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Parametric state feedback design for linear infinite-dimensional systems

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***Abstract*— In this paper, a parametric approach for state feedback controllers for a class of linear infinite-dimensional systems is presented. Therein, besides the closed-loop eigenvalues, the corresponding parameter vectors are introduced as design parameters. This allows not only to assign finitely many eigenvalues, but also the corresponding eigenvectors. The presented approach is applied to solve a partial eigenstructure assignment problem for linear time-delay systems with point-delays. An example for the usefulness of the proposed design method is then given by a partial input-output decoupling of a simple time-delay system.**

I. INTRODUCTION

A closed-form solution of the eigenvalue assignment problem for linear multi-input lumped-parameter systems can be obtained by applying the *parametric approach* (see e.g. [1]). This design method introduces the closed-loop eigenvalues and the corresponding parameter vectors as design parameters. This has the advantage that by the assignment of closed-loop eigenvalues the closed-loop stability is assured and the parameter vectors can be chosen to additionally assign the closed-loop eigenvectors. Thus, the whole eigenstructure of the closed-loop system can be assigned to accommodate various design specifications. In [2] the parametric approach was extended to linear distributed-parameter systems on the Hilbert space H where the system operator is a *Riesz-spectral operator* (see [3]) with at most finitely many eigenvalues in the closed right-half plane and no accumulation point on the imaginary axis. Then a *state feedback* for the exponential stabilization of the given distributed-parameter system can be obtained that shifts only finitely many eigenvalues. As in the case of lumped-parameter systems there are still remaining degrees of freedom for the multi-input case after an eigenvalue assignment. Similar to the case of lumped-parameter systems a closed-loop expression for the feedback operator is obtained in [2] to assign the eigenstructure with respect to the dominant modes. This partial eigenstructure assignment can be used to achieve further design purposes as for example partial input-output decoupling (see [4]) or disturbance rejection (see [5]).

However, there are distributed-parameter systems, whose system operator is not a Riesz-spectral operator or where its Riesz-spectral property is hard to prove. Furthermore, for time-delay systems this property is rather restrictive.

In this contribution, the parametric approach is extended to a larger class of linear infinite-dimensional systems where no Riesz-basis property of the eigenvectors is needed and the system operator has to be only the infinitesimal generator of a C_0-semigroup. In order to apply the eigenvalue assignment approach for the stabilization, the *spectrum determined growth assumption* is assumed. At first the parametric approach is extended to this class of linear infinite-dimensional systems. Then, these results are used to obtain a parametric solution of the eigenvalue assignment problem for multi-input linear *time-delay systems* with *point-delays*. An advantage of the new approach is the assignment of the closed-loop eigenvectors in addition to the corresponding eigenvalues. This is not possible, when using *finite spectrum assignment* (see [6]), *continuous pole placement* (see [7]) or the *modal approach* given in [8]. Different from these methods also the closed-loop eigenvectors in addition to the closed-loop eigenvalues can be assigned with the new approach. Consequently, only stabilization problem can be solved with these methods. In contrast the new approach also allows to specify the partial closed-loop eigenstructure of the linear time-delay systems. However, the resulting state feedback cannot be directly applied since it contains distributed delays. Therefore, an approach is proposed for approximately implementing the corresponding state feedback for time-delay systems with *commensurate point-delays*. The basic idea is to derive a representation of the feedback in form of a non-minimal continuous-time filter (see also [6]). By time-discretization of this system a pole-zero cancellation can be performed to obtain a FIR-filter which is easy to implement. The results of the contribution are illustrated by input-output decoupling a simple time-delay system.

II. PROBLEM FORMULATION

Considered are linear infinite-dimensional systems described by the abstract initial value problem

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t), \quad t > 0, \quad x(0) = x_0 \in H \qquad (1)$$

with the state $x(t) \in H$, $\forall t \geq 0$, and the inputs $u(t) \in \mathbb{R}^p$. The state space is an infinite-dimensional and complex Hilbert space H with inner product $\langle \cdot, \cdot \rangle$. The compact input operator $\mathcal{B} : \mathbb{C}^p \to H$ is given by

$$\mathcal{B}u(t) = \sum_{i=1}^{p} b_i u_i(t), \quad b_i \in H \qquad (2)$$

with p linearly independent inputs, which means that b_i, $i = 1, \ldots, p$, are linearly independent over $\mathbb{C}$. The operator $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ is assumed to be the infinitesimal generator of a C_0-semigroup. This class of infinite-dimensional systems contains many distributed-parameter systems as well as time-delay systems. Furthermore, it is assumed that the operator $\mathcal{A}$ fulfills the *spectrum determined growth assumption*,

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which means that the stability of the system (1) is determined by the spectrum of $\mathcal{A}$. In particular there exists a positive constant $M_{\mathcal{A}}$ such that the C_0-semigroup $\mathcal{T}_{\mathcal{A}}(t)$ generated by $\mathcal{A}$ fulfills

$$\|\mathcal{T}_{\mathcal{A}}(t)\| \leq M_{\mathcal{A}} e^{\omega_0 t}, \quad t \geq 0 \tag{3}$$

with the *growth bound*

$$\omega_0 = \sup_{\lambda \in \sigma(\mathcal{A})} \operatorname{Re} \lambda, \tag{4}$$

where $\sigma(\mathcal{A})$ is the *spectrum* of $\mathcal{A}$. Then the system is exponentially stable, if $\omega_0 < 0$. In order to *β-stabilize* the system, which means to shift the growth bound ω_0 to $\beta < \omega_0$, some further assumptions are needed. Therefore, the spectrum is decomposed into two distinct parts

$$\sigma_\beta^+(\mathcal{A}) = \sigma(\mathcal{A}) \cap \overline{\mathbb{C}_\beta^+}, \quad \mathbb{C}_\beta^+ = \{\lambda \in \mathbb{C} \,|\, \operatorname{Re} \lambda > \beta\} \tag{5}$$

$$\sigma_\beta^-(\mathcal{A}) = \sigma(\mathcal{A}) \cap \mathbb{C}_\beta^-, \quad \mathbb{C}_\beta^- = \{\lambda \in \mathbb{C} \,|\, \operatorname{Re} \lambda < \beta\}. \tag{6}$$

First of all, it is assumed that $\sigma_\beta^+(\mathcal{A})$ consists of finitely many isolated eigenvalues of finite multiplicity. Obviously $\sigma_\beta^+(\mathcal{A})$ consists of a pure point spectrum without residual spectra or continuous spectra. The number of eigenvalues in $\sigma_\beta^+(\mathcal{A})$ is denoted by n. In the following the eigenvalues λ_i, $i = 1, \ldots, n$, of $\sigma_\beta^+(\mathcal{A})$ are ordered with respect to their decreasing real parts, i.e. $\operatorname{Re} \lambda_1 \geq \operatorname{Re} \lambda_2 \geq \ldots$. With $\sigma_\beta^+(\mathcal{A})$ as assumed above the *spectrum decomposition assumption* at β holds for $\mathcal{A}$ (for details see [3]). Thus, the system (1) can be decomposed by the *modal projection*

$$\mathcal{P}_n = \sum_{i=1}^{n} \phi_i \langle \cdot, \psi_i \rangle \tag{7}$$

and its complementary projection $\mathcal{P}_R = I - \mathcal{P}_n$, where ϕ_i, $i = 1, \ldots, n$, are the linear independent eigenvectors of $\mathcal{A}$ to eigenvalues λ_i, $i = 1, \ldots, n$, and ψ_i, $i = 1, \ldots, n$, denote the linear independent eigenvectors of the adjoint operator $\mathcal{A}^*$ to $\overline{\lambda}_i$, $i = 1, \ldots, n$. The eigenvectors ϕ_i and ψ_i are scaled such that they are *biorthonormal*, i.e.

$$\langle \phi_i, \psi_j \rangle = \delta_{ij} = \begin{cases} 1 \,:\, i = j \\ 0 \,:\, i \neq j \end{cases}. \tag{8}$$

The projection (7) induces a decomposition $H = H_n \oplus H_R$ of the state space, where $H_n = \operatorname{span}\{\phi_1, \ldots, \phi_n\}$ and $H_R = \operatorname{span}\{\psi_1, \ldots, \psi_n\}^\perp$. The decomposition $x(t) = x_n(t) + x_R(t)$, with $x_n(t) = \mathcal{P}_n x(t)$, $x_R(t) = \mathcal{P}_R x(t)$, then provides the decoupled systems

$$\dot{x}_n(t) = \mathcal{A}_n x_n(t) + \mathcal{B}_n u(t), \quad x_{0n} = \mathcal{P}_n x_0, \tag{9}$$

$$\dot{x}_R(t) = \mathcal{A}_R x_R(t) + \mathcal{B}_R u(t), \quad x_{0R} = \mathcal{P}_R x_0, \tag{10}$$

where $\mathcal{A}_n = \mathcal{A}|_{H_n}$, $\mathcal{A}_R = \mathcal{A}|_{H_R}$, $\mathcal{B}_n = \mathcal{P}_n \mathcal{B}$, $\mathcal{B}_R = \mathcal{P}_R \mathcal{B}$. Thereby, the fact that H_n and H_R are *$\mathcal{A}$-invariant* was used. As a consequence the state space H_n is finite-dimensional and the eigenvectors ϕ_i, $i = 1, \ldots, n$, form a basis in H_n. Thus, the spectrum determined growth assumption is fulfilled for $\mathcal{A}_n$ and its spectrum is given by $\sigma(\mathcal{A}_n) = \sigma_\beta^+(\mathcal{A})$. The system operator $\mathcal{A}_R$ has the spectrum $\sigma(\mathcal{A}_R) = \sigma_\beta^-(\mathcal{A})$ and is the infinitesimal generator of the C_0-semigroup $\mathcal{T}_{\mathcal{A}_R}(t)$ (see Lemma 2.5.7 in [3]). In general, the spectrum determined growth assumption is not satisfied for operator $\mathcal{A}_R$. Therefore, in the sequel it is assumed that the growth bound ω_{0R} of $\mathcal{T}_{\mathcal{A}_R}(t)$ fulfills $\omega_{0R} < \beta$, which for example is given for Riesz-spectral operators with pure point spectrum and finitely many eigenvalues in $\sigma_\beta^+(\mathcal{A})$ and for linear time-delay sytems with point-delays. Then it is sufficient to β-stabilize the subsystem (9) and to leave the dynamical behaviour of the residual subsystem (10) unchanged (for details see [3]). If the subsystem (9) is controllable, this can be achieved with the state feedback

$$u(t) = -\mathcal{K} x(t), \tag{11}$$

where the compact feedback operator $\mathcal{K} : H \to \mathbb{C}^p$ can be represented by

$$\mathcal{K} = \begin{bmatrix} \langle \cdot, k_1 \rangle \\ \vdots \\ \langle \cdot, k_p \rangle \end{bmatrix}, \quad i = 1, \ldots p, \tag{12}$$

with

$$k_i = \sum_{j=1}^{n} k_{ij}^* \psi_j, \quad k_{ij}^* \in \mathbb{C}. \tag{13}$$

Since then with (7) and (12)–(13)

$$u(t) = -\mathcal{K} x(t) = \mathcal{K} \mathcal{P}_n x(t) = -\mathcal{K} x_n(t) \tag{14}$$

is fulfilled for (11), the controlled system in decoupled representation (9)–(10) takes form

$$\dot{x}(t) = \tilde{\mathcal{A}}_d x(t) \tag{15}$$

with

$$\tilde{\mathcal{A}}_d = \begin{bmatrix} \mathcal{A}_n - \mathcal{B}_n \mathcal{K} & 0 \\ -\mathcal{B}_R \mathcal{K} & \mathcal{A}_R \end{bmatrix} = \begin{bmatrix} \tilde{\mathcal{A}}_n & 0 \\ -\mathcal{B}_R \mathcal{K} & \mathcal{A}_R \end{bmatrix}, \tag{16}$$

$$x(t) = \begin{bmatrix} x_n(t) \\ x_R(t) \end{bmatrix} \in H = H_n \oplus H_R, \tag{17}$$

where it can be seen that the dynamical behaviour of the subsystem (9) is modified, whereas the dynamical behaviour of the residual system (10) remains unchanged as well as its spectrum. For notational convenience the state of (1) and (15) are denoted with $x(t)$. Since $\mathcal{B}$ and $\mathcal{K}$ are compact operators $\tilde{\mathcal{A}}_d$ is the generator of a C_0-semigroup (see Theorem 3.2.1 in [3]) In the following the problem of determing the coefficents k_{ij}^* in (13) such that

1) n desired eigenvalues $\tilde{\lambda}_i$ are assigned to closed-loop system (15) and
2) that the closed-loop system (15) has n desired closed-loop eigenvectors $\tilde{\phi}_i$.

This means that a partial eigenstructure assignment for the infinite-dimensional system (1) has to be achieved. After allocating the desired closed-loop eigenvalues there are still $n(p-1)$ remaining degrees of freedom. These degrees can be used to additionally assign the closed-loop eigenvectors $\tilde{\phi}_i$. In the next section a parametric expression for the coefficients k_{ij}^* is introduced that parameterizes the remaining degrees of freedom after eigenvalue assignment. Thereby the n eigenvalues can independently be assigned.

III. PARAMETRIC EXPRESSION OF THE STATE FEEDBACK

By applying the state feedback (11) to system (1) the closed-loop system takes the form

$$\dot{x}(t) = \tilde{\mathcal{A}}x(t), \quad t > 0, \quad x(0) = x_0 \in H \tag{18}$$

where

$$\tilde{\mathcal{A}} = \mathcal{A} - \mathcal{B}\mathcal{K}, \quad D(\tilde{\mathcal{A}}) = D(\mathcal{A}). \tag{19}$$

The eigenvectors $\tilde{\phi}_i$ with respect to the eigenvalues $\tilde{\lambda}_i$, $i = 1, \ldots, n$, satisfy

$$\tilde{\mathcal{A}}\tilde{\phi}_i = \tilde{\lambda}_i\tilde{\phi}_i, \quad i = 1, \ldots, n. \tag{20}$$

With (19) the eigenvalue problem (20) can be written as

$$\mathcal{A}\tilde{\phi}_i - \mathcal{B}\mathcal{K}\tilde{\phi}_i = \tilde{\lambda}_i\tilde{\phi}_i, \quad i = 1, \ldots, n. \tag{21}$$

By introducing the *parameter vectors*

$$p_i = \mathcal{K}\tilde{\phi}_i, \quad i = 1, \ldots, n \tag{22}$$

equation (21) becomes

$$(\tilde{\lambda}_i I - \mathcal{A})\tilde{\phi}_i = -\mathcal{B}p_i, \quad i = 1, \ldots, n. \tag{23}$$

It is assumed that the k_{ij}^* are assigned such that the eigenvalues $\tilde{\lambda}_i$, $i = 1, \ldots, n$, of $\tilde{\mathcal{A}}$ are distinct and are an element of the *resolvent set* $\rho(\mathcal{A})$ of $\mathcal{A}$, i.e. $\tilde{\lambda}_i \in \rho(\mathcal{A})$, $i = 1, \ldots, n$. Thus (23) can be solved for $\tilde{\phi}_i$ (see e.g. [3]). In this way the eigenvectors $\tilde{\phi}_i$ are parameterized in terms of the eigenvalues $\tilde{\lambda}_i$ and the parameter vectors p_i by

$$\tilde{\phi}_i = (\mathcal{A} - \tilde{\lambda}_i I)^{-1}\mathcal{B}p_i, \quad i = 1, \ldots, n. \tag{24}$$

In the sequel a parameterization for the feedback operator $\mathcal{K}$ respectively the coefficients k_{ij} is derived. To this end, the decomposition $\tilde{\phi}_i = \tilde{\phi}_{i,n} + \tilde{\phi}_{i,R}$, with $\tilde{\phi}_{i,n} = \mathcal{P}_n\tilde{\phi}_i$ and $\tilde{\phi}_{i,R} = \mathcal{P}_R\tilde{\phi}_i$ is considered leading to the equation

$$\tilde{\mathcal{A}}_d \begin{bmatrix} \tilde{\phi}_{i,n} \\ \tilde{\phi}_{i,R} \end{bmatrix} = \tilde{\lambda}_i \begin{bmatrix} \tilde{\phi}_{i,n} \\ \tilde{\phi}_{i,R} \end{bmatrix} \tag{25}$$

with $\tilde{\mathcal{A}}_d$ in (16). As the eigenvalues $\tilde{\lambda}_i$ satisfy $\tilde{\lambda}_i \notin \sigma(\mathcal{A})$ and consequently $\tilde{\lambda}_i \notin \sigma(\mathcal{A}_R)$, $i = 1, \ldots, n$, the projection $\tilde{\phi}_{i,n}$ on H_n fulfills

$$\tilde{\mathcal{A}}_n\tilde{\phi}_{i,n} = \tilde{\lambda}_i\tilde{\phi}_{i,n}, \quad i = 1, \ldots, n \tag{26}$$

with $\tilde{\phi}_{i,n} \neq 0$ and thus $\tilde{\phi}_{i,n}$ is the eigenvector of $\tilde{\mathcal{A}}_n$ to the corresponding eigenvalue $\tilde{\lambda}_i$. The projection $\tilde{\phi}_{i,R}$ on H_R is then given by

$$\tilde{\phi}_{i,R} = (\tilde{\lambda}_i I - \mathcal{A}_R)^{-1}\mathcal{B}_R\mathcal{K}\tilde{\phi}_{i,n}, \quad i = 1, \ldots, n. \tag{27}$$

The projection of (21) on H_n with $\mathcal{P}_n$ results in

$$(\tilde{\lambda}_i I - \mathcal{A}_n)\tilde{\phi}_{i,n} = -\mathcal{B}_n\mathcal{K}\tilde{\phi}_{i,n}, \quad i = 1, \ldots, n, \tag{28}$$

where $\mathcal{P}_n\mathcal{A}\mathcal{P}_R = 0$ and (14) was used. Thus, with (14) there is also a parameterization for the eigenvectors

$$\tilde{\phi}_{i,n} = (\mathcal{A}_n - \tilde{\lambda}_i I)^{-1}\mathcal{B}_n p_i, \quad i = 1, \ldots, n, \tag{29}$$

in terms of the eigenvalues $\tilde{\lambda}_i$ and parameter vectors p_i. Since $\tilde{\phi}_{i,n} \in H_n$, these eigenvectors can be represented by

$$\tilde{\phi}_{i,n} = \Phi\tilde{v}_i, \quad i = 1, \ldots, n, \tag{30}$$

with

$$\tilde{v}_i = \begin{bmatrix} \langle\tilde{\phi}_i, \psi_1\rangle & \cdots & \langle\tilde{\phi}_i, \psi_n\rangle \end{bmatrix}^T, \tag{31}$$

$$\Phi = \begin{bmatrix} \phi_1 & \cdots & \phi_n \end{bmatrix} \tag{32}$$

so that (22) with (12)–(13) becomes

$$p_i = \mathcal{K}\tilde{\phi}_n = \mathcal{K}\tilde{\phi}_{i,n} = \begin{bmatrix} \langle\tilde{\phi}_i, k_1\rangle \\ \vdots \\ \langle\tilde{\phi}_i, k_p\rangle \end{bmatrix} = \sum_{j=1}^{n} \begin{bmatrix} \overline{k_{1j}^*}\langle\tilde{\phi}_i, \psi_j\rangle \\ \vdots \\ \overline{k_{pj}^*}\langle\tilde{\phi}_i, \psi_j\rangle \end{bmatrix} = \overline{K^*}\tilde{v}_i \tag{33}$$

with

$$\overline{K^*} = \begin{bmatrix} \overline{k_{11}^*} & \cdots & \overline{k_{1n}^*} \\ \vdots & & \vdots \\ \overline{k_{p1}^*} & \cdots & \overline{k_{pn}^*} \end{bmatrix}. \tag{34}$$

Writing (33) in matrix form, one gets the parameterization

$$\begin{bmatrix} p_1 & \cdots & p_n \end{bmatrix} = \overline{K^*} \begin{bmatrix} \tilde{v}_1 & \cdots & \tilde{v}_n \end{bmatrix} \tag{35}$$

for the feedback coefficients k_{ij}^*. The eigenvectors $\tilde{\phi}_i$, $i = 1, \ldots, n$, as well as the eigenvectors $\tilde{\phi}_{i,n}$, $i = 1, \ldots, n$, are linearly independent due to the fact that the closed-loop eigenvalues are chosen mutually different (see e.g. [9]). Consequently, in view of (30) the vectors $\tilde{v}_i$, $i = 1, \ldots, n$ are also linearly independent. As a consequence (35) can be solved for $\overline{K^*}$ giving

$$\overline{K^*} = \begin{bmatrix} p_1 & \cdots & p_n \end{bmatrix} \begin{bmatrix} \tilde{v}_1 & \cdots & \tilde{v}_n \end{bmatrix}^{-1}, \tag{36}$$

in which a parameterization of the vectors $\tilde{v}_i$ is needed. To obtain this parameterization one uses the spectral representation of the resolvent operator (see e.g. [3])

$$(\lambda I - \mathcal{A}_n)^{-1} = \sum_{i=1}^{n} \frac{\langle\cdot, \psi_i\rangle}{\lambda - \lambda_i}\phi_i, \quad \lambda \in \rho(\mathcal{A}_n) \tag{37}$$

and the representation of the input operator

$$\mathcal{B}_n = \Phi \begin{bmatrix} \langle\mathcal{B}, \psi_1\rangle \\ \vdots \\ \langle\mathcal{B}, \psi_n\rangle \end{bmatrix} = \Phi B^*. \tag{38}$$

Then (30) can be written with (29) as

$$\Phi\tilde{v}_i = \sum_{j=1}^{n} \frac{\langle\Phi, \psi_j\rangle}{\tilde{\lambda}_i - \lambda_j}\phi_j B^* p_i = \sum_{j=1}^{n} \frac{\phi_j e_j^T}{\tilde{\lambda}_i - \lambda_j} B^* p_i = \Phi(\Lambda - \tilde{\lambda}_i I)^{-1}B^* p_i \tag{39}$$

with

$$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n). \tag{40}$$

Thus, the parameterization for $\tilde{v}_i$ is given by

$$\tilde{v}_i = (\Lambda - \tilde{\lambda}_i I)^{-1} B^* p_i \tag{41}$$

since the eigenvectors ϕ_i, $i = 1, \ldots, n$, are linearly independent. As (41) is well known from the parametric approach for linear time-invariant lumped-parameter systems (Λ, B^*), the controllability of (Λ, B^*) assures that for arbitrary but distinct numbers $\tilde{\lambda}_i$ there exist vectors p_i such that the vectors $\tilde{v}_i$ in (41) are linearly independent (see e.g. [1]) and thus $\overline{K^*}$ can be obtained with (36). The controllability of (Λ, B^*) coincides with the assumption of the controllability of system (9), since the decoupled system (9) could also be described by a lumped-parameter system with pair (Λ, B^*) in $\mathbb{C}^n$ instead of H_n (see e.g. [10]). The parameter vectors p_i obviously contain the $n(p-1)$ remaining degrees of freedom after assigning n eigenvalues. This can be verified by (36), in which the matrix of parameter vectors with pn elements appears. As a multiplication of p_i with a non-zero constant would not change the result for (36), there are $n(p-1)$ remaining degrees of freedom in the matrix of parameter vectors. If the eigenvalues $\tilde{\lambda}_i$ are chosen mututally different from the spectrum $\sigma(\mathcal{A})$, and hence mutually different from the eigenvalues of Λ, the eigenvectors $\tilde{v}_i$ in (41) parameterized by $\tilde{\lambda}_i$ and p_i are unique. Consequently, if $\tilde{\lambda}_i$ and p_i is chosen such that the eigenvectors $\tilde{v}_i$ are linearly independent, $\overline{K^*}$ in (36) and thus $\mathcal{K}$ is completely parameterized by the set of closed-loop eigenvalues and the corresponding parameter vectors. For the closed-loop finite-dimensional system with operator $\mathcal{A}_n - \mathcal{B}_n K$ the spectrum determined growth assumption is still fulfilled and thus the closed-loop system (15) is β-exponentially stable if the eigenvalues satisfy $\max_{i=1,\ldots,n} \operatorname{Re} \tilde{\lambda}_i < \beta$ (see (16)). It remains to show that the parametric design of the state feedback controllers for considered linear systems can be obtained by (36).

Theorem 1: **(Parametric expression for the state feedback)** Consider the system (1) with the assumptions made in Section II. Given a set of different numbers $\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_n$ with $\tilde{\lambda}_i \in \rho(\mathcal{A})$, $i = 1, \ldots, n$, and corresponding $p \times 1$ vectors $p_1, p_2, \ldots, p_n$. Further, assume that the $\tilde{\lambda}_i$ and the p_i have been chosen such that the vectors (41) are linearly independent. Then, the feedback (11)–(13) and $\overline{K^*}$ in (34) given by (36) assigns the eigenvalues $\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_n$ and the corresponding parameter vectors $p_1, p_2, \ldots, p_n$ to the closed-loop system (15). Then

$$\tilde{\phi}_i = (\mathcal{A} - \tilde{\lambda}_i I)^{-1} \mathcal{B} p_i, \quad i = 1, 2, \ldots, n, \tag{42}$$

are eigenvectors of the closed-loop system. Furthermore, there exists a positive constant $M_{\tilde{\mathcal{A}}_d}$ such that the C_0-semigroup $\mathcal{T}_{\tilde{\mathcal{A}}_d}(t)$ generated by $\tilde{\mathcal{A}}_d$ in (15) satisfies $\|\mathcal{T}_{\tilde{\mathcal{A}}_d}(t)\| \leq M_{\tilde{\mathcal{A}}_d} e^{\omega_0 t}$, $t \geq 0$, with the growth bound

$$\omega_0 = \max(\beta, \max_{i=1,\ldots,n} \operatorname{Re} \tilde{\lambda}_i). \tag{43}$$

Conversely, each matrix $\overline{K^*}$ can be expressed by (36) if it assigns distinct closed-loop eigenvalues $\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_n$ which are mutually different from the eigenvalues λ_i of $\mathcal{A}$ and corresponding parameter vectors $p_1, p_2, \ldots, p_n$.

Proof: First it is shown that (42) are eigenvectors of the closed-loop system with respect to $\tilde{\lambda}_i$. Therefore consider

$$(\tilde{\lambda}_i I - \tilde{\mathcal{A}})\tilde{\phi}_i = (\tilde{\lambda}_i I - \mathcal{A})\tilde{\phi}_i + \mathcal{B}\mathcal{K}\tilde{\phi}_i, \tag{44}$$

where (19) was used. Since $\mathcal{P}_n \tilde{\phi}_i = \tilde{\phi}_{i,n} = \Phi \tilde{v}_i$ as shown in (28)–(30) and $\mathcal{K}\Phi\tilde{v}_i = \overline{K^*}\tilde{v}_i$ as shown in (33) equation (44) can be written with (22) as

$$(\tilde{\lambda}_i I - \tilde{\mathcal{A}})\tilde{\phi}_i = (\tilde{\lambda}_i I - \mathcal{A})\tilde{\phi}_i + \mathcal{B}\overline{K^*}\tilde{v}_i. \tag{45}$$

With (36) one gets

$$\overline{K^*}\tilde{v}_i = \begin{bmatrix} p_1 & \ldots & p_n \end{bmatrix} \begin{bmatrix} \tilde{v}_1 & \ldots & \tilde{v}_n \end{bmatrix}^{-1} \tilde{v}_i = p_i. \tag{46}$$

Thus (45) simplifies with (42) to

$$(\tilde{\lambda}_i I - \tilde{\mathcal{A}})\tilde{\phi}_i = -\mathcal{B}p_i + \mathcal{B}p_i = 0, \tag{47}$$

which shows that (42) are closed-loop eigenvectors for the closed-loop eigenvalues $\tilde{\lambda}_i$ in view of (26). In a similar way it could be shown that the eigenvectors (29) are the eigenvectors corresponding to the eigenvectors $\tilde{\lambda}_i$ of the finite-dimensional operator $\tilde{\mathcal{A}}_n$. Equation (46) also shows that the closed-loop system has the parameter vectors p_i. With the preceding deriviations this proves that the growth-bound of the closed-loop C_0-semigroup fulfills $\max(\beta, \max_{i=1,\ldots,n} \operatorname{Re} \tilde{\lambda}_i)$. The second part of Theorem 1 can be obtained applying the results given above. ∎

This results extend the work in [2] to a larger class of infinite-dimensional systems, where the eigenvectors need not form a basis in the state space. This is of special interest for linear time-delay systems, where the eigenvector-basis property is rather restrictive. Therefore, the results of this paragraph are applied to linear time-delay systems in the sequel.

IV. LINEAR TIME-DELAY SYSTEMS

A. Parametric approach

In the following linear time-delay systems with *point-delays* will be considered. This class of system can be described by

$$\dot{\chi}(t) = A_0\chi(t) + \sum_{i=1}^{N} A_i \chi(t - T_i) + Bu(t), \quad t > 0, \tag{48}$$

$$\chi(0) = \chi_0, \tag{49}$$

$$\chi(\theta) = \chi_0(\theta), \quad -T_N \leq \theta < 0 \tag{50}$$

with $\chi(t) \in \mathbb{C}^{n_\chi}$, $u(t) \in \mathbb{R}^p$ and $y(t) \in \mathbb{R}^m$. The point-delays are characterised by $0 < T_0 < T_1 < \ldots < T_N$. System (48)–(50) can be formulated as an abstract initial value problem (1) with the operators

$$\mathcal{A} \begin{bmatrix} h_1 \\ h_2(\theta) \end{bmatrix} = \begin{bmatrix} A_0 h_1 + \sum_{i=1}^{N} A_i h_2(-T_i) \\ \frac{dh_2(\theta)}{d\theta} \end{bmatrix}, \tag{51}$$

$$D(\mathcal{A}) = \left\{ \begin{bmatrix} h_1 \\ h_2(\theta) \end{bmatrix} \in H \mid h_2 \text{ abs. cont.}, \right.$$

$$\left. \frac{dh_2(\theta)}{d\theta} \in L_2([-T_N, 0], \mathbb{C}^{n_\chi}),\ h_2(0) = h_1 \right\}, \tag{52}$$

$$\mathcal{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} \tag{53}$$

on the Hilbert space $H = \mathbb{C}^{n_\chi} \oplus L_2([-T_N, 0], \mathbb{C}^{n_\chi})$ with inner product

$$\left\langle \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle = \langle u_1, v_1 \rangle_{\mathbb{C}^{n_\chi}} + \langle u_2, v_2 \rangle_{L_2} \tag{54}$$

(see e.g. [3]). The initial value x_0 of the abstract initial value problem is

$$x_0 = \begin{bmatrix} \chi_0 \\ \chi_0(\theta) \end{bmatrix}, \quad -T_N < \theta < 0. \tag{55}$$

According to this the state history $\chi(t+\theta)$, $-T_N < \theta < 0$, is stored in the distributed state $x_2(\theta)$ of the abstract state $x(t) = \begin{bmatrix} x_1^T & x_2^T(\theta) \end{bmatrix}^T$. The spectrum of $\mathcal{A}$ is discrete and can be obtained by solving the *characteristic equation* $\det \Delta(s) = 0$ with

$$\Delta(s) = sI - A_0 - \sum_{i=1}^{N} A_i \, \mathrm{e}^{-sT_i}. \tag{56}$$

The corresponding eigenvector of the adjoint operator $\mathcal{A}^*$ to eigenvalue $\overline{\lambda_i}$ is given by

$$\psi_i = \begin{bmatrix} \psi_{1,i} \\ \sum_{i=1}^{N} 1_{[-T_i,0]}(\theta) A_i^* \, \mathrm{e}^{-\overline{\lambda_i}(\theta+T_i)} \, \psi_{1,i} \end{bmatrix} \tag{57}$$

with $\psi_{1,i}$ solving $\Delta^*(\lambda_i)\psi_{1,i} = 0$. The function $1_{[-T_i,0]}(\theta)$ in (57) equals the identity on $[-T_i, 0]$ and zero elsewhere. In general the eigenvectors ϕ_i and ψ_i do not form a Riesz-Basis in H. However, the spectrum determined growth assumption is fulfilled and there are only finitely many eigenvalues in every right half plane $\mathbb{C}_\beta^+$, $\beta \in \mathbb{R}$. Furthermore, after a decompostion of the time-delay system as described in Section II the residual subsystem is β-exponentially stable as required. To β-exponentially stabilize the system, the subsystem (9) has to be controllable, which is satisfied for

$$\operatorname{rk} \begin{bmatrix} \Delta(s) & B \end{bmatrix} = n_\chi, \quad \forall s \in \overline{\mathbb{C}_\beta^+}. \tag{58}$$

Then, the feedback (11) could be designed as proposed in Section III. For a realization of the feedback (11), the operator $\mathcal{K}$ with (12), (13) and (34) can be written as

$$\mathcal{K} = \overline{K^*} \begin{bmatrix} \langle \cdot, \psi_1 \rangle \\ \vdots \\ \langle \cdot, \psi_n \rangle \end{bmatrix}. \tag{59}$$

Thus, the *modal states*

$$x_i^*(t) = \langle x(t), \psi_i \rangle, \quad i = 1, \ldots, n \tag{60}$$

have to be determined. This becomes

$$x_i^*(t) = \chi^T(t)\overline{\psi_{1,i}} + \xi_i(t) \tag{61}$$

with

$$\xi_i(t) = \sum_{j=1}^{N} \overline{\psi_{1,i}}^T A_j \int_{T_j}^{0} \chi(t+\theta) \, \mathrm{e}^{-\lambda_i(\theta+T_j)} \, d\theta, \tag{62}$$

where (54) and (57) as well as the relation $x_1 = \chi(t)$ and $x_2(\theta) = \chi(t+\theta)$ were used. A similar state feedback as in (59) can be obtained in [8] where also finitely many eigenvalues are shifted. The difference to the present approach is that an eigenstructure assignment is not achieved.

B. Implementation of the state feedback

As a consequence of the integrals appearing in (62), the complete state history $\chi(t+\theta)$, $-T_N < \theta < 0$, is needed. This cannot be implemented in technical application as an exact storage of the state history is in general not possible. Additionally, for the exact evaluation of (62) the state history has to be described in a closed analytic expression, which is also not given normally. Thus, it is required to approximate the expression (62). A substitution $\tau = t + \theta$ in every integral in (62) and a derivation of (62) with respect to time using the *Leibniz's rule* for the deriviation of parameter integrals yields

$$\begin{aligned} \dot{\xi}_i(t) &= \sum_{j=1}^{N} \overline{\psi_{1,i}}^T A_j \left(\lambda_i \int_{-T_j}^{0} \chi(t+\theta) \, \mathrm{e}^{-\lambda_i(\theta+T_j)} \, d\theta \right. \\ &\quad \left. + \chi(t) \, \mathrm{e}^{-\lambda_i T_j} - \chi(t - T_j) \right) \\ &= \lambda_i \xi_i(t) + \sum_{j=1}^{N} \overline{\psi_{1,i}}^T A_j \left(\chi(t) \, \mathrm{e}^{-\lambda_i T_j} - \chi(t - T_j) \right) \end{aligned} \tag{63}$$

after some manipulations (see also [6]). This is a non-minimal representation of (62) in form of a continuous-time filter. In order to abtain a minimal filter a time-discretization of (63) is considerd. This leads to a minimal and stable FIR-Filter realizing (11). For this purpose it is assumed that the point-delays of the time-delay system (48) are *commensurate*, which means $T_i = c_i T$ with $c_0 < c_1 < \ldots < c_N \in \mathbb{N}$ and $T > 0$. This restriction to linear time-delay systems with *commensurate point-delays* is made to dicretisize every point-delay in (63) with one *sample time* $T_s = \frac{T}{k}$, $k \in \mathbb{N}$ and a zero-order hold at input $\chi(t)$. A z-transformation of the time-discretisized filter (63) with zero-order hold at the input then results in

$$F_{s,i}(z) = -\frac{1 - \mathrm{e}^{\lambda_i T_s}}{\lambda_i (z - \mathrm{e}^{\lambda_i T_s})} \sum_{j=1}^{N} \overline{\psi_{1,i}}^T A_j \left(\mathrm{e}^{-\lambda_i c_j k T_s} - z^{-c_j k} \right). \tag{64}$$

It could be seen that the transfer function (64) has a pole as well as a zero at $\mathrm{e}^{\lambda_i T_s}$. Different to the continuous-time filter (63) now a pole-zero cancellation is possible using $a^k - b^k = (a-b)\sum_{l=0}^{k-1} a^l b^{k-1-l}$. Then, after some manipulations one obtains the *FIR-filter*

$$\begin{aligned} F_{s,i}(z) &= \frac{1 - \mathrm{e}^{-\lambda_i T_s}}{\lambda_i} \\ &\quad \sum_{j=1}^{N} \overline{\psi_{1,i}}^T A_j \left(\sum_{l=1}^{c_j k} \mathrm{e}^{-\lambda_i (c_j k - l) T_s} \, z^{-l} \right) \\ &= \frac{1 - \mathrm{e}^{-\lambda_i T_s}}{\lambda_i} \overline{\psi_{1,i}}^T \sum_{l=1}^{c_N k} \left(\sum_{\substack{j=1, \\ l \le c_j k}}^{N} A_j \, \mathrm{e}^{-\lambda_i (c_j k - l) T_s} \right) z^{-l} \end{aligned} \tag{65}$$

from (64), which can be implemented with input $\chi(t)$ to approximate (62) and thus the feedback (11). The choice of k and thus the order $c_N k$ of the FIR-filter depends on the dynamic of the system. An inaccurate choice of the sample time T_s can even destabilize the system (see [11] or [12]).

V. EXAMPLE

The proposed approach is illustrated by partial input-output decoupling a simple time-delay system (48) with one point-delay $T_1 = 1$ and an additional output $y(t) = C\chi(t)$. The system matrices are given by

$$A_0 = \begin{bmatrix} -3 & 6 & -4 \\ -2 & -5 & -5 \\ 5 & -4 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.2 & 0.1 & 0.1 \\ 0.2 & 0.4 & 0.5 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}, \quad (66)$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix}. \quad (67)$$

For partial decoupling with state feedback the control law (11) is extended with the input matrix M and the reference input $w(t)$ to $u(t) = -\mathcal{K}x(t) + Mw(t)$. The eigenvalues of this system are approximated with the MATLAB-toolbox DDE-BIFTOOL (see [13]). For the purpose of partial decoupling the design method proposed in [4] for distributed-parameter systems is adopted in a similar way for time-delay systems. In this way 14 eigenvalues were shifted and their corresponding eigenvectors were assigned. Therein 4 respectively 5 eigenvalues $\tilde{\lambda}_i$ and parameter vectors p_i were assigned to fulfill

$$-F(\tilde{\lambda}_i)p_i = e_j \quad (68)$$

with $F(s)$ the transfer matrix of the system and $j = 1$ respectively $j = 2$. These eigenvalues then only affect the output $y_1(t)$ respectively $y_2(t)$. The 5 remaining eigenvalues $\tilde{\lambda}_i$ and corresponding parameter vectors p_i were chosen fulfilling

$$F(\tilde{\lambda}_i)p_i = 0 \quad (69)$$

and thus are not modal observable at any output. Then, the corresponding eigenvectors could be determined by (24) and the matrix $\overline{K^*}$ in (59) can be obtained by (36). The input matrix M is chosen as

$$M = (\mathcal{C}(-\mathcal{A} + \mathcal{B}\mathcal{K})^{-1}\mathcal{B})^{-1} \quad (70)$$

to assure a vanishing steady state error. In Figure 1 the response of the controlled system is compared with the response of the uncontrolled system with $u(t) = Mw(t)$. Obviously, an input-output decoupling with a good control performance and low residual couplings between the outputs is achieved. This example also demonstrates that the solution of the considered input-output decoupling problem requires the assignment of both the closed-loop eigenvalues and their eigenvectors.

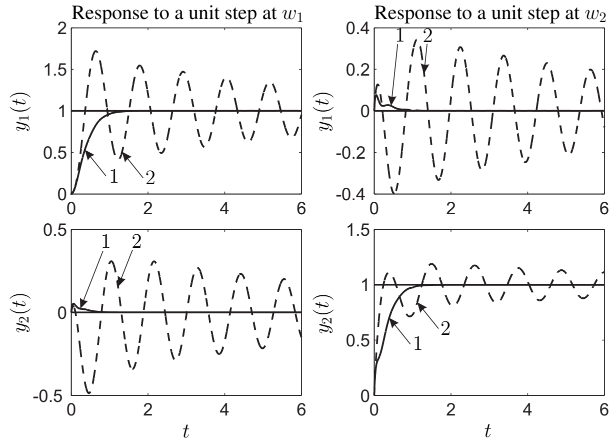

Fig. 1. Reference response of the closed- and open-loop system to reference step inputs $w_1(t) = \sigma(t)$ and $w_2(t) = 0$ as well as $w_1(t) = 0$ and $w_2(t) = \sigma(t)$. 1: closed-loop reference transfer behaviour, 2: open-loop transfer behaviour

VI. CONCLUDING REMARKS

Due to duality, the proposed approach can also be used for the design of infinite-dimensional state observers. The stability of the digital implementation with FIR-filters has not been investigated in detail and is an interesting topic for further research. The order of FIR-filters can be decreased by using multirate-filtering, which reduces the implementation effort.

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