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A solution to the tracking problem using distributed predictive control

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***Abstract*— A Distributed Predictive Control (DPC) algorithm for tracking (piecewise) constant output reference signals is presented in this paper. The overall system is assumed to be composed by a number of discrete-time linear subsystems interconnected through the states and/or the inputs. The algorithm is non-cooperative, based on neighbor-to-neighbor communication, and does not require an iterative exchange of information among neighbors. Unfeasible reference signals, that cannot be reached due to state and control constraints, can also be considered by computing the nearest feasible reference value. A simulation example is reported.**

I. INTRODUCTION

Motivated by the increasing complexity of industrial systems and infrastructures, the design of distributed control algorithms based on Model Predictive Control (MPC) has been recently widely considered, see for example [4], [8], [13], [3] and the review papers [14], [11]. Most of the contributions in this field are referred to the so-called regulation problem, i.e. asymptotically steering the state of all the subsystems to zero by coordinating local control actions with a minimum amount of information transmitted among the subsystems. On the contrary, minor attention (see, e.g., [6], where a cooperative distributed MPC scheme for tracking is proposed) has been placed to the design of distributed control schemes for the asymptotic tracking of constant reference output signals. Indeed, in a distributed setting, the standard approach, based on the reformulation of the tracking problem as a regulation one by computing at any set-point change of the output the corresponding state and control target values, cannot be followed due to the decentralization constraint.

For these reasons, in this paper a new distributed control method for the solution of the tracking problem is proposed. It relies on the state-feedback Distributed Predictive Control (DPC) algorithm developed in [5] for the solution of the regulation problem. Preliminary results on the use of DPC for the solution of the tracking problem are reported in [5], although the approach there adopted is different from the one described in this paper. DPC assumes that the future state and control reference trajectories are transmitted by each subsystem to its neighbors, and the differences between these reference trajectories and the true ones are interpreted as disturbances to be rejected. In addition, the proposed algorithm also allows one to consider the presence of possible unfeasible reference signals, i.e. of set-points that cannot be reached due to state and/or control constraints.

The paper is organized as follows. In Section II the overall system made by a number of interacting subsystems is introduced together with some preliminary assumptions. Section III describes the distributed DPC algorithm for tracking, together with its convergence properties, while Section IV illustrates a simulation example. Some conclusions are drawn in Section V. The proofs of the technical results can be found in [2].

Notation. A matrix is Schur if all its eigenvalues lie in the interior the unit circle. The short-hand $\mathbf{v} = (v_1, \dots, v_s)$ denotes a column vector with s (not necessarily scalar) components $v_1, \dots, v_s$. The symbol $\oplus$ denotes the Minkowski sum and $\bigoplus_{i=1}^{M} A_i = A_1 \oplus \cdots \oplus A_M$. A generic p-norm ball center at the origin in the $\mathbb{R}^{dim}$ space is defined as follows

$$\mathscr{B}_{p,\varepsilon}^{(dim)}(0) := \{x \in \mathbb{R}^{dim} : \|x\|_p \leq \varepsilon\}$$

For a discrete-time signal s_t and $a, b \in \mathbb{N}$, $a \leq b$, $(s_a, s_{a+1}, \dots, s_b)$ is denoted with $s_{[a:b]}$. Finally, $\lambda_M(\cdot)$ and $\lambda_m(\cdot)$ are the maximum and the minimum eigenvalue of a matrix, respectively.

II. INTERACTING SUBSYSTEMS

Consider the set of M dynamically interacting subsystems which, according to the notation used in [9], are described by

$$x_{t+1}^{[i]} = A_{ii} x_t^{[i]} + B_{ii} u_t^{[i]} + E_i s_t^{[i]} \tag{1a}$$

$$y_t^{[i]} = C_{ii} x_t^{[i]} \tag{1b}$$

$$z_t^{[i]} = C_{zi} x_t^{[i]} + D_{zi} u_t^{[i]} \tag{1c}$$

where $x^{[i]} \in \mathbb{R}^{n_i}$ and $u_t^{[i]} \in \mathbb{R}^{m_i}$ are the state and input vectors, respectively, of the i-th subsystem, while $y_t^{[i]} \in \mathbb{R}^{m_i}$ is its output vector. In line with the interaction-oriented models introduced in [9], the coupling input and output vectors $s_t^{[i]}$ and $z_t^{[i]}$, respectively, are defined to characterize the interconnections among the subsystems; in a collective form, they are defined as $\mathbf{s}_t = (s_t^{[1]}, \dots, s_t^{[M]})$, $\mathbf{z}_t = (z_t^{[1]}, \dots, z_t^{[M]})$, and the interconnections among subsystems are described by means of the algebraic equation

$$\mathbf{s}_t = \mathbf{L}\mathbf{z}_t \tag{2}$$

where $\mathbf{L}$ is denoted *interconnection matrix*. More specifically, the *coupling input* $s_t^{[i]}$ to subsystem i depends on the *coupling output* $z_t^{[j]}$ of the j-th subsystem according to

$$s_t^{[i]} = \sum_{j=1}^{M} L_{ij} z_t^{[j]} \tag{3}$$

We say that subsystem j is a *dynamic neighbor* of subsystem i if and only if $L_{ij} \neq 0$, and we denote as $\mathscr{N}_i$ the set of dynamic neighbors of subsystem i (which excludes i).

The input and state variables are subject to the constraints $u_t^{[i]} \in \mathbb{U}_i \subseteq \mathbb{R}^{m_i}$ and $x_t^{[i]} \in \mathbb{X}_i \subseteq \mathbb{R}^{n_i}$, respectively, where the

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sets $\mathbb{U}_i$ and $\mathbb{X}_i$ are convex.

Collecting the subsystems (1) for all $i = 1,\dots,M$, we obtain the collective dynamical model

$$\mathbf{x}_{t+1} = \mathbf{A}\,\mathbf{x}_t + \mathbf{B}\mathbf{u}_t \tag{4a}$$

$$\mathbf{y}_t = \mathbf{C}\,\mathbf{x}_t \tag{4b}$$

where $\mathbf{x}_t = (x_t^{[1]},\dots,x_t^{[M]}) \in \mathbb{R}^n$, $n = \sum_{i=1}^M n_i$, $\mathbf{u}_t = (u_t^{[1]},\dots,u_t^{[M]}) \in \mathbb{R}^m$, $m = \sum_{i=1}^M m_i$, and $\mathbf{y}_t = (y_t^{[1]},\dots,y_t^{[M]}) \in \mathbb{R}^m$ are the collective state, input, and output vectors, respectively. The state transition matrices $A_{11} \in \mathbb{R}^{n_1 \times n_1}$, …, $A_{MM} \in \mathbb{R}^{n_M \times n_M}$ of the M subsystems are the diagonal blocks of $\mathbf{A}$, whereas the dynamic coupling terms between subsystems correspond to the non-diagonal blocks of $\mathbf{A}$, i.e., $A_{ij} = E_i L_{ij} C_{zj}$, with $j \neq i$. We also define $\mathbf{A}^* =\mathrm{diag}(A_{11},\dots,A_{MM})$. Correspondingly, B_{ii}, $i = 1,\dots,M$, define the direct influence of input $u_t^{[i]}$ upon the state $x_t^{[i]}$, and are the diagonal blocks of $\mathbf{B}$, whereas the influence of the input of a subsystem upon the state of different subsystems is represented by the off-diagonal terms of $\mathbf{B}$, i.e., $B_{ij} = E_i L_{ij} D_{zj}$, with $j \neq i$. The collective output matrix is defined as $\mathbf{C} =\mathrm{diag}(C_{11},\dots,C_{MM})$.

We define $\mathbb{X} = \prod_{i=1}^M \mathbb{X}_i \subseteq \mathbb{R}^n$ and $\mathbb{U} = \prod_{i=1}^M \mathbb{U}_i \subseteq \mathbb{R}^m$, which are convex by convexity of $\mathbb{X}_i$ and $\mathbb{U}_i$, respectively.

Concerning system (4a) and its partition, the following main assumption on decentralized stabilizability is introduced:

Assumption 1: There exists a block-diagonal matrix $\mathbf{K} =\mathrm{diag}(K_1,\dots,K_M)$, with $K_i \in R^{m_i \times n_i}$, $i = 1,\dots,M$ such that: (i) $\mathbf{F} = \mathbf{A} + \mathbf{B}\mathbf{K}$ is Schur, (ii) $F_{ii} = (A_{ii} + B_{ii}K_i)$ is Schur, $i = 1,\dots,M$.

Remark 1: The design of the stabilizing matrix $\mathbf{K}$ can be performed according to the procedure proposed in [5] or by resorting to an LMI formulation, see [1], based on well known results in decentralized control, see e.g. [15].

Moreover, in order to solve the tracking problem for constant reference signals, the following standard assumption is made.

Assumption 2: The input-output system (4) has no invariant zeros in 1, i.e., $\mathrm{rank}(\mathbf{S}) = n + m$, where

$$\mathbf{S} = \begin{bmatrix} I_n - \mathbf{A} & -\mathbf{B} \\ \mathbf{C} & 0 \end{bmatrix}$$

III. THE DISTRIBUTED PREDICTIVE CONTROL ALGORITHM FOR TRACKING

We want to design a distributed state-feedback control law, based on MPC, for the tracking of a given constant set-point signal $y_{set-point}^{[i]} \in \mathbb{R}^{m_i}$, i.e., that asymptotically steers the system output $y_t^{[i]}$ to the desired value $y_{set-point}^{[i]}$ for all $i = 1,\dots,M$. The main idea behind the proposed algorithm is sketched in the following.

The proposed control architecture is composed in three different layers (see Figure 1).

1) The reference output trajectory management layer. For each subsystem $i = 1,\dots,M$, a local reference trajectory management unit is required, which defines the reference trajectory $\tilde{y}_{t+k}^{[i]}$ of the output $y_{t+k}^{[i]}$. Although it would be natural to take $\tilde{y}_{t+k}^{[i]} = y_{set-point}^{[i]}$ for all $k \geq 0$, this choice could easily lead to infeasible standard MPC optimization problems, even in the centralized framework. Furthermore, in the distributed context this point is particularly critical, since too rapid changes in the output reference trajectory for a given subsystem could greatly affect the performance and the behavior of the other subsystems. Therefore $\tilde{y}^{[i]}$ will be regarded as an argument of an optimization problem rather than a fixed parameter, and constraints limiting the time variation of the local reference signals will be defined and computed. This layer is completely decentralized, i.e., the transmission of information between local reference trajectory management units is not needed.

2) The reference state and input trajectory layer. For each subsystem $i = 1,\dots,M$ assume that at any time instant t the future reference trajectories $\tilde{y}_k^{[i]}$, $k = t,\dots,t+N-1$, are available for all $i = 1,\dots,M$. In order to define the reference trajectories $\tilde{x}_t^{[i]}$, $\tilde{u}_t^{[i]}$, and $\tilde{z}_t^{[i]}$ of the corresponding state, input, and coupling output variables, we design a suitable observer, using the output reference information as data.

3) The robust MPC layer. For each subsystem $i = 1,\dots,M$, a robust MPC unit is designed to drive the real state and input trajectories $x_t^{[i]}$ and $u_t^{[i]}$ as close as possible to the reference ones $\tilde{x}_t^{[i]}$, $\tilde{u}_t^{[i]}$, while respecting the constraints on the same variables. As in the case of the reference state and input trajectory layer, information is required to be transmitted from reference trajectory management units of neighboring regulators, in a neighbor-to-neighbor fashion.

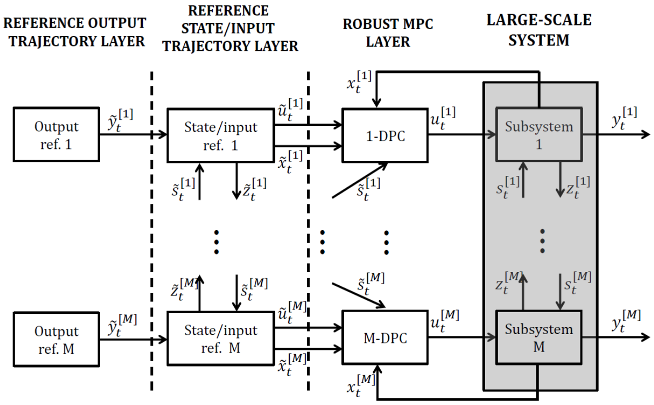

Fig. 1. Control system architecture

A. *The reference output trajectory management layer*

As discussed, one of the main requirements for guaranteeing good performance and constraint satisfaction of our control scheme is to limit the rate of variation in time of the output reference signals. More specifically, by adding suitable constraints to the MPC problem formulation we will guarantee that, for all $i = 1,\dots,M$, for all $t \geq 0$ (and for any norm p in the definition of a ball, see the notation section)

$$\tilde{y}_{t+1}^{[i]} \in \tilde{y}_t^{[i]} \oplus \mathscr{B}_{p,\varepsilon}^{(m_i)}(0) \tag{5}$$

Denote

$$\mathscr{A}_{ij} = \begin{cases} \begin{bmatrix} A_{ii} & B_{ii} \\ 0 & I_{m_i} \end{bmatrix} & \text{if } j = i \\ \begin{bmatrix} A_{ij} & B_{ij} \\ 0 & 0 \end{bmatrix} & \text{if } j \neq i \end{cases}, \mathscr{C}_i = \begin{bmatrix} C_{ii} & 0 \end{bmatrix} \qquad (6)$$

Define, for all $i = 1, \dots, M$ and for all $t \geq 0$, $\chi_t^{[i]ss} = (x_t^{[i]ss}, u_t^{[i]ss})$ where $x_t^{[i]ss}$ and $u_t^{[i]ss}$ are the steady-state state and input values corresponding to the reference outputs $\tilde{y}_t^{[i]}$ and satisfy the following steady-state equations

$$\begin{array}{rcl} \chi_t^{[i]ss} & = & \mathscr{A}_{ii}\chi_t^{[i]ss} + \sum_{j\in\mathscr{N}_i}\mathscr{A}_{ij}\chi_t^{[j]ss} \\ \tilde{y}_t^{[i]} & = & \mathscr{C}_i\chi_t^{[i]ss} \end{array} \qquad (7)$$

It is easy to verify that Assumption 2 guarantees that a solution to the system (7) exists and is unique. In view of this, letting $\mathbf{x}_t^{ss} = (x_t^{[1]ss}, \dots, x_t^{[M]ss})$, $\mathbf{u}_t^{ss} = (u_t^{[1]ss}, \dots, u_t^{[M]ss})$, and $\tilde{\mathbf{y}}_t = (\tilde{y}_t^{[1]}, \dots, \tilde{y}_t^{[M]})$, from (7) one has

$$\begin{bmatrix} \mathbf{x}_t^{ss} - \mathbf{x}_{t+1}^{ss} \\ \mathbf{u}_t^{ss} - \mathbf{u}_{t+1}^{ss} \end{bmatrix} \in -\mathbf{S}^{-1} \begin{bmatrix} 0 \\ I_m \end{bmatrix} \prod_{i=1}^{M} \mathscr{B}_{p,\varepsilon}^{(m_i)}(0) \qquad (8)$$

from which it follows that, for all $i = 1, \dots, M$, there exists a set Δ_i^{ss} such that, for all $t \geq 0$

$$\chi_t^{[i]ss} - \chi_{t+1}^{[i]ss} \in \Delta_i^{ss} \qquad (9)$$

B. The reference state and input trajectory layer

An observer is designed to provide an estimate $\chi_t^{[i]} = (\tilde{x}_t^{[i]}, \tilde{u}_t^{[i]})$ of the collective variable $\chi_t^{[i]ss}$ using the output reference information as data. We let

$$\tilde{s}_t^{[i]} = \sum_{j\in\mathscr{N}_i} L_{ij}\tilde{z}_t^{[j]} \qquad (10)$$

The dynamics of the variable $\chi_t^{[i]}$ is defined by the following dynamical system

$$\chi_{t+1}^{[i]} = \mathscr{A}_{ii}\chi_t^{[i]} + \sum_{j\in\mathscr{N}_i}\mathscr{A}_{ij}\chi_t^{[j]} + \mathscr{G}_i(\tilde{y}_{t+1}^{[i]} - \mathscr{C}_i\chi_t^{[i]}) \qquad (11)$$

where $\mathscr{G}_i = \begin{bmatrix} G_i^x \\ G_i^u \end{bmatrix}$ are gains to be determined as follows: denoting by $\mathscr{A}$ the matrix whose block elements are $\mathscr{A}_{ij}$, $\mathscr{C} = diag(\mathscr{C}_i)$, and $\mathscr{G} = diag(\mathscr{G}_i)$, the following assumption must be fulfilled

Assumption 3: The matrix $\mathscr{A} - \mathscr{G}\mathscr{C}$ is Schur

Note that the synthesis of the $\mathscr{G}_i$s can be performed according to the procedures proposed in Remark 1.

From (6)-(7), it follows that

$$\begin{aligned} \chi_{t+1}^{[i]} - \chi_{t+1}^{[i]ss} &= (\mathscr{A}_{ii} - \mathscr{G}_i\mathscr{C}_i)(\chi_t^{[i]} - \chi_t^{[i]ss}) \qquad (12) \\ &+ \sum_{j\in\mathscr{N}_i}\mathscr{A}_{ij}(\chi_t^{[j]} - \chi_t^{[j]ss}) + (\mathscr{A}_{ii} - \mathscr{G}_i\mathscr{C}_i)(\chi_t^{[i]ss} - \chi_{t+1}^{[i]ss}) \\ &+ \sum_{j\in\mathscr{N}_i}\mathscr{A}_{ij}(\chi_t^{[j]ss} - \chi_{t+1}^{[j]ss}) \end{aligned}$$

In view of (9) we can rewrite (12) as

$$\begin{aligned} \chi_{t+1}^{[i]} - \chi_{t+1}^{[i]ss} = \; & (\mathscr{A}_{ii} - \mathscr{G}_i\mathscr{C}_i)(\chi_t^{[i]} - \chi_t^{[i]ss}) \\ & + \textstyle\sum_{j\in\mathscr{N}_i}\mathscr{A}_{ij}(\chi_t^{[j]} - \chi_t^{[j]ss}) + \tilde{w}_t^{[i]} \end{aligned} \qquad (13)$$

where

$$\tilde{w}_t^{[i]} = (\mathscr{A}_{ii} - \mathscr{G}_i\mathscr{C}_i)(\chi_t^{[i]ss} - \chi_{t+1}^{[i]ss}) + \sum_{j\in\mathscr{N}_i}\mathscr{A}_{ij}(\chi_t^{[j]ss} - \chi_{t+1}^{[j]ss}) \in \tilde{\mathbb{W}}_i$$

can be regarded as a bounded disturbance in view of (9) and

$$\tilde{\mathbb{W}}_i = (\mathscr{A}_{ii} - \mathscr{G}_i\mathscr{C}_i)\Delta_i^{ss} \oplus (\bigoplus_{j\in\mathscr{N}_i}\mathscr{A}_{ij}\Delta_j^{ss}) \qquad (14)$$

Under Assumption 3, for the system (13) there exists a possibly non-rectangular Robust Positive Invariant (RPI) set $\mathbf{\Delta}^{xu}$ such that, if $(\tilde{\mathbf{x}}_t - \mathbf{x}_t^{ss}, \tilde{\mathbf{u}}_t - \mathbf{u}_t^{ss}) \in \mathbf{\Delta}^{xu}$, then it is guaranteed that $(\tilde{\mathbf{x}}_{t+1} - \mathbf{x}_{t+1}^{ss}, \tilde{\mathbf{u}}_{t+1} - \mathbf{u}_{t+1}^{ss})) \in \mathbf{\Delta}^{xu}$. This, in turn, implies that there exist sets Δ_i^{xu}, $i = 1, \dots, M$, such that, for any initial condition $(\tilde{\mathbf{x}}_0 - \mathbf{x}_0^{ss}, \tilde{\mathbf{u}}_0 - \mathbf{u}_0^{ss}) \in \mathbf{\Delta}^{xu}$, it is possible to guarantee that

$$(\chi_t^{[i]} - \chi_t^{[i]ss}) \in \Delta_i^{xu} \qquad (15)$$

for all $t \geq 0$. This, in turn implies that, denoting $\Delta_i^y = -\mathscr{C}_i\Delta_i^{xu}$, $i = 1, \dots, M$, it is possible to guarantee that, for all $t \geq 0$,

$$\tilde{y}_t^{[i]} - C_{ii}\tilde{x}_t^{[i]} = -\mathscr{C}_i(\chi_t^{[i]} - \chi_t^{[i]ss}) \in \Delta_i^y \qquad (16)$$

C. The robust MPC layer

As discussed, the goal of the MPC layer is to compute a suitable control signal $u_t^{[i]}$, for all time steps $t \geq 0$ and for all subsystems $i = 1, \dots, M$, such that the real state and input trajectories $x_t^{[i]}$ and $u_t^{[i]}$, respectively are steered on the reference ones, i.e., $\tilde{x}_t^{[i]}$ and $\tilde{u}_t^{[i]}$, respectively. To do so we use robust "tube-based" MPC [10]. Specifically, adding suitable constraints to the MPC problem formulation, for each subsystem and for all $t \geq 0$ we will be able to guarantee that the actual coupling output trajectories lie in specified time-invariant neighborhoods of the reference trajectories, i.e, $z_t^{[i]} \in \tilde{z}_t^{[i]} \oplus \mathscr{Z}_i$, where $0 \in \mathscr{Z}_i$ which, in view of (2) and (10), implies that $s_t^{[i]} \in \tilde{s}_t^{[i]} \oplus \mathscr{S}_i$, where $\mathscr{S}_i = \bigoplus_{j\in\mathscr{N}_i} L_{ij}\mathscr{Z}_j$. In this way, (1a) can be written as

$$x_{t+1}^{[i]} = A_{ii}x_t^{[i]} + B_{ii}u_t^{[i]} + E_i\tilde{s}_t^{[i]} + E_i(s_t^{[i]} - \tilde{s}_t^{[i]}) \qquad (17)$$

where $E_i(s_t^{[i]} - \tilde{s}_t^{[i]})$ is a bounded disturbance and the term $E_i\tilde{s}_{t+k}^{[i]}$ can be interpreted as an input, known in advance over the prediction horizon $k = 0, \dots, N-1$.

For the statement of the individual MPC sub-problems, henceforth called i-DPC problems, we define the i-th subsystem nominal model associated to equation (17)

$$\hat{x}_{t+1}^{[i]} = A_{ii}\hat{x}_t^{[i]} + B_{ii}\hat{u}_t^{[i]} + E_i\tilde{s}_t^{[i]} + G_i^x(\tilde{y}_{t+1}^{[i]} - C_{ii}\tilde{x}_t^{[i]}) \qquad (18)$$

and let

$$\hat{z}_t^{[i]} = C_{zi}\hat{x}_t^{[i]} + D_{zi}\hat{u}_t^{[i]} \qquad (19)$$

The control law for the i-th subsystem (17), for all $t \geq 0$, is assumed to be given by

$$u_t^{[i]} = \hat{u}_t^{[i]} + K_i(x_t^{[i]} - \hat{x}_t^{[i]}) \qquad (20)$$

where K_i satisfies Assumption 1. Letting $\varepsilon_t^{[i]} = x_t^{[i]} - \hat{x}_t^{[i]}$ from (17), (18) and (20) we obtain

$$\varepsilon_{t+1}^{[i]} \;\; = \;\; F_{ii}\varepsilon_t^{[i]} + w_t^{[i]} \qquad (21)$$

where

$$w_t^{[i]} = E_i(s_t^{[i]} - \tilde{s}_t^{[i]}) - G_i^x(\tilde{y}_{t+1}^{[i]} - C_{ii}\tilde{x}_t^{[i]}) \qquad (22)$$

is a bounded disturbance since $s_t^{[i]} - \tilde{s}_t^{[i]} \in \mathscr{S}_i$ and, in view of (5) and (16)

$$\tilde{y}_{t+1}^{[i]} - C_{ii}\tilde{x}_t^{[i]} = \tilde{y}_{t+1}^{[i]} - \tilde{y}_t^{[i]} + \tilde{y}_t^{[i]} - C_{ii}\tilde{x}_t^{[i]} \in \mathscr{B}_{p,\varepsilon}^{(m_i)}(0) \oplus \Delta_i^y$$

It follows that

$$w_t^{[i]} \in \mathbb{W}_i = E_i\mathscr{S}_i \oplus (-G_i^x)(\mathscr{B}_{p,\varepsilon}^{(m_i)}(0) \oplus \Delta_i^y) \qquad (23)$$

Since $w_t^{[i]}$ is bounded and F_{ii} is Schur, there exists a RPI $\mathscr{E}_i$ for (21) such that, for all $\varepsilon_t^{[i]} \in \mathscr{E}_i$, then $\varepsilon_{t+1}^{[i]} \in \mathscr{E}_i$. Therefore at time $t+1$, in view of (1c) and (19), it holds that $z_{t+1}^{[i]} - \hat{z}_{t+1}^{[i]} = (C_{zi} + D_{zi}K_i)\varepsilon_{t+1}^{[i]} \in (C_{zi} + D_{zi}K_i)\mathscr{E}_i$.

In order to guarantee that, at time $t+1$, $z_{t+1}^{[i]} - \tilde{z}_{t+1}^{[i]} \in \mathscr{Z}_i$ can be still verified by adding suitable constraints to the optimization problems, the following assumption must be fulfilled.

Assumption 4: For all $i = 1, \ldots, M$, there exists a positive scalar ρ_i such that

$$(C_{zi} + D_{zi}K_i)\mathscr{E}_i \oplus \mathscr{B}_{p,\rho_i}(0) \subseteq \mathscr{Z}_i \qquad (24)$$

If Assumption 4 is fulfilled, we define, for all $i = 1, \ldots, M$, the sets Δ_i^z satisfying

$$\Delta_i^z \subseteq \mathscr{B}_{p,\rho_i}(0) \qquad (25)$$

and we consider the constraint $\hat{z}_{t+1}^{[i]} - \tilde{z}_{t+1}^{[i]} \in \Delta_i^z$, in such a way that

$$z_{t+1}^{[i]} - \tilde{z}_{t+1}^{[i]} = z_{t+1}^{[i]} - \hat{z}_{t+1}^{[i]} + \hat{z}_{t+1}^{[i]} - \tilde{z}_{t+1}^{[i]} \in (C_{zi} + D_{zi}K_i)\mathscr{E}_i \oplus \Delta_i^z \subseteq \mathscr{Z}_i \qquad (26)$$

as required at all time steps $t \geq 0$.

D. *i-DPC problems*

It is now possible to state the Distributed Predictive Control problem for the *i*-th subsystem, henceforth called *i*-DPC problem, to be solved at any time instant. The overall design problem is then composed by a preliminary centralized off-line design and an on-line solution of the M *i*-DPC problems. These two steps are detailed in the following.

1) Off-line design: The off-line design consists of the following procedure:

1) compute the matrices $\mathbf{K}$ and $\mathscr{G}$ satisfying Assumptions 1 and 3 [see Remark 3];
2) define $\mathscr{B}_{p,\varepsilon}^{(m_i)}(0)$, compute Δ_i^{ss} with (8), (9), $\tilde{\mathbb{W}}_i$ with (14) and $\tilde{\mathbb{W}} = \prod_{i=1}^{M} \tilde{\mathbb{W}}_i$;
3) compute $\mathbf{\Delta}^{xu}$ (a RPI for the collection of subsystems (13)), Δ_i^{xu} and Δ_i^y with (16) [for the computation of RPIs see [12]];
4) compute the RPI sets $\mathscr{E}_i$ for the subsystems (21) and the sets Δ_i^z satisfying (25) and (26);
5) compute $\hat{\mathbb{X}}_i \subseteq \mathbb{X}_i \ominus \mathscr{E}_i$, $\hat{\mathbb{U}}_i \subseteq \mathbb{U}_i \ominus K_i\mathscr{E}_i$, the positively invariant set Σ_i for the equation

$$\delta x_{t+1} = F_{ii}\delta x_t \qquad (27)$$

such that

$$(C_{zi} + D_{zi}K_i)\Sigma_i \subseteq \Delta_i^z \qquad (28)$$

and the convex sets $\mathbb{Y}_i$ such that

$$H_i\mathbf{S}^{-1}\begin{bmatrix}0\\ I_m\end{bmatrix}\prod_{j=1}^{M}\mathbb{Y}_j \oplus \Delta_i^{xu} \oplus \begin{bmatrix}I_{n_i}\\ K_i\end{bmatrix}\Sigma_i \subseteq \hat{\mathbb{X}}_i \times \hat{\mathbb{U}}_i \qquad (29)$$

where H_i is the matrix, of suitable dimensions, that selects the vector $(x^{[i]}, u^{[i]})$ out of $(\mathbf{x}, \mathbf{u})$, $i = 1, \ldots, M$.

2) On-line design: Having reformulated the distributed control problem as a robustness one, we rely on the robust "tube-based" MPC algorithm presented in [10]. On the other hand, for the definition of the reference trajectories, a different approach that the one used in [7] is adopted, since the reference output management layer does not use information about the current state of the system for defining $\tilde{y}_t^{[i]}$, $i = 1, \ldots, M$.

The reference output trajectory management layer. The reference output trajectory management layer solves, at each time step t, the following minimization problem.

$$\min_{\bar{y}_{t+N}^{[i]}} V_i^y(\bar{y}_{t+N}^{[i]}, t) \qquad (30)$$

subject to

$$\bar{y}_{t+N}^{[i]} - \bar{y}_{t+N-1}^{[i]} \in \mathscr{B}_{p,\varepsilon}^{(m_i)}(0) \qquad (31)$$

$$\bar{y}_{t+N}^{[i]} \in \mathbb{Y}_i \qquad (32)$$

where

$$V_i^y(\bar{y}_{t+N}^{[i]}) = \gamma\|\bar{y}_{t+N}^{[i]} - \bar{y}_{t+N-1}^{[i]}\|^2 + \|\bar{y}_{t+N}^{[i]} - y_{set-point}^{[i]}\|_{T_i}^2$$

The weight T_i must verify the inequality

$$T_i > \gamma I_{m_i} \qquad (33)$$

while γ is an arbitrarily small positive constant.

At time t, $\bar{y}_{t+N|t}^{[i]}$ is the solution to the optimization problem (30).

Remark 2: In the present implementation, coupling constraints are not included for simplicity. However it is possible to include, in the problem, constraints involving the state of more than one subsystems. This, for instance, can be handled by imposing suitable constraints at the reference output trajectory management layer level. This implies that trasmission of information must be scheduled between local output trajectory management units. This issue will be explored more in details in future works.

The robust MPC layer. The *i*-DPC problem solved by the *i*-th robust MPC layer unit is defined as follows:

$$\min_{\hat{x}_t^{[i]}, \hat{u}_{[t:t+N-1]}^{[i]}} V_i^N(\hat{x}_t^{[i]}, \hat{u}_{[t:t+N-1]}^{[i]}) \qquad (34)$$

where

$$V_i^N(\hat{x}_t^{[i]}, \hat{u}_{[t:t+N-1]}^{[i]}) = \sum_{k=t}^{t+N-1} \|\hat{x}_k^{[i]} - \tilde{x}_k^{[i]}\|_{Q_i}^2 + \|\hat{u}_k^{[i]} - \tilde{u}_k^{[i]}\|_{R_i}^2 + \|\hat{x}_{t+N}^{[i]} - \bar{x}_{t+N}^{[i]}(\bar{y}_{t+N|t}^{[i]})\|_{P_i}^2 \qquad (35)$$

subject to (18),

$$\bar{x}^{[i]}_{t+N}(\bar{y}^{[i]}_{t+N|t}) = A_{ii}\tilde{x}^{[i]}_{t+N-1} + B_{ii}\tilde{u}^{[i]}_{t+N-1} + E_i\tilde{s}^{[i]}_{t+N-1} + G^x_i(\bar{y}^{[i]}_{t+N|t} - C_{ii}\tilde{x}^{[i]}_{t+N-1}) \tag{36}$$

and, for $k = t,\dots,t+N-1$,

$$x^{[i]}_t - \hat{x}^{[i]}_t \in \mathscr{E}_i \tag{37a}$$

$$\hat{z}^{[i]}_k - \tilde{z}^{[i]}_k \in \Delta^z_i \tag{37b}$$

$$\hat{x}^{[i]}_k \in \hat{\mathbb{X}}_i \tag{37c}$$

$$\hat{u}^{[i]}_k \in \hat{\mathbb{U}}_i \tag{37d}$$

and to the terminal constraint

$$\hat{x}^{[i]}_{t+N} - \bar{x}^{[i]}_{t+N}(\bar{y}^{[i]}_{t+N|t}) \in \Sigma_i \tag{38}$$

Remark 3: The weights Q_i and R_i in the performance index (35) must be taken as positive definite matrices of appropriate dimensions while, in order to prove the convergence properties of the proposed approach, it is advisable to select the matrices P_i as the solutions of the (fully independent) Lyapunov equations

$$F^T_{ii}P_iF_{ii} - P_i = -(Q_i + K^T_i R_i K_i) \tag{39}$$

At time t, the tuple $(\hat{x}^{[i]}_{t|t}, \hat{u}^{[i]}_{[t:t+N-1]|t}, \bar{y}^{[i]}_{t+N|t})$ is the solution to the i-DPC problem and $\hat{u}^{[i]}_{t|t}$ is the input to the nominal system (18).

It is important to remark that the problems (30) and (34) are independent from each other. In fact, on the one hand, it is easy to see that (30) does not depend on $\hat{x}^{[i]}_t$ and $\hat{u}^{[i]}_{[t:t+N-1]}$. On the other hand note that, both in the definition (35) of the cost function V^N_i and in the contraints (37), the difference $\hat{x}^{[i]}_{t+N} - \bar{x}^{[i]}_{t+N}(\bar{y}^{[i]}_{t+N|t})$ appears (and not the two terms alone); in view of (18) and (36)

$$\hat{x}^{[i]}_{t+N} - \bar{x}^{[i]}_{t+N}(\bar{y}^{[i]}_{t+N|t}) = A_{ii}(\hat{x}^{[i]}_{t+N-1} - \tilde{x}^{[i]}_{t+N-1}) + B_{ii}(\hat{u}^{[i]}_{t+N-1} - \tilde{u}^{[i]}_{t+N-1})$$

which is independent of $G^x_i(\bar{y}^{[i]}_{t+N|t} - C_{ii}\tilde{x}^{[i]}_{t+N-1})$, since the latter is additive to both $\hat{x}^{[i]}_{t+N}$ and $\bar{x}^{[i]}_{t+N}(\bar{y}^{[i]}_{t+N|t})$.

Then, according to (20), the input to the system (1a) is

$$u^{[i]}_t = \hat{u}^{[i]}_{t|t} + K_i(x^{[i]}_t - \hat{x}^{[i]}_{t|t}) \tag{40}$$

and it is set $\tilde{y}^{[i]}_{t+N} = \bar{y}^{[i]}_{t+N|t}$: note, in fact, that the optimal value of $\bar{y}^{[i]}_{t+N}$ will be used as reference for the output variable of subsystem i at instant $t+N$. Denoting by $\hat{x}^{[i]}_{k|t}$ the state trajectory of system (18) stemming from $\hat{x}^{[i]}_{t|t}$ and $\hat{u}^{[i]}_{[t:t+N-1]|t}$, at time t it is also possible to compute $\hat{x}^{[i]}_{t+N|t}$ and

$$\hat{u}^{[i]}_{t+N|t} = \tilde{u}^{[i]}_{t+N} + K_i(\hat{x}^{[i]}_{t+N|t} - \tilde{x}^{[i]}_{t+N}) \tag{41}$$

where $\tilde{x}^{[i]}_{t+N}$ and $\tilde{u}^{[i]}_{t+N}$ are computed with (11) once $\tilde{y}^{[i]}_{t+N}$ is given.

E. Convergence properties

The convergence properties of the proposed distributed MPC algorithm for tracking can now be summarized in the following result (the proof is reported in [2]).

Theorem 1: Let Assumptions (2)-(4) be verified and the tuning parameters be selected as described in sections (III-D.1), (III-D.2). Then if at time $t = 0$ a feasible solution to (34), (30) exists for all $i = 1,\dots,M$, the resulting MPC controller asymptotically steers the i-th system to the admissible set-point $y^{[i]}_{feas.set-point}$, while respecting the constraints $(x^{[i]}_t, u^{[i]}_t) \in \mathbb{X}_i \times \mathbb{U}_i$ for all $t \geq 0$ and for all $i = 1,\dots,M$, where $y^{[i]}_{feas.set-point}$ is the solution of

$$y^{[i]}_{feas.set-point} = \underset{y^{[i]}\in\mathbb{Y}_i}{\operatorname{argmin}} \|y^{[i]} - y^{[i]}_{set-point}\|^2_{T_i} \tag{42}$$

IV. SIMULATION EXAMPLE

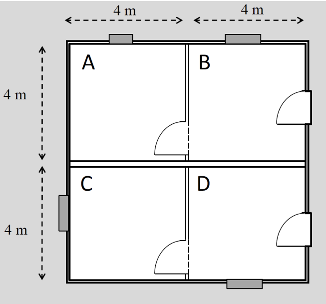

Fig. 2. Schematic representation of a building with two apartments

Consider the problem of controlling the temperature of the apartment depicted in Figure 2 and constituted by two parts both with two rooms: rooms A and B belong to the first one, while rooms C and D to the second one. Each room is characterized by its own temperature (T_A, T_B, T_C and T_D) and is endowed with its own radiator (supplying heats q_A, q_B, q_C and q_D). Heat exchanges are possible between rooms A and C, B and D (with transmittance coefficient $k^t_1 = 1\,W/m^2K$), between rooms A and B, C and D (with transmittance coefficient $k^t_2 = 2.5\,W/m^2K$) and between the rooms and the external environment (with transmittance coefficient $k^t_e = 0.5\,W/m^2K$), where the temperature is $T_E = 0\,°C$. For simplicity we neglect solar radiation. The rooms have volume $V = 48\,m^3$. Furthermore, the wall surfaces between the rooms are equal to $s_r = 12\,m^2$, while those between the rooms and the environment are equal to $s_e = 24\,m^2$. The considered nominal working point is $q_i = \bar{q} = 20 s_e k^t_e\,W$, with $T_i = \bar{T} = 20\,°C$, $i = A,B,C,D$. Let $\delta T_i = T_i - \bar{T}$ and $\delta q_i = (q_i - \bar{q})/c\rho V$, $i = A,B,C,D$. In this way, denoting $\sigma_1 = s_r k^t_1/c\rho V$, $\sigma_2 = s_r k^t_2/c\rho V$, $\sigma_e = s_e k^t_e/c\rho V$, $\sigma_{ol} = \sigma_1 + \sigma_2 + \sigma_e$, $\mathbf{x} = (\delta T_A, \delta T_B, \delta T_C, \delta T_D)$ and $\mathbf{u} = (\delta q_A, \delta q_B, \delta q_C, \delta q_D)$, the continuous-time model is $\dot{\mathbf{x}}(t) = \mathbf{A_c}\mathbf{x}(t) + \mathbf{B_c}\mathbf{u}(t)$ where $\mathbf{B_c} = I_4$ and

$$\mathbf{A_c} = \begin{bmatrix} -\sigma_{ol} & \sigma_2 & \sigma_1 & 0 \\ \sigma_2 & -\sigma_{ol} & 0 & \sigma_1 \\ \sigma_1 & 0 & -\sigma_{ol} & \sigma_2 \\ 0 & \sigma_1 & \sigma_2 & -\sigma_{ol} \end{bmatrix}$$

The discrete-time system is obtained by zero-order-hold discretization with sampling time $T = 30\,s$. The partition of inputs, outputs and states is $x^{[1]} = (x_1,x_2)$, $u^{[1]} = (u_1,u_2)$, $y^{[1]} = (y_1,y_2)$, $x^{[2]} = (x_3,x_4)$, $u^{[2]} = (u_3,u_4)$, and $y^{[2]} = (y_3,y_4)$. The matrices K_i and $\mathscr{G}_i$ fulfilling Assumption 1 and Assumption 2 have been computed as suggested in Remark 1 and $Q_1 = Q_2 = I_2$, $R_1 = R_2 = I_2$, $T_1 = T_2 = I_2$, $\gamma = 10^{-6}$, $N = 3$. In the simulations, the reference trajectories for $y^{[2]}_{set-point}$ are both always equal to zero, as well as the one related to T_B, while the first output of the first subsystem, T_A, should track a piece-wise constant reference trajectory, which values are 2 and -1. The results achieved are depicted in Figure 3, while the trajectories of the input variables are shown in Figure 4.

In both these figures, a comparison between the outputs obtained with DPC and with centralized MPC is provided: it is possible to see that the transients obtained with DPC are slower, due to the limitation imposed to the set-point variations by constraint (31).

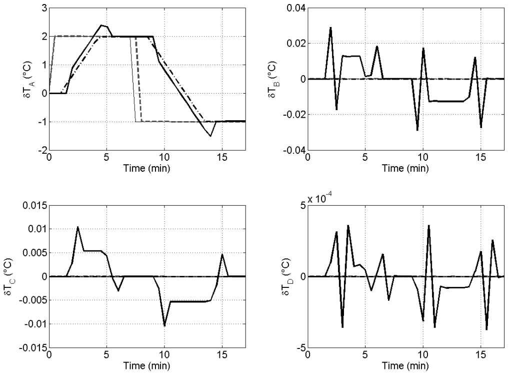

Fig. 3. Trajectories of the output variables $y^{[1]}$ (above) and $y^{[2]}$ (below) obtained with DPC (black solid lines) and with cMPC (dashed gray lines). Think black lines: desired set-points $y^{[1,2]}_{set-point}$; black dash-dot lines: reference trajectories $\tilde{y}^{1,2}$.

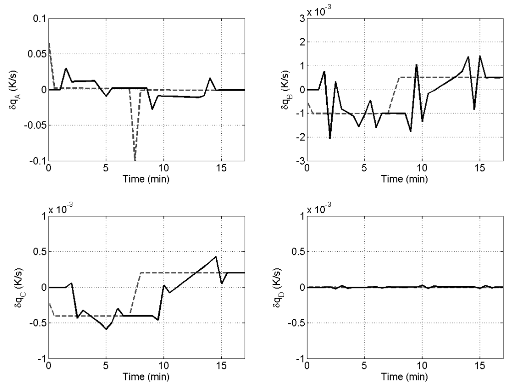

Fig. 4. Trajectories of the inputs variables $u^{[1]}$ (above) and $u^{[2]}$ (below) obtained with DPC (black solid lines) and with cMPC (dashed gray lines).

V. CONCLUSIONS

A new distributed predictive control algorithm for tracking constant output reference signals has been presented. The proposed method is based on the reformulation of the control problem in a robustness framework and can also deal with unfeasible set-points. Its convergence properties have been established and a simulation example has been reported to discuss its potentialities and limitations. To this regard, it has been shown how some of the constraints included in the distributed optimization problems to guarantee convergence can limit the overall tracking performance and the set of reachable reference signals. Therefore, further work is required to weaken these constraints.

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