2013 European Control Conference (ECC)
July 17-19, 2013, Zürich, Switzerland.

Almost global attitude stabilization of a rigid body for both internal and external actuation schemes

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***Abstract*— Recent developments in the attitude control of a rigid body include the development of almost globally stabilizing coordinate-free control laws. However, all of these results focus on external actuation. In this paper, we observe that the frequently used way of stabilization by error functions can also be given a Hamiltonian interpretation. We show that there exists a class of control laws which are applicable for both external and internal actuation. We also construct such control laws, motivated by the potential energy of a spinning top.**

I. Introduction

Attitude stabilization of a rigid body is a classical and important problem and has been widely studied in the context of controlling spacecraft or underwater vehicles. Even though the state space for this problem is a nonlinear manifold, classically this problem has been analyzed by making use of local representations such as Euler angles or global but redundant representations like quaternions, resulting in differential equations on a Euclidean space. It is only recently that the problem has been analyzed in a 'coordinate-free' framework ([1], [2], [3], [4], [5], [6]) where one deals with the dynamics directly on the nonlinear state space, namely the tangent or the cotangent bundle of $SO(3)$. The motivation for doing this, rather than use a coordinate dependent approach, has been documented well in the references cited above.

One of the main advantages of the coordinate-free approach is that it allows a global analysis of the designed feedback control law. However, since the rigid body dynamics evolves on a fiber bundle over a compact manifold, global stabilization is not achievable through continuous feedback [7]. The notion of *almost global asymptotic stability* (AGAS) is the best one can hope to achieve in this situation, where the region of attraction of the desired equilibrium point is a dense set in the state space. Such control laws have been indeed developed for the case of external actuation [1], [3], [4], [5], [6].

Since the existing body of work addresses the case of external actuation alone, in this work we address the problem of almost global asymptotic stabilization of a fully actuated rigid body where the torque is applied through internal rotors (such as reaction wheels). In fact, we show that there exist control laws that achieve AGAS with external actuation and achieve the same with internal actuation, with only a sign change.

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We first note in section III that the Riemannian-geometric error function based approach that is mostly used in the literature for the design of externally applied stabilizing control torques, can also be given a Hamiltonian interpretation where the error function acts like a potential function with a minimum at the desired equilibrium. In section IV, we then show how the same potential can be used to design stabilizing internal torques. Keeping this in mind, in section V we show how to derive a stabilizing potential motivated by the classical heavy top potential, which finally leads to the often used error function, namely the modified trace function. We present simulation results in section VI.

II. Preliminaries

The vector space of skew-symmetric matrices $\mathfrak{so}(3)$ can be identified with $\mathbb{R}^3$ using the standard 'skew-symmetrizer' map $\mathcal{S} : \mathbb{R}^3 \longrightarrow \mathfrak{so}(3)$. We can use this map to identify $\mathfrak{so}(3)^*$ with $\mathbb{R}^3$ as follows (see for example [8]). Define $p : \mathfrak{so}(3)^* \longrightarrow (\mathbb{R}^3)^*$, $p = \mathcal{S}^*$, the adjoint of $\mathcal{S}$. If we identify $(\mathbb{R}^3)^*$ with $\mathbb{R}^3$ using the inner product, then $p : \mathfrak{so}(3)^* \longrightarrow \mathbb{R}^3$ is given by:

$$p(\rho) \cdot \eta := \langle \rho, \mathcal{S}(\eta) \rangle, \tag{1}$$

for any $\rho \in \mathfrak{so}(3)^*$. Here the $(\cdot)$ on the left hand side denotes the standard inner product on $\mathbb{R}^3$ and the angles-bracket on the right hand side denotes the dual action of $\mathfrak{so}(3)^*$ on $\mathfrak{so}(3)$. We often denote $\hat{\eta} = \mathcal{S}(\eta)$, for $\eta \in \mathbb{R}^3$, to avoid notational complexity.

We consider the dynamics of the rigid body on $T^*SO(3)$, the cotangent bundle of $SO(3)$. We make use of the *left trivialization* of $T^*SO(3)$, $\lambda : T^*SO(3) \longrightarrow SO(3) \times \mathfrak{so}(3)^*$ given by $\lambda(\alpha_R) = (R, T_e L_R^* \alpha_R)$ to describe the equations of motion. Thus, R represents the attitude matrix which transforms the vectors expressed in the body frame into the inertial frame and $\mathfrak{so}(3)^*$, identified with $\mathbb{R}^3$ using the p-map, represents the set of *body frame angular momenta* of the rigid body.

III. Stabilization by external actuation: The Hamiltonian interpretation and the error functions

The equations of motion of a rigid body with external torque u are given by,

$$\dot{R} = R\mathcal{S}\left(I^{-1}\Pi\right), \tag{2}$$
$$\dot{\Pi} = \Pi \times I^{-1}\Pi + u, \tag{3}$$

where I is the moment of inertia matrix. Our focus is on a feedback control law which can be derived from a scalar valued smooth function $V : SO(3) \longrightarrow \mathbb{R}$.

The paper by Koditschek [1] is among the first ones to analyze coordinate-free, globally defined control laws, which are derived from what he calls as *navigation functions*. A navigation function is a Morse function on $SO(3)$ and the stabilization properties of the resulting control law were studied using the Riemannian geometry structure that can be given to $SO(3)$. Bullo and Lewis also derive stabilizing control laws from globally defined *error functions* on manifolds using a Riemannian-geometric framework [2].

In this section we observe that on $SO(3)$, the stability of a system with a navigation or an error function can also be interpreted as the stability of a Hamiltonian system on $T^*SO(3)$ by resorting to the Lagrange-Dirichlet criterion [9]. While this approach does not offer any significant advantages over the Riemannian geometric framework already present in the literature, this point is worth noting in our opinion since Hamiltonian systems also form an important class of mechanical systems. This also sets the perspective for the ensuing sections.

A. Hamiltonian equations of motion

Let $V : SO(3) \longrightarrow \mathbb{R}$ be a smooth function, which we shall call the potential function. In (R, Π) co-ordinates defined in section II, the Hamiltonian for the rigid body under the potential V is,

$$H(R, \Pi) = \frac{1}{2}\Pi^T I^{-1} \Pi + V(R). \tag{4}$$

The manifold $SO(3) \times \mathfrak{so}(3)^*$ can be given a symplectic structure through the pull-back by λ of the canonical two form $\Omega_{T^*SO(3)}$ on $T^*SO(3)$. The explicit form of this was derived by Cushman [10] (also see [11], proposition 4.4.1). It turns out that the Hamiltonian vector field corresponding to the Hamiltonian (4) on $SO(3) \times \mathfrak{so}(3)^*$ (identified with $SO(3) \times \mathbb{R}^3$) with this symplectic structure is given by

$$X_H(R, \Pi) = \left(\mathcal{S}(I^{-1}\Pi),\ \Pi \times I^{-1}\Pi - p\left(T_e^* L_R\ dV\right)\right). \tag{5}$$

Remark 3.1: For a free rigid body, $V = 0$ and therefore, the equations of motion are

$$\begin{aligned} \dot{R} &= R\,\mathcal{S}\left(I^{-1}\Pi\right), \\ \dot{\Pi} &= \Pi \times I^{-1}\Pi. \end{aligned}$$

We stress the difference between the usual interpretation of the above equations and the present interpretation. The second equation above is generally seen as the Lie-Poisson reduced dynamics on $\mathfrak{so}(3)^*$ ([9], [8]) and the first equation then is the reconstruction equation. However, here we treat them both together has the Hamiltonian dynamics corresponding to the Hamiltonian $H(R, \Pi) = 1/2\Pi^T I^{-1}\Pi$.

B. Stability and relation with error functions

Now suppose $R_e \in SO(3)$ is a critical point of V, that is, $dV(R_e) = 0$. Thus, $(R_e, 0)$ is an equilibrium point for the vector field (5). To state the Lagrange-Dirichlet criterion for stability, we need the concept of the Hessian.

Definition 3.2: Suppose $V : M \longrightarrow \mathbb{R}$ is a smooth function and $\varphi : R^k \longrightarrow M$ is a local parameterization around $x \in M$. Then, the *Hessian* of V at x is defined as the second variation of $\tilde{V} : R^k \longrightarrow \mathbb{R}$, where $\tilde{V} := V \circ \varphi$.
The Lagrange-Dirichlet theorem states the condition for stability or instability of a Hamiltonian system. We can specialize it to the present case as follows.

Theorem 3.3 (Lagrange-Dirichlet): Consider the vector field (5). Suppose $R_e \in SO(3)$ is a critical point of V. Then, the equilibrium point $(R_e, 0)$ is stable (unstable) if the Hessian of V at R_e is positive (negative) definite.

Thus, to stabilize a rigid body with external torque [equations (2) – (3)] about a point $(R_e, 0)$, we can find a function $V : SO(3) \longrightarrow \mathbb{R}$, which has a positive definite Hessian at the critical point R_e and apply the feedback torque $u(R) := -p\left(T_e^* L_R\ dV(R)\right)$.

In fact, these are precisely the conditions defining the error functions [2]. A function $\Psi : SO(3) \longrightarrow \mathbb{R}$ is called an error function about $R_e \in SO(3)$ if:

- $\Psi(R_e) = 0$,
- $d\Psi(R_e) = 0$, and
- The Hessian of Ψ is positive definite around R_e.

It is then shown that the feedback derived from Ψ, which depends on $d\Psi$, stabilizes the dynamics about $(R_e, 0)$. Thus, the error function framework of stabilization with external torques can also be given a Hamiltonian interpretation.

C. Damping and almost global asymptotic stability

Suppose we introduce damping to the control by defining

$$u(R, \Pi) = -CI^{-1}\Pi - p\left(T_e^* L_R\ dV(R)\right), \tag{6}$$

where C is a positive definite matrix. The closed loop system becomes

$$\dot{R} = R\,\mathcal{S}\left(I^{-1}\Pi\right), \tag{7}$$

$$\dot{\Pi} = \Pi \times I^{-1}\Pi - CI^{-1}\Pi - p\left(T_e^* L_R\ dV(R)\right). \tag{8}$$

By exploiting the Riemannian structure, it is now possible to show that $(R_e, 0)$ is locally asymptotically stable [2]. For AGAS, the potential function should have the following additional properties:

Assumption 3.4: Suppose $V : SO(3) \longrightarrow \mathbb{R}$ satisfies the following properties:

- V has isolated critical points,
- the Hessian of V is positive definite only at a single critical point R_e, and non-singular at the other critical points.

Modified trace functions [12], which we shall discuss in section V, can be constructed so that they satisfy this assumption. To show AGAS, one can proceed as follows. It is possible to linearize the equations about the equilibrium points using the Riemannian structure and it turns out that the linearization is unstable about all the equilibrium points except $(R_e, 0)$. This allows to conclude almost global asymptotic stability. Thus, the following result follows, the proof of which has been derived in different contexts in [1], [3], [4]:

Theorem 3.5: Consider the system (7)–(8). Suppose V satisfies assumption 3.4. Then the equilibrium point $(R_e, 0)$ is AGAS.

IV. Stabilization by internal actuation

Now we turn to the stabilization of a rigid body with internal rotors. Our aim in this section is to derive a control law that achieves AGAS for this system.

We assume that the rotors are mounted along the principle axis of the body. Since the actuation is internal, the total inertial angular momentum of the body μ is a constant of motion. Let I_r be the moment of inertial of the rotors and let I be the locked inertia tensor of the rigid body with rotors. Let Ω be the body angular velocity of the rigid body. Define $I_s = I - I_r$ and $\Pi = I_s\Omega$. Then, it can be shown that the equations of motion are given by

$$\dot{R} = R\mathcal{S}\left(I_s^{-1}\Pi\right), \tag{9}$$

$$\dot{\Pi} = R^T\mu \times I_s^{-1}\Pi - u, \tag{10}$$

where u is the torque applied to the rotors. We omit the derivation of these equations due to the lack of space. We refer the reader to [13] for a derivation.

A. Lyapunov stability

We first show that if a function $V : SO(3) \longrightarrow \mathbb{R}^3$ is such that $u(R) = -p\left(T_e^* L_R\ dV(R)\right)$ it stabilizes (in the sense of Lyapunov) an equilibrium point for system (2) – (3), then $u(R) = p\left(T_e^* L_R\ dV(R)\right)$ stabilizes the same equilibrium point for system (9) – (10).

The resulting closed loop system, defines a vector field X over $SO(3) \times \mathbb{R}^3$. Explicitly,

$$X := \left(R\mathcal{S}\left(I_s^{-1}\Pi\right), R^T\mu \times I_s^{-1}\Pi - p\left(T_e^* L_R\ dV(R)\right)\right). \tag{11}$$

The equilibrium points of the above vector field are $(R_c, 0)$ where R_c is a critical point of V. Even though the feedback torque is derived from a potential, the system (11) is not a Hamiltonian system, since we have eliminated the rotor variables in arriving at these equations. Therefore the Lagrange-Dirichlet criterion cannot be used for analyzing stability. However, using the special form of the feedback torque, we can analyze the local and global stability as we show in theorem 4.1 and theorem 4.5 respectively.

Theorem 4.1: Suppose R_e is a critical point of V and that the Hessian of V is positive definite about R_e. Then the system (11) is Lyapunov stable about the equilibrium point $(R_e, 0)$.

Proof: Define the function

$$H(R, \Pi) = \frac{1}{2}\Pi^T I_s^{-1}\Pi + V(R). \tag{12}$$

Since the Hessian of V is positive definite about R_e and since I_s^{-1} is a positive definite matrix, it follows that H is locally positive definite about $(R_e, 0)$. It also follows that $\dot{H} = 0$ along the trajectories of X. Thus, H is a Lyapunov function for X and the Lyapunov stability follows. ∎

The natural next step is to introduce dissipation to the system so that one can expect to achieve asymptotic stability. As we saw in section III-C, asymptotic stability analysis for external actuation depended crucially on linearization. However, it is not possible to linearize the present system in a Riemannian setting as done in [2] and [4], since it possesses no such structure. We present the linearization relevant to the present context in the following. Using this linearization, we can resort to the strategy explained in section III-C to prove almost global asymptotic stability.

B. Linearization

Let X be a smooth vector field on M, which is a smooth embedded submanifold of $\mathbb{R}^k$ for some $k \in \mathbb{Z}^+$. and let $X(x_0) = 0$. To determine the stability at x_0, we can resort to linearization of the vector field expressed in a local chart around x_0. To be more precise, let $U \subset M$ be a neighbourhood around x_0 and let $\phi : U \longrightarrow \mathbb{R}^n$ be a coordinate chart such that $\phi(x_0) = 0$. Let $\varphi = \phi^{-1}$ on the image of ϕ, thus $\varphi : \phi(U) \longrightarrow U$ is a local parameterization of M around U. Note that both ϕ and φ are local diffeomorphisms. Now let Y be the local representative of X in the coordinate chart ϕ, which is basically the pull-back of X by the diffeomorphism φ. That is, in a sufficiently small neighbourhood of $0 \in \mathbb{R}^n$,

$$Y(\eta) := \left(D\varphi(\eta)\right)^{-1} X(\varphi(\eta))$$

is well defined. Thus, Y is a vector field on $\phi(U) \subset \mathbb{R}^n$. We can linearize Y and analyze the stability of the resulting linear system. We can see that if the equilibrium point 0 of Y is stable or unstable, then the same conclusion holds for the equilibrium point x_0 of X. The linearization of Y at η_0 in the direction of η can be computed as

$$DY(\eta_0)\ \eta = \left.\frac{d}{dt}\right|_{t=0} Y(\eta_0 + t\eta).$$

We call the linear vector field $DY(\eta_0)$ the linearization of X at x_0.

C. Damping and almost global asymptotic stability

Let us introduce damping to the system by changing the feedback control law to

$$u(R, \Pi) = CI_s^{-1}\Pi + p\left(T_e^* L_R\ dV(R)\right),$$

where C is a positive definite matrix. Thus, the closed loop system becomes

$$\dot{R} = R\mathcal{S}\left(I_s^{-1}\Pi\right), \tag{13}$$

$$\dot{\Pi} = R^T\mu \times I_s^{-1}\Pi - CI_s^{-1}\Pi - p\left(T_e^* L_R\ dV(R)\right). \tag{14}$$

Thus we get a modified vector field X_d on $SO(3) \times \mathbb{R}^3$:

$$X_d := \Big(R\mathcal{S}\left(I_s^{-1}\Pi\right), \\ R^T\mu \times I_s^{-1}\Pi - CI_s^{-1}\Pi - p\left(T_e^* L_R\ dV(R)\right)\Big).$$

Note that the equilibrium points remain the same as those of the vector field X.

Suppose $R_0 \in SO(3)$ is a critical point of V and we want to analyze the stability of X_d at the equilibrium point

$(R_0, 0)$ by linearization. We have a special choice for local parameterization of $SO(3)$ around R_0, using the exponential map. The exponential map $\exp : \mathfrak{so}(3) \longrightarrow SO(3)$ coincides with the usual matrix exponential $\mathrm{expm} : \mathbb{R}^{3\times3} \longrightarrow \mathbb{R}^{3\times3}$, restricted to the space of skew-symmetric matrices.

The derivative of the matrix exponential function $\mathrm{expm} : \mathbb{R}^{3\times3} \longrightarrow \mathbb{R}^{3\times3}$ at $0 \in \mathbb{R}^{3\times3}$ is the identity map between the corresponding tangent spaces [14]. Since the exponential map $\exp$ that we are interested is a restriction of expm, it follows that $D\exp(0) = \mathrm{id} : \mathfrak{so}(3) \longrightarrow \mathfrak{so}(3)$.

We define $\varphi_1 : \mathbb{R}^3 \longrightarrow SO(3)$ as

$$\varphi_1 := L_{R_0} \circ \exp \circ \mathcal{S}, \tag{15}$$

where $L_{R_0} : SO(3) \longrightarrow SO(3)$ is the diffeomorphism of the left action of R_0. This is a local parameterization of $SO(3)$ on $\mathbb{R}^3$ around R_0. Define $\varphi : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow SO(3) \times \mathbb{R}^3$,

$$\varphi = (\varphi_1, \mathrm{id}_{\mathbb{R}^3}). \tag{16}$$

It follows that this is a local parameterization of $SO(3) \times \mathbb{R}^3$ around $(R_0, 0)$.

Proposition 4.2: Let (η, Π) be the coordinates of $SO(3) \times \mathbb{R}^3$ around $(R_0, 0)$ in the chart defined by the parameterization φ. Then, the linearization of X_d at $(R_0, 0)$, in these coordinates is given by

$$\begin{pmatrix} \dot\eta \\ \dot\Pi \end{pmatrix} = \begin{pmatrix} 0 & I_s^{-1} \\ -K & \mathcal{S}\left(R_0^T \mu\right) - C I_s^{-1} \end{pmatrix} \begin{pmatrix} \eta \\ \Pi \end{pmatrix}, \tag{17}$$

where $K = \delta^2 \tilde{V}(0)$ and $\tilde{V} : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is given by $\tilde{V} = V \circ \varphi_1$.

Proof: The proof has been omitted due to space restrictions. ■

Let us denote

$$A_d := \begin{pmatrix} 0 & I_s^{-1} \\ -K & \mathcal{S}\left(R_0^T \mu\right) - C I_s^{-1} \end{pmatrix}. \tag{18}$$

The linearized system can be written in the form of a 'second-order equation', which becomes a linear gyroscopic system with damping:

$$I_s \ddot\eta + \left(C - \mathcal{S}\left(R_0^T \mu\right)\right) \dot\eta + K\eta = 0.$$

The stability of this type of a system has been analyzed extensively. The theorem by Kelvin-Tait-Chatayev [15] is the most relevant in our context:

Theorem 4.3 (Kelvin-Tait-Chetaev): The origin of the system $(\eta, \omega) = (0, 0)$ is

- asymptotically stable if both K and C are positive definite,
- unstable if K is not positive definite and C is positive definite.

Remark 4.4: When K is not positive definite and C is positive definite, even though the theorem asserts that the system (18) is unstable, the possibility of zero eigenvalues is not ruled out. However, if we further assume that K is nonsingular, it can be shown that A_d does not have purely imaginary eigenvalues. This is rather important since a purely imaginary eigenvalue does not allow us conclude about the original system by analyzing the linearized system.

With these facts, we are ready to state the following result.

Theorem 4.5: Consider the system (13)–(14). Suppose V satisfies assumption 3.4. Then the equilibrium point $(R_e, 0)$ is AGAS.

Proof: Because of the assumption on the nature of the Hessian of V, it follows from the theorem 4.3 that the linearization at $(R_e, 0)$ is asymptotically stable and the linearization at the other equilibria are unstable. Further, from remark 4.4 it follows that the same conclusion holds locally for the original dynamics at the corresponding equilibria. It also follows that the stable manifolds for the unstable equilibria are lower-dimensional.

To analyze global behaviour, consider the function H defined in (12). It can be shown that $\dot{H} = -(I_s^{-1}\Pi)^T C I_s^{-1}\Pi \leq 0$. The largest invariant set where $\dot{H}$ vanishes contains only the equilibrium points. Thus, by LaSalle's invariance principle ([2], theorem 6.19), the trajectories of the system tend to one of the equilibrium points. But since only is $(R_e, 0)$ asymptotically stable, and since the unstable equilibria have lower-dimensional stable manifolds, it follows that the region of attraction of $(R_e, 0)$ is a dense set in $SO(3) \times \mathbb{R}^3$. This proves the result. ■

Thus, AGAS with internal actuation depends crucially on the nature of the Hessian of V at the critical points, just the same way that the external actuation depended on it. Comparing with theorem 3.5, we are now ready to state the main result:

Theorem 4.6: Suppose $V : SO(3) \longrightarrow \mathbb{R}$ satisfies assumption 3.4. Then,

- The control law of the form
  $$u_{\text{ext}}(R, \Pi) = u(R) := -CI^{-1}\Pi - p\left(T_e^* L_R \; dV(R)\right)$$
  almost globally satisfies the equilibrium point $(R_e, 0)$ for the externally actuated system (2) – (3).
- The control law of the form
  $$u_{\text{int}}(R, \Pi) = C I_s^{-1}\Pi + p\left(T_e^* L_R \; dV(R)\right)$$
  almost globally satisfies the equilibrium point $(R_e, 0)$ for the internally actuated system (9) – (10).

Note that even though the term Π represents different physical quantities for external and internal actuation, the terms $I^{-1}\Pi$ for external actuation and $I_s^{-1}\Pi$ in case of internal actuation represent the same physical quantity, namely the body angular velocity of the rigid body.

V. The heavy top potential and the modified trace function

In the analysis so far, we have shown that the same potential can be used for constructing stabilizing control laws for both external and internal actuation cases. We now show how we can build a stabilizing potential motivated by the heavy top potential and how this leads to one of the potential functions frequently used in the literature: the modified trace function ([1],[2],[3]). The modified trace function was proposed by Chillingworth et al. [12] and was used in the context of rigid body stabilization by Koditschek [1]. On the other hand, the heavy top has been discussed thoroughly in the mechanics literature. We refer the reader to [9] and [8] for a geometric treatment. In what follows, we shall see how one can analyze the stability of heavy top equations using the theory built in section III, and how one can try to 'fix' the instability by altering the heavy top potential which finally leads to a modified trace function.

Modified trace function

Given any $P \in \mathrm{sym}(3\times 3)$, the set of 3×3 symmetric matrices, the modified trace function $\mathrm{trm}_P : SO(3) \longrightarrow \mathbb{R}$ is defined as:

$$\mathrm{trm}_P(R) = \mathrm{trace}(PR). \tag{19}$$

In [12], the authors prove the following result.

Lemma 5.1 (Chillingworth, Marsden and Wan): If P is a symmetric 3×3 matrix with distinct eigenvalues π_1, π_2, π_3 and if

$$(\pi_1+\pi_2)(\pi_2+\pi_3)(\pi_3+\pi_1) \neq 0,$$

then there are exactly four critical points of trm_P.
It is shown in [2] that these critical points are precisely

$$e, \exp(\pi\hat{e}_1), \exp(\pi\hat{e}_2), \exp(\pi\hat{e}_3),$$

where e_1, e_2, e_3 are the (distinct) unit eigenvectors of P. Note that e_1, e_2, e_3 can be chosen to be orthonormal since P is symmetric. It further holds ([1], [2]) that the Hessian of trm_P is positive definite at e and has negative eigenvalues but nonsingular at the other critical points.

Heavy top equations of motion and stability

The heavy top is a rigid body, rotating with respect to a fixed point on the body (the pivot) under the potential due to gravity. The coordinate-free version of the heavy top equations are

$$\dot{R} = R\mathcal{S}(I^{-1}\Pi), \tag{20}$$
$$\dot{\Pi} = \Pi \times I^{-1}\Pi - mg\chi \times R^T k, \tag{21}$$

where, I is the moment of inertia matrix of the heavy top, Π is the body angular momentum, k is along the direction of the gravity taken in the opposite sense, χ is the vector from the fixed point to the center of mass, expressed in the body frame. The energy H of the heavy top is the sum of kinetic and potential energy:

$$H(R,\Pi) = \frac{1}{2}\Pi^T I^{-1}\Pi + mgk\cdot R\chi. \tag{22}$$

To see that the heavy top equations of motion (20) – (21) are indeed Hamiltonian equations of motion for H, let us define

$$V_{\mathrm{HT}}(R) = mgk\cdot R\chi.$$

It can be shown that $p(T_e L_R^* dV_{\mathrm{HT}}(R)) = mgk \times R^T\chi$. Comparing with vector field (5), we see that the heavy top equations of motion are indeed Hamiltonian equations for H defined in (22).

Now suppose the body frame is so chosen that $\chi = k$, and hence the heavy top potential changes to $V_1(R) = mgk\cdot Rk$. Note that in this case, only the symmetric part of R contributes to V. However, in this case also, it can be shown that $p(T_e L_R^* dV(R)) = mgk\times R^T k$.

Let us compute the Hessian of V_1 at e in the parameterization $\varphi_1 : \mathbb{R}^3 \longrightarrow SO(3)$, $\varphi_1(\eta) = \exp(\hat{\eta})$. Thus, let $\tilde{V}_1 = V_1\circ\varphi_1$ and $K = \delta^2\tilde{V}_1(0)$. Then, it can be shown that

$$K\eta = \left.\frac{d}{dt}\right|_{t=0} p\left(T_e^* L_{\exp(t\hat{\eta})}\, dV_1|_{\exp(t\hat{\eta})}\right) = mg\hat{k}\hat{k}\eta.$$

Now, $\eta^T K\eta = mg\eta^T\hat{k}\hat{k}\eta = mg\|k\times\eta\|^2$. It follows that $\tilde{V}_1(0)$ is positive semidefinite at identity, hence the stability of the heavy top equations cannot be established using the Lagrange-Dirichlet criterion.

Modification of the heavy top potential

The above analysis indicates how we can think of modifying the potential function V_1 so that it gives rise to a dynamics which is stable at the identity. Indeed, if $k_1, k_2 \in \mathbb{R}^3$ are linearly independent and if we define V_2 as

$$V_2(R) = k_1\cdot Rk_1 + k_2\cdot Rk_2,$$

it follows from a similar analysis that

$$\eta^T\delta^2\tilde{V}_2(0)\eta = \|k_1\times\eta\|^2 + \|k_2\times\eta\|^2.$$

Thus the Hessian of V_2 is positive definite at identity, which guarantees stability at identity. However, this function is nothing but a modified trace function, as the following result shows.

Proposition 5.2: If $k_1, k_2 \in \mathbb{R}^3$ are linearly independent and not orthonormal, then there exists a $P\in\mathrm{sym}(3\times 3)$ such that

$$V_2(R) = k_1\cdot Rk_1 + k_2\cdot Rk_2 = \mathrm{trace}(PR),$$

where P is diagonalizable to $\mathrm{diag}(c_1^2/2, c_2^2/2, 0)$, where

$$c_1 = \|k_1+k_2\|,\ c_2 = \|k_1-k_2\|.$$

Proof: The proof has been omitted due to space restrictions. ■

Thus, V_2 is a modified trace function. The eigenvalues of the matrix SDS^T are $c_1^2/2, c_2^2/2, 0$ and satisfy the condition specified by the lemma 5.1. Thus, V_2 can be used as a stabilizing potential about the identity. Also, since at the other three critical points the Hessian is nonsingular with some negative eigenvalues, V_2 also qualifies for an almost-global stabilizing potential.

Remark 5.3: We can modify V_2 and define for any $R_d\in SO(3)$,

$$V_{R_d}(R) = k_1\cdot R_d^T Rk_1 + k_2\cdot R_d^T Rk_2. \tag{23}$$

The resulting control will stabilize the rigid body around R_d.

Remark 5.4: If k_1 and k_2 are orthonormal, then it follows that c_1, c_2 as defined above are equal. In such cases, the critical points of V_2 are all the rotations around the axis $k_1\times k_2$. See [2], proposition 11.31 for details.

Remark 5.5: In [6], for stabilizing a rigid body by external actuation, the authors consider a control law of the form

$$u_1(R) = \sum_{i=1}^{3} a_i e_i \times R_d^T Re_i,$$

where $a_i, i = 1, \dots, 3$ are distinct positive real numbers. The modified trace function $\mathrm{trm}_P(R) = \mathrm{trace}(PR_d^T R)$, where $P = \mathrm{diag}(a_1, a_2, a_3)$, will yield the following control law

$$u_2(R) = -\sum_{i=1}^{3} a_i e_i \times R^T R_d e_i.$$

It can be shown that both of these control laws have the same stabilizing properties.

VI. Simulation results

In this section we present simulation results for the stabilization with internal rotors, for the following control law:

$$u(R, \Pi) = C\Pi + c_1 e_1 \times R^T R_d e_1 + c_2 e_2 \times R^T R_d e_2.$$

The proportional part of this control, as we discussed in section V, comes from the modified trace function T_P, where $P = \mathrm{diag}(c_1, c_2, 0)$. We choose the model parameters $I = \mathrm{diag}(40, 45, 42.5)\ kg\ m^2$, $I_r = \mathrm{diag}(0.01, 0.01, 0.01)\ kg\ m^2$ and the control parameters $C = 10 I_{3\times3}$, $c_1 = 1$, $c_2 = 1.2$.

The simulation is carried out for the case when $R_0 = R_x(\pi/6) R_y(\pi/8) R_z(5\pi/12)$, $R_d = e$ and $\mu = (1, 1.5, -2)$. Figure 1 plots the error norm of R against R_d and the rigid body body angular velocity, which shows asymptotic convergence to the desired value. Figure 2 shows the torque applied at the rotors. Figure 3 shows the angular velocity of the three rotors, indicating that the inertial angular momentum is absorbed at the rotors.

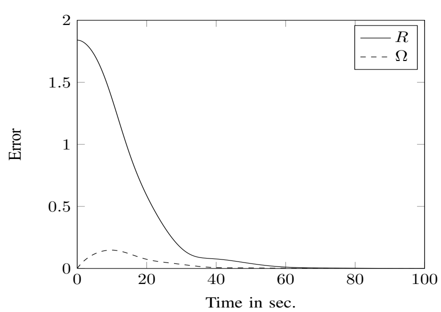

Fig. 1: Error norm of R and the rigid body angular velocity Ω

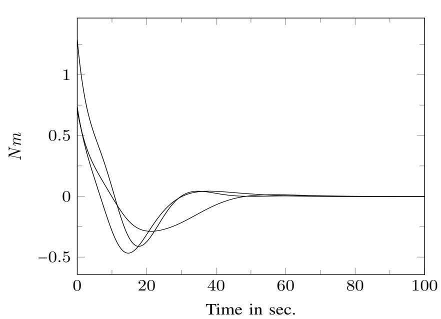

Fig. 2: Torque in Nm applied at the three rotors

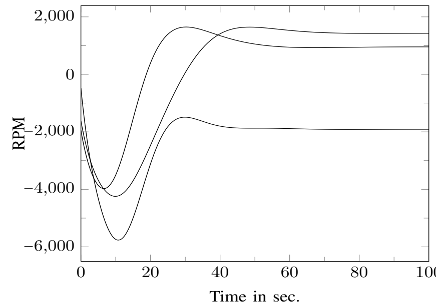

Fig. 3: Angular velocity in RPM of the three rotors

VII. Conclusion

We showed that the frequently used error function formalism for rigid body stabilization with external torques can also be given a Hamiltonian interpretation. We then showed that if an externally applied control torque derived from a potential can stabilize a desired equilibrium, then the negative of the same torque applied internally through rotors can stabilize the same equilibrium. We also showed that the classical heavy top potential can be used as a motivation for deriving a stabilizing potential, which leads to the modified trace function.

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