows. In Section II, the discrete-time version of δISS as well as invariant sets for the states of δISS systems are introduced. The application of this idea to the framework of Robust MPC is presented in Section III. An example showing the usability of the idea is given in Section IV. In the end a conclusion is given.

Notation. A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class $\mathcal{K}$ function if $\alpha(s) > 0$ for all $s > 0$, it is strictly increasing, and $\alpha(0) = 0$. It is furthermore a class $\mathcal{K}_\infty$ function if $\alpha(s) \to \infty$ for $s \to \infty$. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class $\mathcal{KL}$ function if $\beta(s,t)$ is a class $\mathcal{K}$ function with respect to s for all t, it is strictly decreasing in t for all $s > 0$, and $\beta(s,t) \to 0$ as $t \to \infty$ for all $s > 0$. By id, the identity function is denoted.

A set is called a C-set if it is compact, convex, and contains the origin.

The Minkovski set addition is defined by $X \oplus Y := \{x+y : x \in X, y \in Y\}$; the Pontryagin set difference is defined as $X \ominus Y := \{z \in \mathbb{R}^n : z + y \in X,\ \forall y \in Y\}$.

II. DISCRETE-TIME INCREMENTAL ISS

We consider the nonlinear time-varying system

$$x(k+1) = f(k, x(k), w(k)), \qquad x(0) = \xi \tag{1}$$

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where $w(k)$ and $x(k)$ are the input and the system state, respectively, at the discrete time instance $k \in \mathbb{Z}_{\geq 0}$. The initial state is denoted ξ. We assume that $f(k, 0, 0) = 0$, i.e., $\xi = 0$ is an equilibrium point of the unforced system for all times. We assume w to be bounded in a C-set $\mathbb{W}$ with

$$w(k) \in \mathbb{W} \subset \mathbb{R}^q, \quad \forall k \in \mathbb{Z}_{\geq 0}.$$

Later we will consider w as a disturbance to our system, or in other words an uncontrollable input from the environment to the system. A short word concerning the notation: By $\boldsymbol{w} = \{w(0), w(1), \ldots, w(k-1)\}$ a sequence of k disturbances with $w(i) \in \mathbb{W}$, $i \in [0, k-1]$ is described. By $x(k, \xi, \boldsymbol{w})$, the state of system (1) at time instance k subject to the initial state ξ and the sequence of k disturbances $\boldsymbol{w}$ is determined.

First, we introduce the notion of *incremental global asymptotic stability* (δ*GAS*) in discrete time.

Definition 1 (δGAS): A system (1) is called *incrementally globally asymptotically stable* (δGAS), if there exists a $\mathcal{KL}$ function β such that for all $k \in \mathbb{Z}_{\geq 0}$, any initial states ξ_1, ξ_2 and any disturbance sequence $\boldsymbol{w}$,

$$|x(k, \xi_1, \boldsymbol{w}) - x(k, \xi_2, \boldsymbol{w})| \leq \beta(|\xi_1 - \xi_2|, k) \tag{2}$$

holds true.

The definition of δISS for systems in discrete time is introduced as follows.

Definition 2 (δISS): A system (1) is called *incrementally Input-to-State Stable* (δISS), if there exists a $\mathcal{KL}$ function β and a $\mathcal{K}_\infty$ function γ such that for any $k \in \mathbb{Z}_{\geq 0}$, any initial states ξ_1, ξ_2 and any couple of disturbance sequences $\boldsymbol{w}_1$, $\boldsymbol{w}_2$,

$$|x(k, \xi_1, \boldsymbol{w}_1) - x(k, \xi_2, \boldsymbol{w}_2)| \leq \beta(|\xi_1 - \xi_2|, k) + \gamma(\|\boldsymbol{w}_1 - \boldsymbol{w}_2\|)$$

holds true.

In the following, we abbreviate $x(k, \xi_i, \boldsymbol{w}_i)$ by x_i.

Definition 3 (δISS Lyapunov function): A function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is called a δISS Lyapunov function, if

$$\alpha_1(|x_1 - x_2|) \leq V(x_1, x_2) \leq \alpha_2(|x_1 - x_2|) \tag{3}$$

for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and there exists a $\kappa \in \mathcal{K}_\infty$ such that for any $w_1, w_2 \in \mathbb{W}$ and any x_1, x_2

$$\kappa(|x_1 - x_2|) \geq |w_1 - w_2| \Rightarrow \tag{4}$$

$$V\big(f(k, x_1, w_1), f(k, x_2, w_2)\big) - V(x_1, x_2) \leq -\rho(|x_1 - x_2|),$$

with $\rho \in \mathcal{K}_\infty$ positive definite for all $k \in \mathbb{Z}_{\geq 0}$.

We can provide the following sufficient condition for a discrete-time system (1) to be δISS, by following steps analogous to those described in [11] in continuous time.

Theorem 1: If system (1) admits a time-invariant δISS Lyapunov function, then it is δISS.

The structure of the proof of Theorem 1 follows the proof of *Input-to-State-Stability* in discrete-time presented in [11].

Proof: Assume that $V(x_1, x_2)$ is a time-invariant δISS Lyapunov function for system (1). Let $\alpha_1, \alpha_2, \alpha_3$ and κ be as defined in Definition 3. Condition (4) can be written as

$$\begin{aligned} &V\big(f(k, x_1, w_1), f(k, x_2, w_2)\big) - V(x_1, x_2) \\ &\quad \leq -\alpha_4(|x_1 - x_2|) + \sigma(|w_1 - w_2|), \end{aligned} \tag{5}$$

where α_4 is any $\mathcal{K}_\infty$–function and σ is any $\mathcal{K}$–function such that $(\alpha_4 - \sigma \circ \kappa) \in \mathcal{K}_\infty$, satisfying $(\alpha_4 - \sigma \circ \kappa)(r) = \rho(r)$. Thus, we can rewrite (5)

$$\begin{aligned} &V\big(f(k, x_1, w_1), f(k, x_2, w_2)\big) - V(x_1, x_2) \\ &\quad \leq -\alpha_5\big(V(x_1, x_2)\big) + \sigma(|w_1 - w_2|), \end{aligned} \tag{6}$$

where $\alpha_5 = \alpha_4 \circ \alpha_2^{-1}$ ($\in \mathcal{K}_\infty$ by definition). Without loss of generality, we assume that $\mathrm{id} - \alpha_5 \in \mathcal{K}_\infty$ (see Lemma B.1 in [11]). Now, we fix two points ξ_1, ξ_2 and pick two arbitrary inputs w_1 and w_2, both within the set $\mathbb{W}$. We consider the set D defined by

$$D = \{(\xi_1, \xi_2) : V(\xi_1, \xi_2) \leq b\}, \tag{7}$$

where $b = \alpha_5^{-1} \circ \nu^{-1} \circ \sigma(|w_1 - w_2|)$, with $\nu \in \mathcal{K}_\infty$ chosen such that $\mathrm{id} - \nu \in \mathcal{K}_\infty$.

Claim 1: If there exists a $k_0 \in \mathbb{Z}_{\geq 0}$ such that $(x_1(k_0), x_2(k_0)) \in D$, then $(x_1(k), x_2(k)) \in D$ for all $k \geq k_0$.

Proof: Assume that $(x_1(k_0), x_2(k_0)) \in D$. Then $V\big(x_1(k_0), x_2(k_0)\big) \leq b$, and from (7) we can derive $\nu \circ \alpha_5(V(x_1(k_0), x_2(k_0))) \leq \sigma(\|w_1 - w_2\|)$. Using (6), we can write

$$\begin{aligned} V(x_1(k_0+1), x_2(k_0+1)) \leq &(\mathrm{id} - \alpha_5)V(x_1(k_0), x_2(k_0)) \\ &+ \sigma(\|w_1 - w_2\|) \\ \leq &(\mathrm{id} - \alpha_5)(b) + \sigma(\|w_1 - w_2\|). \end{aligned}$$

The second inequality holds as $\mathrm{id} - \alpha_5 \in \mathcal{K}_\infty$. Since $\sigma(\|w_1 - w_2\|) = \nu \circ \alpha_5(b)$, it follows

$$\begin{aligned} V(x_1(k_0+1), x_2(k_0+1)) &\leq (\mathrm{id} - \alpha_5)(b) + \sigma(\|w_1 - w_2\|) \\ &\leq -(\mathrm{id} - \nu) \circ \alpha_5(b) + b \leq b \end{aligned}$$

and by induction, one can conclude that $V(x_1(k_0 + j), x_2(k_0 + j)) \leq b$ for all $j \in \mathbb{Z}_{\geq 0}$. ■

Now, we introduce $j_0 = \min\{k \in \mathbb{Z}_{\geq 0} : x(k) \in D\} \leq \infty$. From the conclusion above, we can state $V(x_1(k), x_2(k)) \leq \tilde{\gamma}(|w_1 - w_2|)$ for all $k \geq j_0$, where $\tilde{\gamma}(r) = \alpha_5^{-1} \circ \nu^{-1} \circ \sigma(r)$. Due to the definition of b, $\nu \circ \alpha_5(V(x_1(k), x_2(k))) > \sigma(|w_1 - w_2|)$ holds true for all $k < j_0$. Using this relation and (6), one can see that

$$\begin{aligned} &V(f(k, x_1(k), w_1), f(k, x_2(k), w_2)) - V(x_1(k), x_2(k)) \\ &\quad \leq -(\mathrm{id} - \nu) \circ \alpha_5(V(x_1(k), x_2(k)). \end{aligned}$$

By comparison lemma (see, e.g., [12]), there exists a $\mathcal{KL}$ function $\tilde{\beta}$ such that $V(x_1(k), x_2(k)) \leq \tilde{\beta}(V(x_1(0), x_2(0)), k)$ for all $0 \leq k \leq j_0 + 1$. Finally, we can conclude

$$\begin{aligned} &V(x_1(k), x_2(k)) \\ &\leq \max\Big\{\tilde{\beta}(V(x_1(0), x_2(0)), k), \tilde{\gamma}(\|w_1 - w_2\|)\Big\}, \forall k \in \mathbb{Z}_{\geq 0}. \end{aligned}$$

The statement of Definition 2 follows with $\beta(s, t) = \alpha_1^{-1}\left(\tilde{\beta}\left(\alpha_2(|x_1(0) - x_2(0)|), t\right)\right)$ and $\gamma(s) = \alpha_1^{-1} \circ \tilde{\gamma}(s)$. ■

We have shown that the existence of a δISS Lyapunov function for a discrete-time system implies δISS for this system. In the following, statements about systems that satisfy the δISS condition are given. By these statements, some intuition for δISS systems should be provided.

Proposition 1: The class of δISS systems is a strict subset of the class of systems that are ISS.

Even though not all ISS systems are δISS, there are some classes of systems that can be shown to be δISS under certain conditions. One of the most widely used systems, is the class of linear time-invariant (LTI) systems. For LTI systems, δISS follows from asymptotic stability. Another class of systems for which a general conclusion can be drawn is the class of globally Lipschitz continuous systems. Assume that system (1) is globally Lipschitz, that is, $|f(x_1, w) - f(x_2, w)| \leq L|x_1 - x_2|$ for all x_1, x_2 and for all $w \in \mathbb{W}$, where $0 < L < \infty$ is the Lipschitz constant. Then the system can be shown to be δISS if $0 < L < 1$ holds. Both statements can be proven by choosing a Lyapunov candidate function $V(x_1, x_2) = (x_1 - x_2)^T P(x_1 - x_2)$ with an appropriate P.

For the case of Lipschitz systems a dissipativity condition is presented in [13] which can be seen as a special case of the presented Lyapunov condition.

In the following, we discuss how to derive robustly positively invariant sets for the difference of two states of system (1) under disturbances. These sets are called *incremental robustly positively invariant sets* or, in short, δRPI.

Definition 4 (δRPI): A set $\mathbb{E} \subset \mathbb{R}^n \times \mathbb{R}^n$ is said to be δRPI under the dynamics of system (1), if $(\xi_1, \xi_2) \in \mathbb{E}$ implies

$$\big(x(k, \xi_1, \boldsymbol{w}_1), x(k, \xi_2, \boldsymbol{w}_2)\big) \in \mathbb{E} \tag{8}$$

for all $k \in \mathbb{Z}_{\geq 0}$ and for all $\boldsymbol{w}_1, \boldsymbol{w}_2 \in \mathbb{W}$.

δISS is introduced as a gain notion. However, one can use the gains β and γ in Definition 2 in order to derive explicit bounds for the difference of the states. Using the δISS Lyapunov function and assuming that the disturbance is bounded within a certain set, we can state a set for the state difference according to the following proposition.

Proposition 2: Let Theorem 1 hold, then

$$\mathbb{E} = \{(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n : V(x_1, x_2) \leq b^*\} \tag{9}$$

is a δRPI set with

$$b^* = \max_{w_1, w_2 \in \mathbb{W}} \alpha_5^{-1} \circ \nu^{-1} \circ \sigma(|w_1 - w_2|) \tag{10}$$

and $V(x_1, x_2)$ satisfying (3).

Proof: It is known from (7) and Claim 1 that the provided set

$$\mathbb{E} = \{(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n : V(x_1, x_2) \leq b\}, \tag{11}$$

is incrementally robustly positively invariant under the given disturbances for

$$b = \sup_{\tau} \alpha_5^{-1} \circ \nu^{-1} \circ \sigma(|w_1(\tau) - w_2(\tau)|),\ \forall\, w(\tau) \in \mathbb{W}. \tag{12}$$

As long as the initial conditions (ξ_1, ξ_2) lie within $\mathbb{E}$, all future state differences will stay within $\mathbb{E}$.

The bound b^* should be determined such that incremental robustly positive invariance is guaranteed for all possible combinations of input sequences. Since it is given that $\alpha_5 \in \mathcal{K}_\infty$, $\nu \in \mathcal{K}_\infty$, and $\sigma \in \mathcal{K}$, the maximum for b^* in (10) can be determined by using the maximal distance between two points within the disturbance set $\mathbb{W}$ and applying it to the composition of class $\mathcal{K}_\infty$ and $\mathcal{K}$ functions, respective. ∎

III. The Framework of Robust MPC

In this section, it is shown that the set theoretic interpretation of δISS systems leads to an efficient solution for Robust MPC problems.

The discrete-time uncertain model

$$x(k+1) = f(x(k), u(k), w(k)), \quad x(0) = \xi, \tag{13}$$

is taken into consideration in the following, to which in extension to (1) an additional input $u(k)$ is added. The input $u(k)$ is seen as a time-dependent parameter in the following. In contrast to the "disturbance input" $w(k)$, the input $u(k)$ can be influenced in order to control the system. For this system, we want to solve the (optimal control) problem

$$(P) \begin{cases} \min\limits_{\boldsymbol{u}} \sum\limits_{k=0}^{\infty} \ell\,(x(k), u(k)) \\ \text{s.t. } x(k+1) = f(x(k), u(k), w(k)), \quad x(0) = \xi \\ \quad\ x(k) \in \mathbb{X}, \quad \forall k, \\ \quad\ u(k) \in \mathbb{U}, \quad \forall k, \\ \quad\ w(k) \in \mathbb{W},\ \forall k, \end{cases}$$

where x, u, and w are subject to hard constraints. The constraints on the state $\mathbb{X}$, the input $\mathbb{U}$, and the disturbance $\mathbb{W}$ are polytopic C-sets. The concepts presented in the following hold in general for all C-sets and are not necessarily limited to polytopic sets. However, in accordance to the usual framework in MPC, we only consider polytopic sets. The cost function $\ell(\cdot, \cdot)$ is positive definite for all $x \neq 0$ and $u \neq 0$, and, without loss of generality, $\ell(0, 0) = 0$.

In the following, we want to replace problem (P) by a nominal problem, that is, a problem without consideration of the disturbances. By neglecting the disturbance, the *nominal system* can be introduced as

$$\bar{x}(k+1) = \bar{f}(\bar{x}(k), \bar{u}(k)) := f(\bar{x}(k), \bar{u}(k), 0). \tag{14}$$

The question we deal with in the next section is how the nominal trajectory may look like such that the original constraints on the disturbance-affected system are still satisfied. By bar, all variables related to the nominal system are denoted. According to the standard definition in the literature, see e.g. [8], we introduce the error representing the difference of the uncertain system to the nominal system as

$$e(k) = x(k) - \bar{x}(k). \tag{15}$$

A. Tightened Constraints for the Nominal Problem

In the following, we want to derive the constraints for the nominal system, $\bar{\mathbb{X}}$ and $\bar{\mathbb{U}}$, such that by guaranteeing those, implicitly satisfaction of the original constraints by the disturbed system is ensured.

In a first consideration, we state the following assumption for the model dynamics (13) providing that the system is δISS.

Assumption 1: There exists a δISS Lyapunov function $V(x_1, x_2)$ such that it holds for all $u \in \mathbb{U}$

$$\begin{aligned} V\big(f(x_1, u, w_1), f(x_2, u, w_2)\big) - V(x_1, x_2) \\ \leq -\alpha\big(|x_1 - x_2|\big) + \sigma(|w_1 - w_2|), \end{aligned}$$

where α is a $\mathcal{K}_\infty$–function and σ is a $\mathcal{K}$–function.

Considering the constraints for the input, the input sequence $\bar{\boldsymbol{u}}$ derived from the MPC of the nominal system is applied to the uncertain system as well. Therefore, one can use all input capability for the nominal system, that is, $\bar{\mathbb{U}} = \mathbb{U}$.

In the general case of an existing Lyapunov function V satisfying Theorem 1, the tightened constraint for the system state can be given by

$$\bar{\mathbb{X}} = \{\bar{x} \in \mathbb{X} : \forall \tilde{x} \text{ s.t. } V(\bar{x}, \tilde{x}) \leq b \Rightarrow \tilde{x} \in \mathbb{X}\} \tag{16}$$

This is the exact solution for the tightened state constraint. Unfortunately, this set is difficult to derive since one might need to check for a large number of points. However, due to the definition of the δISS Lyapunov function in (3), an lower bound for the Lyapunov function is given depending only on the norm of the difference of the states. This lower bound provides an outer approximation of the δRPI $\mathbb{E}$ introduced in (9). Due to the equivalence of norms, a lower bound can be given for any norm. Thus, one can state

$$\alpha_1(|e|) = \alpha_1(|x - \bar{x}|) \leq V(x, \bar{x}),$$

with α_1 defined in (3). Now, the following lemma holds.

Lemma 1: Consider the set

$$\mathbb{E}_{\alpha_1} = \{e \in \mathbb{R}^n : \alpha_1(|e|) \leq b\}, \tag{17}$$

with

$$b = \max_{w \in \mathbb{W}} \alpha_5^{-1} \circ \nu^{-1} \circ \sigma(|w|). \tag{18}$$

Let Assumption 1 hold and let $x(k)$ and $\bar{x}(k)$ be the trajectories of the system (1) with initial conditions ξ and $\bar{\xi}$, respectively, satisfying $(\xi, \bar{\xi}) \in \mathbb{E}$ and generated by the disturbance sequences $\boldsymbol{w}_1$ and $\boldsymbol{w}_2 \equiv 0$. Then the error $e(k) = x(k) - \bar{x}(k)$ satisfied $e(k) \in \mathbb{E}_{\alpha_1}$ for all $k \geq 0$.
Note that that the bound b is derived from (10) taking $\boldsymbol{w}_2 \equiv 0$.

Proof: By Assumption 1 there exists a δISS Lyapunov function. It is know from Proposition 2 that $\mathbb{E}$ is a δRPI, that is, $(x(k), \bar{x}(k)) \in \mathbb{E}$ for all times. From Definition 3 follows the lower bound for the Lyapunov function $\alpha_1(|x - \bar{x}|) \leq V(x, \bar{x})$. Since $V(x, \bar{x}) \leq b$, with b defined in (18), $\alpha_1(|x - \bar{x}|) \leq b$. Since for the states holds $(x, \bar{x}) \in \mathbb{E}$ for all $k \geq 0$, the errors satisfy $e(k) \in \mathbb{E}_{\alpha_1}$ for all times. ■

Corollary 1: If the initial error $e(0) = \xi - \bar{\xi}$ lies in the convex set

$$\mathbb{E}_{\alpha_2} = \{e \in \mathbb{R}^n : \alpha_2(|e|) \leq b\}, \tag{19}$$

then $e(k) \in \mathbb{E}_{\alpha_1}$ for all $k \geq 0$.

Proof: From Definition 3 follows the upper bound for the Lyapunov function $V(x, \bar{x}) \leq \alpha_2(|x - \bar{x}|)$. If $\alpha_2(|\xi - \bar{\xi}|) = \alpha_2(|e(0)|) \leq b$, then $V(\xi, \bar{\xi}) \leq b$. With Proposition 2 and Lemma 1, the statement holds. ■

Please note that the set provided by $\mathbb{E}_{\alpha_1}$ is a convex unit ball depending on the choice of the norm in Definition 3. Convexity follows from the definition of $\mathcal{K}$–functions. This set $\mathbb{E}_{\alpha_1}$ is not robustly invariant in the classical sense (see, e.g., [6]) since an error dynamics cannot be explicitly derived. It is only known that the two states are in an δRPI set and therefore one can derive that the error will remain within a certain set for all times being introduced by means of the same condition as the δRPI for the states. Moreover, please note that $\mathbb{E} \subset \mathbb{R}^n \times \mathbb{R}^n$ whereas $\mathbb{E}_{\alpha_1} \subset \mathbb{R}^n$, only. When the error set is given by $\mathbb{E}_{\alpha_1}$, an inner approximation of the nominal set constraint $\bar{\mathbb{X}}$ can be derived by the Pontryagin set difference $\mathbb{X}_{\alpha_1} = \mathbb{X} \ominus \mathbb{E}_{\alpha_1} \subseteq \bar{\mathbb{X}}$.

Theorem 2: Consider the uncertain system (13) with a known and bounded disturbance set $\mathbb{W}$. Let Assumption 1 hold and let (13) admit a δISS Lyapunov function. And, let the initial offset $e(0) = \xi - \bar{\xi}$ be bounded within the set $\mathbb{E}_{\alpha_2}$.
Assume there exists an input sequence $\boldsymbol{u} = \{u(0), u(1), \dots\}$, with $u(k) \in \mathbb{U} = \bar{\mathbb{U}}, \forall k \geq 0$, such that $\bar{x}(k) \in \mathbb{X}_{\alpha_1}$, $\forall k \geq 0$. Then, $x(k) \in \mathbb{X}$, $\forall k \geq 0$.

Proof: Under Assumption 1, due to the existence of a lower bound on the Lyapunov function, and by guaranteeing $e(0) = \xi - \bar{\xi} \in \mathbb{E}_{\alpha_2}$, we can derive from Lemma 1 and Corollary 1 that the error $e(k) \in \mathbb{E}_{\alpha_1}$ for all times. Assuming a feasible input sequence which provides $\bar{x}(k) \in \mathbb{X}_{\alpha_1}$ for all $k \geq 0$, it follows by Minkovski set addition that $x(k) \in \mathbb{X}_{\alpha_1} \oplus \mathbb{E}_{\alpha_1} = \mathbb{X}$ for all $k \geq 0$. ■

B. Error based Lyapunov function

In the following, we consider δISS Lyapunov functions of the form

$$V(x_1, x_2) = V(x_1 - x_2). \tag{20}$$

This is a special case since not all δISS Lyapunov functions need to satisfy this condition allowing us to introduce several simplifications. The interested reader is referred to [14] for a discussion of the necessity of the "$V(x_1, x_2)$-construction". Even though not all δISS Lyapunov functions satisfy (20), a large subclass of functions do, such that it is reasonable to discuss this case in detail. Moreover, in the remainder of the paper, we assume that all Lyapunov functions satisfy (20).

Using (15), one can state

$$V(x, \bar{x}) = V(x - \bar{x}) = V(e). \tag{21}$$

Now, the following lemma holds.

Lemma 2: Consider the set

$$\mathbb{E}_V = \{e \in \mathbb{R}^n : V(e) \leq b\}, \tag{22}$$

with

$$b = \max_{w \in \mathbb{W}} \alpha_5^{-1} \circ \nu^{-1} \circ \sigma(|w|). \tag{23}$$

Let Assumption 1 hold with a Lyapunov function of the form (21) and let $x(k)$ and $\bar{x}(k)$ be the trajectories of the system (1) with initial conditions ξ and $\bar{\xi}$, respectively, and generated by the disturbance sequences $\boldsymbol{w}_1$ and $\boldsymbol{w}_2 \equiv 0$. Then the error $e(k) = x(k) - \bar{x}(k)$ satisfies $e(k) \in \mathbb{E}_V$ for all $k \geq 0$.

Proof: By Assumption 1 there exists a δISS Lyapunov function. It is know from Proposition 2 that $\mathbb{E}$ is a δRPI, that is, $(x(k), \bar{x}(k)) \in \mathbb{E}$ for all times. Since (21) holds in the special case, (9) can be rewritten

$$\mathbb{E} = \{(x, \bar{x}) \in \mathbb{R}^n \times \mathbb{R}^n : V(x - \bar{x}) \leq b\}. \tag{24}$$

Thus, $(x(k), \bar{x}(k)) \in \mathbb{E}$ for all times if and only if $e(k) = x(k) - \bar{x}(k) \in \mathbb{E}_V$ for all times. ■

Remark 1: In general, the result stated by Lemma 2 provides a less conservative statement than the bounds given in Lemma 1.

By the special structure of the Lyapunov function, $\mathbb{E}_V$ is an invariant set. It contains the differences of all states $(x, \bar{x}) \in \mathbb{E}$, and, vice versa, for all errors contained in $\mathbb{E}_V$, there exists at least one difference of states $(x, \bar{x}) \in \mathbb{E}$.

When the error set is given by the results of Lemma 2, the nominal set constraint $\bar{\mathbb{X}}$ can be derived by the Pontryagin set difference $\bar{\mathbb{X}} = \mathbb{X} \ominus \mathbb{E}_V$. In case of $V(e)$ being a quasi-convex function, the set $\mathbb{E}_V$ is convex. For the special case of polytopic constraints, the Pontryagin difference can be derived directly and for many special cases the tightening can be performed efficiently.

C. Solving the Problem with Tightened Constraints

In the following, we solve the problem with the tightened constraints. Therefore, we will return to the concept of MPC.

In general, it is impossible to solve the infinite-horizon optimal control problem (P). However, an approximation of this problem can be made by a finite-horizon control problem. Please note that according to [15] the optimal cost achieved with this approximation is always different to the infinite-horizon optimal cost.

Within the framework of tube MPC, using the initial state $\bar{\xi}$ of the nominal system as an additional optimization variable is a common concept (see, e.g., [4]). However, $\bar{\xi} \in (\xi \oplus \mathbb{E}_V) \cap \bar{\mathbb{X}}$ must be introduced as an additional constraint for the initial state in order to guarantee feasibility. We introduce the finite-horizon nominal problem $(\bar{P})$

$$(\bar{P}) \begin{cases} \min\limits_{\bar{\xi},\bar{\boldsymbol{u}}} \sum\limits_{k=0}^{N-1} \ell\left(\bar{x}(k), \bar{u}(k)\right) + V_f(\bar{x}(N)) \\ \text{s.t. } \bar{x}(k+1) = \bar{f}(\bar{x}(k), \bar{u}(k)), \qquad \bar{x}(0) = \bar{\xi} \\ \quad \bar{x}(k) \in \bar{\mathbb{X}}, \quad \forall k \in [1, N], \\ \quad \bar{u}(k) \in \bar{\mathbb{U}}, \quad \forall k \in [0, N-1], \\ \quad \bar{x}(N) \in \Omega, \\ \quad \bar{\xi} \in (\xi \oplus \mathbb{E}_V) \cap \bar{\mathbb{X}} \end{cases}$$

where N is the prediction horizon. The cost function $\ell(\cdot,\cdot)$ is the same as for (P), and the constraints $\bar{\mathbb{X}} \subseteq \mathbb{X}$ and $\bar{\mathbb{U}} \subseteq \mathbb{U}$ are tightened constraints according to Section III-B. The terminal set Ω and the terminal cost V_f are chosen such that they satisfy the assumptions for closed-loop asymptotic stability stated in [2]:

A1: $\Omega \subseteq \bar{\mathbb{X}}$, Ω closed, $0 \in \Omega$.
A2: $\exists \kappa_f(\bar{x})$, s.t. $\kappa_f(\bar{x}) \in \bar{\mathbb{U}}$ and $\bar{f}(\bar{x}, \kappa_f(\bar{x})) \in \Omega$, $\forall \bar{x} \in \Omega$.
A3: $V_f(\bar{x}, \kappa_f(\bar{x})) - V_f(\bar{x}) \leq -\ell(\bar{x}, \kappa_f(\bar{x}))$, $\forall \bar{x} \in \Omega$.

Since we assume the system to be δISS, no teightning for the input constraints is needed, that is, $\bar{\mathbb{U}} = \mathbb{U}$. Concerning the notation, we introduce the set $\mathbb{E}_0 = 0 \oplus \mathbb{E}_V$.

By means of the idea of optimizing over the initial condition, Algorithm 1 is introduced.

Algorithm 1

given: initial state $\xi \in \mathbb{X}$
for $i = 0, 1, 2, \ldots$ **do**
solve $(\bar{P})$ $\left(\text{with } \bar{\xi} \in (\xi \oplus \mathbb{E}_V) \cap \bar{\mathbb{X}}\right)$
$x(i+1) = f(\xi, \bar{u}(0), w(i))$, where $\bar{\boldsymbol{u}} = \{\bar{u}(0), \bar{u}(1), \ldots\}$
$\xi = x(i+1)$
end for

Theorem 3: Consider the uncertain system (13) with a known and bounded disturbance set $\mathbb{W}$. Let Assumption 1 hold and let (13) admit a δISS Lyapunov function of the form $V(x, \bar{x}) = V(x - \bar{x})$. And, let the initial offset $e(0) = \xi - \bar{\xi}$ be bounded within the set $\mathbb{E}_V$.

Assume there exists a solution for problem $(\bar{P})$ under the given assumptions A1 - A3. Then, the sequence of states $\boldsymbol{x} = \{\xi, x(1), x(2), \ldots\}$ produced by Algorithm 1 satisfies

$$x(k) \in \mathbb{X}, \ \forall k \geq 0,$$

and the uncertain system converges to the set $\mathbb{E}_0$, that is,

$$\lim_{k\to\infty} d(x(k), \mathbb{E}_0) \to 0,$$

where $d(x, \mathbb{E}_0) = \inf\{d(x, e) : e \in \mathbb{E}_0\}$ and the distance function $d(\cdot,\cdot)$ is a metric on $\mathbb{X}$.

Proof: First, we show recursive feasibility. By the special structure of the Lyapunov function given in (20) and by satisfying Assumption 1, it is known from Lemma 2 that $\bar{x}(k) \in \bar{\mathbb{X}}$ implies $x(k) \in \mathbb{X}$. If a feasible solution is found for $(\bar{P})$ at iteration i, it follows that $\bar{x}^i(1) \in (x^i(1) \oplus \mathbb{E}_V) \cap \bar{\mathbb{X}}$. For iteration $i+1$ at least one feasible trajectory can be found when starting the optimization of iteration $i+1$ at $\bar{\xi} = \bar{x}^i(1)$. Due to the Assumptions A1-A3, a feasible input sequence for the new iteration can be found by $\bar{\boldsymbol{u}}^{i+1} = \{\bar{u}^i(1), \bar{u}^i(2), \ldots, \bar{u}^i(N-1), \kappa_f(\bar{x}^i(N))\}$ providing a feasible state sequence $\bar{\boldsymbol{x}}^{i+1}$. Thus, recursive feasibility is given.

Second, we show convergence to the set $\mathbb{E}_0$. By $V^*(x)$ we denote the cost of the solution of the nominal problem $(\bar{P})$ with an initial state x. It is known that the nominal MPC is stable if $V^*(\bar{\xi}^{i+1}) \leq V^*(\bar{\xi}^i) - \ell(\bar{\xi}^i, \bar{u}_0^i)$ (see, e.g., [2]). Since we are free to choose $\bar{\xi}^{i+1} \in (x^i(1) \oplus \mathbb{E}) \cap \bar{\mathbb{X}}$ and since, by definition, $\bar{x}(1)^i \in (x^i(1) \oplus \mathbb{E}) \cap \bar{\mathbb{X}}$, we can always guarantee

$$V^*(\bar{\xi}^{i+1}) \leq V^*\left(\bar{f}(\bar{\xi}^i, \bar{u}^i(0))\right) = V(\bar{x}^i(1)). \qquad (25)$$

It follows from Assumptions A1-A3 by standard argumentation of Quasi-infinite horizon MPC that $V^*(\bar{x}^i(1)) \leq V(\bar{\xi}^i) - \ell(\bar{\xi}^i, \bar{u}^i(0))$ and by plugging this into (25), the stability condition is satisfied. Hence, the nominal system converges. Moreover, one can see that as soon as $0 \in (x^i(1) \oplus \mathbb{E}) \cap \bar{\mathbb{X}}$, this will always be the optimal solution of the nominal problem $(\bar{P})$. Thus, the structure of the Lyapunov function and Assumption 1 provides that the state of the real system will be within $\mathbb{E}_0$ and stay therein for all times. ■

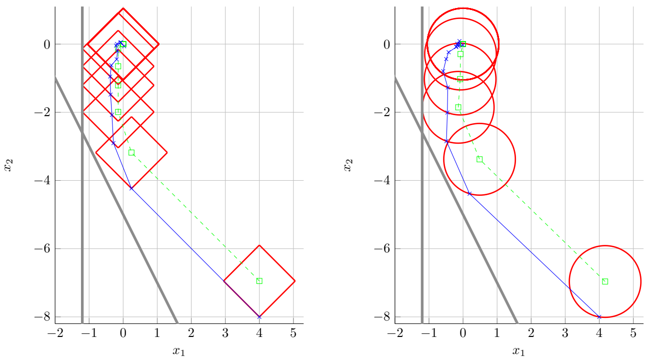
IV. Example

In this section, the idea of combining MPC with δISS is applied to an example from the literature.

Consider the following uncertain state space model of a nonlinear system in discrete time, presented in [16] by

$$\begin{aligned} x_1(k+1) =& 0.55x_1(k) + 0.12x_2(k) \\ &+ \left(0.01 - 0.6x_1(k) + x_2(k)\right)u(k) + w_1(k) \\ x_2(k+1) =& 0.67x_2(k) \\ &+ \left(0.15 + x_1(k) - 0.8x_2(k)\right)u(k) + w_2(k). \end{aligned}$$

The input is constrained to $|u(k)| \leq 0.1$ and the disturbances are given within the set $\mathbb{W} = \{w \in \mathbb{R}^2 : \|w\|_\infty \leq 0.1\}$. Two constraints are imposed upon the states, namely $x_1 \geq -1.2$ and $2x_1 + x_2 \geq -5$. The cost is given by the quadratic cost $l(x, u) = \|(x, u)\|_{Q,R}$, where Q and R are identity matrices of appropriate size and the goal is to stabilize the system around the origin. The terminal set and



(a) $V_1(e) = \|e\|_1$ (b) $V_2(e) = \|e\|_2$

Fig. 1: State space for the system under disturbances. The green dashed line provides the sequence of optimized initial states for the nominal system while the blue solid line represents the system subject to disturbances. The sets $\mathbb{E}_{V_i}$ are given in red. The gray solid lines represent the state constraints.

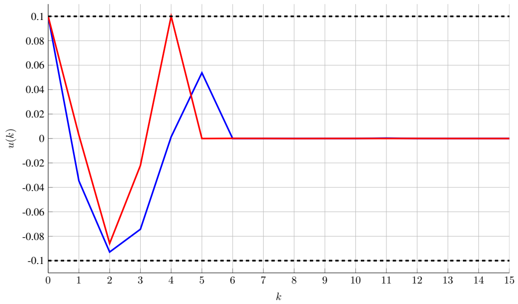


Fig. 2: Inputs for the example with V_1 (blue) and V_2 (red).

the terminal cost are taken from [16]. The prediction horizon is $N = 9$.

For this example, one can find at least two δISS Lyapunov functions. The Lyapunov functions we concentrate on are

$$V_1(x, \bar{x}) = V_1(e) = \|e\|_1 \text{ and } V_2(x, \bar{x}) = V_2(e) = \|e\|_2.$$

We investigate the difference in the results for both functions. The error sets $\bar{\mathbb{E}}$ can be determined for both of them. With some standard calculations, the level sets for the Lyapunov function for which the error e is guaranteed to stay inside are given by

$$\mathbb{E}_{V_1} = \{e \in \mathbb{R}^n : \|e\|_1 \leq b_1 = 1.0526\}$$

and

$$\mathbb{E}_{V_2} = \{e \in \mathbb{R}^n : \|e\|_2 \leq b_2 = 1.0514\},$$

where the values for b_1 and b_2 are bounds as introduced in Lemma 1. In order to solve the nominal NMPC problem, the algorithm presented in [17] is employed, which is also available online.

In Fig. 1, the system behavior for the system under disturbances is displayed for an initial condition of $\xi = (4, -8)^T$. The blue line represents the disturbed system, the green dashed line represents the sequence of optimized initial states for the nominal system. Moreover, one can see that the sets $\mathbb{E}_V$, which are represented by the red squares and circles, respectively, are always within the original constraints, such that the disturbed system will implicitly satisfying all constraints. Furthermore, one can see that the initial states for the nominal system are optimized such that the disturbed states are always on the edge of the sets $\mathbb{E}_{V_i}$. In Fig. 2 one can observe that the inputs satisfy the given constraint on $u(k) = \bar{u}(k)$ for all times.

V. Conclusion

In this paper we have shown that δISS systems can efficiently be used to address the challenges of robust NMPC. It was shown that the existence of a δISS Lyapunov function implies δISS for discrete time systems. Exploiting the set theoretic quantification of the Lyapunov function, the Robust MPC problem could be transferred into a nominal MPC problem where satisfaction of the nominal constraints implicitly satisfied satisfaction of the constraints of the disturbed problem. In the end an example was given presenting that the concept works efficiently.

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