2013 European Control Conference (ECC)
July 17-19, 2013, Zürich, Switzerland.

A $\mathcal{H}_\infty$ approach to robust fault estimation of non-linear discrete-time systems

Marcin Witczak Józef Korbicz Rafał Józefowicz

Abstract— **The paper deals with the problem of robust fault estimation of non-linear discrete-time systems. In particular, it is shown how to employ the unknown input observer approach and the $\mathcal{H}_\infty$ strategy to design a robust fault estimation filter. The proposed approach is designed in such a way that a prescribed disturbance attenuation level is achieved with respect to the fault estimation error while guaranteeing the convergence of the observer. The resulting design procedure is relatively simple and boils down to solving a set of linear matrix inequalities, which can be efficiently achieved with modern computational packages. The final part of the paper presents an illustrative example which exhibits the performance of the proposed approach.**

I. Introduction

A number of books was published in the last decade on the emerging problem of Fault-Tolerant Control (FTC). In particular, the book of [1], which is mainly devoted to fault diagnosis and its applications provides some general rules for the hardware-redundancy-based FTC. On the contrary, [2] introduce the concepts of achieving a passive FTC. They also investigate the problem of performance and stability of FTC under imperfect fault diagnosis. In particular, they consider (under a chain of some, not necessary easy to satisfy assumptions) the effect of a delayed fault detection and imperfect fault identification but the fault diagnosis scheme is treated separately during the design and no real integration of fault diagnosis and FTC is proposed. The FTC is also treated in a very interesting work of [3] where the number of practical case studies of FTC is presented, i.e. a winding machine, a three-tank system, and an active suspension system. Unfortunately, in spite of the incontestable appeal of the proposed approaches neither the FTC integrated with the fault diagnosis nor a systematic approach to non-linear systems are studied. A particular case of non-linear aircraft model is studied by [4] but the above-mentioned integration problem is also neglected.

Nevertheless, there is no doubt that an efficient fault diagnosis [5], [6], [7], [8], [9], [10] and FTC can be achieved only when the three-phase diagnostic procedure is achieved, i.e. fault detection, fault isolation and fault identification.

In this paper a robust fault estimation approach is proposed, which can be efficiently applied to realise the above-mentioned three-step procedure. The proposed approach is based on the general idea of an Unknown Input Observer (UIO) [11], [6], which was initially designed to tolerate a degree of model uncertainty and hence increase the reliability of fault diagnosis. Although the origins of UIOs can be traced back to the early 1970's (cf. the seminal work of Wang et al. [12]) the problem of designing such observers is still of paramount importance both from the theoretical and practical viewpoints. A large amount of knowledge on using these techniques for model-based fault diagnosis has been accumulated through the literature for the last three decades (see [6] and the references therein). Generally, the design problems regarding UIOs for non-linear systems can be divided into the three distinct categories:

- *nonlinear state-transformation-based techniques:* apart from a relatively large class of systems for which they can be applied, even if the nonlinear transformation is possible it leads to another nonlinear system and hence the observer design problem remains open (see [11] and the references therein).
- *linearization-based techniques:* such approaches are based on a similar strategy as that for the Extended Kalman filter (EKF) [5]. In [6] the author proposed an extended unknown input observer for non-linear systems. He also proved that the proposed observer is convergent under certain conditions.
- *observers for particular classes of nonlinear systems:* for example Unknown Input Observers for polynomial and bilinear systems or UIOs for Lipschitz systems [13], [14], [6], [15]).

In this paper, the UIO approach is employed the decouple the effect of a fault from the state estimation error as well as to provide its estimate. Whilst the robustness is achieved through the $\mathcal{H}_\infty$ approach [16]. In particular, the proposed approach is dedicated to a wide class of non-linear systems, which will be subsequently described. In the solutions presented in the literature, regarding the $\mathcal{H}_\infty$ a prescribed disturbance attenuation is achieved with respect to the state estimation error. In this paper the robust observer is designed in such a way that the prescribed disturbance attenuation level is achieved with respect to a fault estimation error while guaranteeing the convergence of the observer.

The paper is organised as follows. Section II presents an introductory background as well as it formulates the problem being undertaken. In Section III, the robust fault estimation approach is proposed while Section IV presents an illustrative example, which empirically exhibits its performance. Finally, the last section concludes the paper.

M. Witczak, J. Korbicz and R. Józefowicz are with the Institute of Control and Computation Engineering, University of Zielona Góra, ul. Podgórna 50, 65–246 Zielona Góra, Poland {M.Witczak,J.Korbicz}@issi.uz.zgora.pl The work was financed as a research project with the science funds for years 2011-2014.

II. Preliminaries

Let us consider a non-linear discrete-time system

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{u}_k + \boldsymbol{g}(\boldsymbol{x}_k) + \boldsymbol{L}\boldsymbol{f}_k + \boldsymbol{W}\boldsymbol{w}_k, \tag{1}$$
$$\boldsymbol{y}_{k+1} = \boldsymbol{C}\boldsymbol{x}_{k+1}, \tag{2}$$

where $\boldsymbol{x}_k \in \mathbb{X} \subset \mathbb{R}^n$ is the state, $\boldsymbol{u}_k \in \mathbb{R}^r$ stands for the input, $\boldsymbol{y}_k \in \mathbb{R}^m$ denotes the output, $\boldsymbol{f}_k \in \mathbb{R}^s$ stands for the fault $\boldsymbol{w}_k \in l_2^n$ is a an exogenous disturbance vector and $\boldsymbol{W} \in \mathbb{R}^{n\times n}$ stands for its distribution matrix while:

$$l_s^n = \left\{\mathbf{w} \in \mathbb{R}^n \mid \|\mathbf{w}\|_{l_2^n} < +\infty\right\}, \tag{3}$$
$$\|\mathbf{w}\|_{l_2^n} = \left(\sum_{k=0}^{\infty} \|\boldsymbol{w}_k\|^2\right)^{\frac{1}{2}}. \tag{4}$$

Moreover, $\boldsymbol{g}(\boldsymbol{x})$ is a non-linear function satisfying [17], [18]:

$$\begin{aligned}(\boldsymbol{g}(\boldsymbol{x}_1) - \boldsymbol{g}(\boldsymbol{x}_2))^T (\boldsymbol{g}(\boldsymbol{x}_1) - \boldsymbol{g}(\boldsymbol{x}_2)) \quad &\leq \\ (\boldsymbol{x}_1 - \boldsymbol{x}_2)^T \boldsymbol{M}^T\boldsymbol{M}(\boldsymbol{x}_1 - \boldsymbol{x}_2), \ \forall \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{X}\end{aligned} \tag{5}$$

where $\boldsymbol{M} \in \mathbb{R}^{n\times n}$. In the subsequent part of this section it will be shown how to deal with such a system representation.

Theorem 1. Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$. There are vectors $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n \in \text{Co}(\boldsymbol{a}, \boldsymbol{b})$, $\boldsymbol{c}_i \neq \boldsymbol{a}$, $\boldsymbol{c}_i \neq \boldsymbol{b}$, $i = 1, \ldots, n$ such that:

$$\boldsymbol{g}(\boldsymbol{a}) - \boldsymbol{g}(\boldsymbol{b}) = \boldsymbol{M}_x(\boldsymbol{a} - \boldsymbol{b}), \tag{6}$$

with

$$\boldsymbol{M}_x = \sum_{i,j=1}^{n,n} \boldsymbol{e}_{n,i}\boldsymbol{e}_{n,j}^T \frac{\partial g_i}{\partial x_i}(\boldsymbol{c}_i) \tag{7}$$

where

$$\mathbb{E}_n = \{\boldsymbol{e}_{n,i} \mid \boldsymbol{e}_{n,i} = [0, \ldots, 0, 1, 0, \ldots, 0]^T, i = 1, \ldots, n\} \tag{8}$$

is the canonical basis of the vectorial space $\mathbb{R}^n$.

Proof. See [15]. Assuming that

$$\bar{a}_{i,j} \geq \frac{\partial g_i}{\partial x_j} \geq \underline{a}_{i,j}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, n, \tag{9}$$

and observing that $\boldsymbol{M}_x$ can be described in an alternative form

$$\boldsymbol{M}_x = \begin{bmatrix} \frac{\partial g_1}{\partial x}(\boldsymbol{c_1}) \\ \vdots \\ \frac{\partial g_n}{\partial x}(\boldsymbol{c_n}) \end{bmatrix}, \tag{10}$$

it is clear that there exists a matrix $\boldsymbol{M}$ bounded with (cf. (9))

$$\bar{a}_{i,j} \geq m_{i,j} \geq \underline{a}_{i,j}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, n, \tag{11}$$

for which $\boldsymbol{M}_x^T\boldsymbol{M}_x \preceq \boldsymbol{M}^T\boldsymbol{M}$. In order to find the upper bound $\boldsymbol{M}^T\boldsymbol{M}$, it is assumed that $\boldsymbol{M}^T\boldsymbol{M} \prec \gamma^2\boldsymbol{I}$ ($\gamma > 0$), which can be expressed (using the Schur complements) by

$$\lambda^* = \max_{\boldsymbol{M}} \min_{\lambda} \lambda, \tag{12}$$
$$\begin{bmatrix} \mathbf{0} & -\boldsymbol{M}^T \\ -\boldsymbol{M} & -I \end{bmatrix} \prec \lambda \begin{bmatrix} \boldsymbol{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{13}$$

with $\lambda = -\gamma^2$. Thus, the task can be settled using the solutions to the generalised eigenvalue optimisation problem [19], i.e. to find the minimal λ under the constraints (11) and (13).

Thus, (5) can be replaced

$$\begin{aligned}&(\boldsymbol{g}(\boldsymbol{x}_1) - \boldsymbol{g}(\boldsymbol{x}_2))^T (\boldsymbol{g}(\boldsymbol{x}_1) - \boldsymbol{g}(\boldsymbol{x}_2)) \leq \\ &\leq \gamma^2 (\boldsymbol{x}_1 - \boldsymbol{x}_2)^T (\boldsymbol{x}_1 - \boldsymbol{x}_2), \ \forall \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{X}\end{aligned} \tag{14}$$

Since the system structure is given then it is possible to develop the fault identification procedure. To tackle this problem, the system (1)–(2) will be transformed into an equivalent one without an unknown input.

III. Fault estimation approach

Let us assume that the system is observable and the following rank conditions are satisfied:

$$\text{rank}(\boldsymbol{CL}) = \text{rank}(\boldsymbol{L}) = s \tag{15}$$

Under the assumption (15) it is possible to calculate $\boldsymbol{H} = (\boldsymbol{CL})^+ = \left[(\boldsymbol{CL})^T\boldsymbol{CL}\right]^{-1}(\boldsymbol{CL})^T$. By multiplying (2) by $\boldsymbol{H}$ and then substituting (1), it can be shown that

$$\boldsymbol{f}_k = \boldsymbol{H}(\boldsymbol{y}_{k+1} - \boldsymbol{CA}\boldsymbol{x}_k - \boldsymbol{CB}\boldsymbol{u}_k - \boldsymbol{Cg}(\boldsymbol{x}_k) - \boldsymbol{CW}\boldsymbol{w}_k). \tag{16}$$

Finally, by substituting (16) into (1) it can be shown that:

$$\boldsymbol{x}_{k+1} = \bar{\boldsymbol{A}}\boldsymbol{x}_k + \bar{\boldsymbol{B}}\boldsymbol{u}_k + \boldsymbol{Gg}(\boldsymbol{x}_k) + \bar{\boldsymbol{L}}\boldsymbol{y}_{k+1} + \bar{\boldsymbol{W}}\boldsymbol{w}_k, \tag{17}$$

where

$$\begin{aligned}&\boldsymbol{G} = (\boldsymbol{I}_n - \boldsymbol{LHC}), \bar{\boldsymbol{A}} = \boldsymbol{GA}, \\ &\bar{\boldsymbol{B}} = \boldsymbol{GB}, \bar{\boldsymbol{L}} = \boldsymbol{LH}, \bar{\boldsymbol{W}} = \boldsymbol{GW}.\end{aligned}$$

In order to estimate (16), i.e. to obtain $\hat{\boldsymbol{f}}_k$ it is necessary to estimate the state of the system, i.e. to obtain $\hat{\boldsymbol{x}}_k$. Consequently, the fault estimate is given as follows

$$\hat{\boldsymbol{f}}_k = \boldsymbol{H}(\boldsymbol{y}_{k+1} - \boldsymbol{CA}\hat{\boldsymbol{x}}_k - \boldsymbol{CB}\boldsymbol{u}_k - \boldsymbol{Cg}(\hat{\boldsymbol{x}}_k)). \tag{18}$$

The corresponding observer structure is

$$\hat{\boldsymbol{x}}_{k+1} = \bar{\boldsymbol{A}}\hat{\boldsymbol{x}}_k + \bar{\boldsymbol{B}}\boldsymbol{u}_k + \boldsymbol{Gg}(\hat{\boldsymbol{x}}_k) + \bar{\boldsymbol{L}}\boldsymbol{y}_{k+1} + \boldsymbol{K}(\boldsymbol{y}_k - \boldsymbol{C}\hat{\boldsymbol{x}}_k), \tag{19}$$

while the state estimation error is given by

$$\begin{aligned}\boldsymbol{e}_{k+1} &= \left(\bar{\boldsymbol{A}} - \boldsymbol{KC}\right)\boldsymbol{e}_k + \boldsymbol{Gs}_k + \bar{\boldsymbol{W}}\boldsymbol{w}_k = \\ &= \boldsymbol{A}_1\boldsymbol{e}_k + \boldsymbol{Gs}_k + \bar{\boldsymbol{W}}\boldsymbol{w}_k,\end{aligned} \tag{20}$$

where

$$\boldsymbol{s}_k = \boldsymbol{g}(\boldsymbol{x}_k) - \boldsymbol{g}(\hat{\boldsymbol{x}}_k). \tag{21}$$

Similarly, the fault estimation error $\boldsymbol{\varepsilon}_{f,k}$ can be defined

$$\boldsymbol{\varepsilon}_{f,k} = \boldsymbol{f}_k - \hat{\boldsymbol{f}}_k = -\boldsymbol{HC}(\boldsymbol{Ae}_k + \boldsymbol{s}_k + \boldsymbol{Ww}_k). \tag{22}$$

The objective of further deliberations is to design the observer (19) in such a way that the state estimation error $\boldsymbol{e}_k$ is asymptotically convergent and the following upper bound is guaranteed

$$\|\boldsymbol{\varepsilon}_f\|_{l_2^n} \leq \mu\|\mathbf{w}\|_{l_2^n} \tag{23}$$

where $\mu > 0$ is a prescribed disturbance attenuation level.

Thus, the problem of $\mathcal{H}_\infty$ observer design [16], [18] is to determine the gain matrix $\boldsymbol{K}$ such that

$$\lim_{k\to\infty} \boldsymbol{e}_k = \boldsymbol{0} \quad \text{for}\, \boldsymbol{w}_k = \boldsymbol{0} \tag{24}$$

$$\|\boldsymbol{\varepsilon}_f\|_{l_2^n} \le \mu \|\mathbf{w}\|_{l_2^n} \quad \text{for}\, \boldsymbol{w}_k \neq \boldsymbol{0},\, \boldsymbol{e}_0 = \boldsymbol{0}. \tag{25}$$

In order to settle the above problem it is sufficient to find a Lyapunov function V_k such that:

$$\Delta V_k + \boldsymbol{\varepsilon}_{f,k}^T \boldsymbol{\varepsilon}_{f,k} - \mu^2 \boldsymbol{w}_k^T \boldsymbol{w}_k < 0,\, k = 0, \ldots \infty, \tag{26}$$

where $\Delta V_k = V_{k+1} - V_k$. Indeed, if $\boldsymbol{w}_k = \boldsymbol{0}$ then (26) boils down to

$$\Delta V_k + \boldsymbol{\varepsilon}_{f,k}^T \boldsymbol{\varepsilon}_{f,k} < 0,\, k = 0, \ldots \infty, \tag{27}$$

and hence $\Delta V_k < 0$, which leads to (24). If $\boldsymbol{w}_k \neq \boldsymbol{0}$ then (26) yields

$$J = \sum_{k=0}^{\infty} \left(\Delta V_k + \boldsymbol{\varepsilon}_{f,k}^T \boldsymbol{\varepsilon}_{f,k} - \mu^2 \boldsymbol{w}_k^T \boldsymbol{w}_k \right) < 0, \tag{28}$$

which can be written as

$$J = V_\infty - V_0 + \sum_{k=0}^{\infty} \boldsymbol{\varepsilon}_{f,k}^T \boldsymbol{\varepsilon}_{f,k} - \sum_{k=0}^{\infty} \mu^2 \boldsymbol{w}_k^T \boldsymbol{w}_k < 0, \tag{29}$$

Knowing that $V_0 = 0$ for $\boldsymbol{e}_0 = 0$ and $V_\infty \ge 0$, (29) leads to (25).

Since the general framework for designing the robust observer is given, then the following form of the Lyapunov function is proposed

$$V_k = \boldsymbol{e}_k^T \boldsymbol{P} \boldsymbol{e}_k + \boldsymbol{s}_k^T \boldsymbol{Q} \boldsymbol{s}_k, \tag{30}$$

where $\boldsymbol{P} \succ \boldsymbol{0}$ and $\boldsymbol{P} \succ \boldsymbol{0}$.

Consequently,

$$\begin{aligned}
&\Delta V_k + \boldsymbol{\varepsilon}_{f,k}^T \boldsymbol{\varepsilon}_{f,k} - \mu^2 \boldsymbol{w}_k^T \boldsymbol{w}_k = \\
&\boldsymbol{e}_k^T \left(\boldsymbol{A}_1^T \boldsymbol{P} \boldsymbol{A}_1 - \boldsymbol{P} - \boldsymbol{A}^T \bar{\boldsymbol{H}} \boldsymbol{A} \right) \boldsymbol{e}_k + \\
&\boldsymbol{e}_k^T \left(\boldsymbol{A}_1^T \boldsymbol{P} \boldsymbol{G} + \boldsymbol{A}^T \bar{\boldsymbol{H}} \right) \boldsymbol{s}_k + \\
&\boldsymbol{e}_k^T \left(\boldsymbol{A}_1^T \boldsymbol{P} \boldsymbol{G} \boldsymbol{W} + \boldsymbol{A}^T \bar{\boldsymbol{H}} \boldsymbol{W} \right) \boldsymbol{w}_k + \\
&\boldsymbol{s}_k^T \left(\boldsymbol{G}^T \boldsymbol{P} \boldsymbol{A}_1 - \bar{\boldsymbol{H}} \boldsymbol{A} \right) \boldsymbol{e}_k + \\
&\boldsymbol{s}_k^T \left(\boldsymbol{G}^T \boldsymbol{P} \boldsymbol{G} + \bar{\boldsymbol{H}} - \boldsymbol{Q} \right) \boldsymbol{s}_k + \\
&\boldsymbol{s}_k^T \left(\boldsymbol{G}^T \boldsymbol{P} \boldsymbol{G} \boldsymbol{W} + \bar{\boldsymbol{H}} \boldsymbol{W} \right) \boldsymbol{w}_k + \\
&\boldsymbol{w}_k^T \left(\boldsymbol{W}^T \boldsymbol{G}^T \boldsymbol{P} \boldsymbol{A}_1 + \boldsymbol{W}^T \bar{\boldsymbol{H}} \boldsymbol{A} \right) \boldsymbol{e}_k + \\
&\boldsymbol{w}_k^T \left(\boldsymbol{W}^T \boldsymbol{G}^T \boldsymbol{P} \boldsymbol{G} + \boldsymbol{W}^T \bar{\boldsymbol{H}} \right) \boldsymbol{s}_k + \\
&\boldsymbol{w}_k^T \left(\boldsymbol{W}^T \boldsymbol{G}^T \boldsymbol{P} \boldsymbol{G} \boldsymbol{W} + \boldsymbol{W}^T \bar{\boldsymbol{H}} \boldsymbol{W} - \mu^2 \boldsymbol{I}_n \right) \boldsymbol{w}_k + \\
&- \boldsymbol{s}_{k+1}^T \boldsymbol{Q} \boldsymbol{s}_{k+1},
\end{aligned} \tag{31}$$

with $\bar{\boldsymbol{H}} = \boldsymbol{C}^T \boldsymbol{H}^T \boldsymbol{H} \boldsymbol{C}$. By defining

$$\boldsymbol{v}_k = \left[\boldsymbol{e}_k^T,\, \boldsymbol{s}_k^T,\, \boldsymbol{w}_k^T,\, \boldsymbol{s}_{k+1}^T \right]^T \tag{32}$$

inequality (31) becomes

$$\Delta V_k + \boldsymbol{\varepsilon}_{f,k}^T \boldsymbol{\varepsilon}_{f,k} - \mu^2 \boldsymbol{w}_k^T \boldsymbol{w}_k = \boldsymbol{v}_k^T \boldsymbol{M}_V \boldsymbol{v}_k, \tag{33}$$

where $\boldsymbol{M}_V$ is given by (34). Additionally, form (5)

$$\beta \boldsymbol{e}_k \boldsymbol{M}^T \boldsymbol{M} \boldsymbol{e}_k - \beta \boldsymbol{s}_k^T \boldsymbol{s}_k \ge 0, \quad \beta > 0, \tag{35}$$

which is equivalent to

$$\boldsymbol{v}_k^T \begin{bmatrix} \beta \boldsymbol{M}^T \boldsymbol{M} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & -\beta \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \boldsymbol{v}_k \ge 0 \tag{36}$$

Similarly, from (5) and by assuming

$$\boldsymbol{P} \succ \alpha \boldsymbol{M}^T \boldsymbol{M}, \tag{37}$$

it can be shown that

$$\begin{aligned} &\alpha \boldsymbol{e}_{k+1}^T \boldsymbol{M}^T \boldsymbol{M} \boldsymbol{e}_{k+1} - \alpha \boldsymbol{s}_{k+1}^T \boldsymbol{s}_{k+1} < \\ &\boldsymbol{e}_{k+1}^T \boldsymbol{P} \boldsymbol{e}_{k+1} - \alpha \boldsymbol{s}_{k+1}^T \boldsymbol{s}_{k+1} \ge 0, \end{aligned} \tag{38}$$

which is equivalent to

$$\boldsymbol{v}_k^T \boldsymbol{M}_Y \boldsymbol{v}_k \ge 0, \tag{39}$$

with

$$\boldsymbol{M}_Y = \begin{bmatrix} \boldsymbol{A}_1^T \boldsymbol{P} \boldsymbol{A}_1 & \boldsymbol{A}_1^T \boldsymbol{P} \boldsymbol{G} & \boldsymbol{A}_1^T \boldsymbol{P} \boldsymbol{G} \boldsymbol{W} & \boldsymbol{0} \\ \boldsymbol{G}^T \boldsymbol{P} \boldsymbol{A}_1 & \boldsymbol{G}^T \boldsymbol{P} \boldsymbol{G} & \boldsymbol{G}^T \boldsymbol{P} \boldsymbol{G} \boldsymbol{W} & \boldsymbol{0} \\ \boldsymbol{W}^T \boldsymbol{G}^T \boldsymbol{P} \boldsymbol{A}_1 & \boldsymbol{W}^T \boldsymbol{G}^T \boldsymbol{P} \boldsymbol{G} & \boldsymbol{W}^T \boldsymbol{G}^T \boldsymbol{P} \boldsymbol{G} \boldsymbol{W} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\alpha \boldsymbol{I} \end{bmatrix} \tag{40}$$

Finally, using (33) along with (36)–(39), the convergence condition of the observer becomes (41). Moreover, by applying the Schur complements, (41) can be transformed into (42). Multiplying (42), from both sites by

$$\text{diag}(\boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}, \boldsymbol{P}), \tag{43}$$

and then substituting

$$\boldsymbol{P} \boldsymbol{A}_1 = \boldsymbol{P} \bar{\boldsymbol{A}} - \boldsymbol{P} \boldsymbol{K} \boldsymbol{C} = \boldsymbol{P} \bar{\boldsymbol{A}} - \boldsymbol{N} \boldsymbol{C}, \tag{44}$$

yields (45).

Note that (45) is a usual Linear Matrix Inequality (LMI), which can be easily solved, e.g. with MATLAB. Thus, the final design procedure is: given a prescribed disturbance attenuation level μ, obtain $\alpha > 0$, $\beta > 0$, $\boldsymbol{P} \succ \boldsymbol{0}$, $\boldsymbol{Q} \succ \boldsymbol{0}$, $\boldsymbol{N}$ by solving (37) and (45). Finally, the gain matrix of (19) is:

$$\boldsymbol{K} = \boldsymbol{P}^{-1} \boldsymbol{N}. \tag{46}$$

It can be also observed that the observer design problem can be treated as an minimization task, i.e.

$$\mu^* = \min_{\mu > 0, \alpha > 0, \beta > 0, \boldsymbol{P} \succ \boldsymbol{0}, \boldsymbol{Q} \succ \boldsymbol{0}, \boldsymbol{N}} \mu \tag{47}$$

under (37) and (45).

$$M_V = \begin{bmatrix} A_1^T P A_1 - P + A^T \bar{H} A & A_1^T P G + A_1^T \bar{H} & A_1^T P G W + A_1^T \bar{H} W & 0 \\ G^T P A_1 + \bar{H} A & G^T P G + \bar{H} - Q & G^T P G W + \bar{H} W & 0 \\ W^T G^T P A_1 + W^T \bar{H} A & W^T G^T P G + W^T \bar{H} & W^T G^T P G W + W^T \bar{H} W - \mu^2 I & 0 \\ 0 & 0 & 0 & Q \end{bmatrix}, \tag{34}$$

$$\begin{bmatrix} 2A_1^T P A_1 - P + A^T \bar{H} A + \beta M^T M & 2A_1^T P G + A_1^T \bar{H} & 2A_1^T P G W + A_1^T \bar{H} W & 0 \\ 2G^T P A_1 + \bar{H} A & 2G^T P G + \bar{H} - Q - \beta I & 2G^T P G W + \bar{H} W & 0 \\ 2W^T G^T P A_1 + W^T \bar{H} A & 2W^T G^T P G + W^T \bar{H} & 2W^T G^T P G W + W^T \bar{H} W - \mu^2 I & 0 \\ 0 & 0 & 0 & Q - \alpha I \end{bmatrix} \prec 0, \tag{41}$$

$$\begin{bmatrix} -P + A^T \bar{H} A + \beta M^T M & A^T \bar{H} & A^T \bar{H} W & 0 & A_1^T \\ \bar{H} A & \bar{H} - Q - \beta I & \bar{H} W & 0 & G^T \\ W^T \bar{H} A & W^T \bar{H} & W^T \bar{H} W - \mu^2 I & 0 & W^T G^T \\ 0 & 0 & 0 & Q - \alpha I & 0 \\ A_1 & G & G W & 0 & -\frac{1}{2} P^{-1} \end{bmatrix} \prec 0. \tag{42}$$

$$\begin{bmatrix} -P + A^T \bar{H} A + \beta M^T M & A^T \bar{H} & A^T \bar{H} W & 0 & \bar{A}^T P - C^T N^T \\ \bar{H} A & \bar{H} - Q - \beta I & \bar{H} W & 0 & G^T P \\ W^T \bar{H} A & W^T \bar{H} & W^T \bar{H} W - \mu^2 I & 0 & W^T G^T P \\ 0 & 0 & 0 & Q - \alpha I & 0 \\ P \bar{A} - N C & P G & P G W & 0 & -\frac{1}{2} P \end{bmatrix} \prec 0. \tag{45}$$

IV. ILLUSTRATIVE EXAMPLE

Let us consider a non-linear system

$$x_{k+1} = A x_k + B u_k + g(x_k) + L f_k + W w_k, \tag{48}$$
$$y_{k+1} = C x_{k+1}, \tag{49}$$

with

$$A = \begin{bmatrix} 0.137 & 0.199 & 0.284 \\ 0.0118 & 0.299 & 0.47 \\ 0.894 & 0.661 & 0.065 \end{bmatrix}, \quad B = \begin{bmatrix} 0.25 \\ 0.6 \\ 0.1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad W = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}$$

and

$$g(x_k) = \left[\frac{0.6 \cos(12 x_{1,k})}{x_{2,k}^2 + 10},\ 0,\ -0.33 \sin(x_{3,k}) \right]^T. \tag{50}$$

Since the system is given, it is straightforward to calculate (10), and then

$$M_x^T M_x = \operatorname{diag}\left(0, \left(\frac{-7.2 \sin(12 x_{2,k})}{x_{2,k}^2 + 10} - \frac{1.2 \cos(12 x_{2,k}) x_{2,k}}{(x_{2,k}^2 + 10)^2} \right)^2, 0.1089 \cos(x_{3,k})^2 \right) \tag{51}$$

Finally, the application of the proposed procedure (cf. (12)–(13)) leads to

$$M^T M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.517 & 0 \\ 0 & 0 & 0.1089 \end{bmatrix} \tag{52}$$

As a result of solving the problem (47), the following couple were obtained:

$$\mu^* = 0.499, \quad K = \begin{bmatrix} 0.4885 & 2.0619 \\ 0 & 0 \\ 0.9724 & 1.0753 \end{bmatrix}. \tag{53}$$

Let the initial condition for the system and the observer be:

$$x_0 = [3, 2, 1]^T, \quad \hat{x}_0 = 0, \tag{54}$$

while the input and the exogenous disturbance are:

$$u_k = \sin(0.002 \pi k), \quad w_k \sim \mathcal{N}(0, 0.1^2 I). \tag{55}$$

Moreover, let us consider the following fault scenario:

$$f_k = \begin{cases} 1, & \text{for } 300 \geq k \geq 200, \\ 0, & \text{otherwise.} \end{cases} \tag{56}$$

First, let us consider the case when $\hat{x}_0 = x_0$ ($e_0 = 0$). Figure 1 clearly indicates that condition (25) is satisfied, which means that an attenuation level $\mu^* = 0.499$ is achieved. Now let use assume that $w_k = 0$ and $\hat{x}_0 \neq x_0$. Figure (2) clearly shows that (24) is satisfied as well. Finally, figure 3 shows the fault and its estimate for the nominal case ($\hat{x}_0 \neq x_0$ and $w_k = 0$). From these results, it can be observed that the proposed approach can be an efficient approach to solving

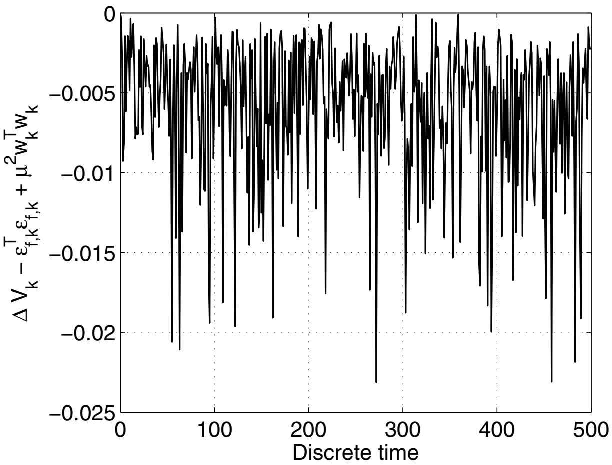

Fig. 1. Evolution of $\Delta V_k + \boldsymbol{\varepsilon}_{f,k}^T \boldsymbol{\varepsilon}_{f,k} - \mu^2 \boldsymbol{w}_k^T \boldsymbol{w}_k$

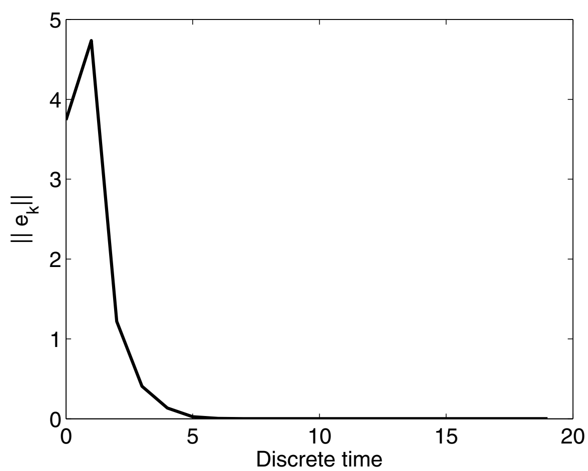

Fig. 2. Evolution of $\|\boldsymbol{e}_k\|$ (for $k = 0, \ldots, 20$)

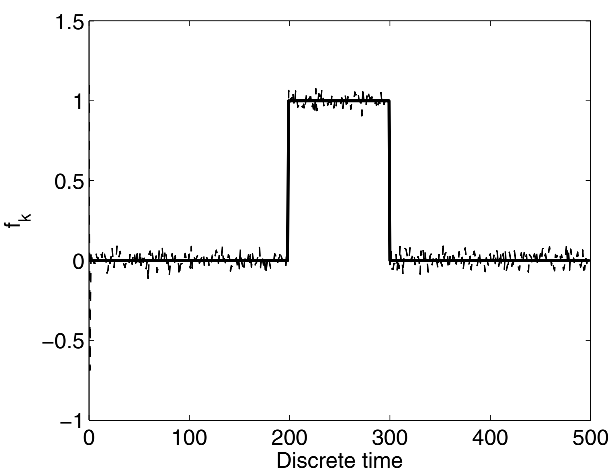

Fig. 3. Fault and its estimate

robust $\mathcal{H}_\infty$-based fault identification of non-linear discrete time systems.

V. Conclusions

The main objective of this paper was to propose a novel approach, which can be used for the estimation of an unknown fault signal for non-linear discrete-time stochastic systems. In particular, the fault estimation is achieved through the general UIO approach while robustness is attained with the $\mathcal{H}_\infty$ approach. In the usual $\mathcal{H}_\infty$ framework, the prescribed disturbance attenuation level is achieved with respect to the state estimation error. The proposed approach is designed in such a way that a prescribed disturbance attenuation level is achieved with respect to the fault estimation error while guaranteeing the convergence of the observer. The proposed design procedure is relatively simple and boils down to solving a set of linear matrix inequalities. The presented results clearly exhibit the performance of the proposed approach.

The main objective of further results is twofold:

1) to design a robust adaptive threshold for the fault estimated obtained with the proposed approach,
2) to design a robust fault-tolerant control scheme which takes into account the fault estimates along with their uncertainties.

References

[1] R. Iserman, *Fault Diagnosis Applications: Model Based Condition Monitoring, Actuators, Drives, Machinery, Plants, Sensors, and Fault-tolerant Systems*. Berlin: Springer-Verlag, 2011.

[2] M. Mahmoud, J. Jiang, and Y. Zhang, *Active Fault Tolerant Control Systems: Stochastic Analysis and Synthesis*. Berlin: Springer-Verlag, 2003.

[3] T. D. P. J. Noura, H. and A. Chamseddine, *Fault-tolerant Control Systems: Design and Practical Applications*. Berlin: Springer-Verlag, 2003.

[4] G. Ducard, *Fault-tolerant Flight Control and Guidance Systems: Practical Methods for Small Unmanned Aerial Vehicles*. Berlin: Springer-Verlag, 2009.

[5] J. Korbicz, J. Kościelny, Z. Kowalczuk, and W. Cholewa (Eds.), *Fault diagnosis. Models, Artificial Intelligence, Applications*. Berlin: Springer-Verlag, 2004.

[6] M. Witczak, *Modelling and Estimation Strategies for Fault Diagnosis of Non-linear Systems*. Berlin: Springer-Verlag, 2007.

[7] S. Ding, *Model-based fault diagnosis techniques: design schemes, algorithms and tools*. Berlin: Springer-Verlag, 2008.

[8] V. Puig, "Fault diagnosis and fault tolerant control using set-membership approaches: Application to real case studies." *International Journal of Applied Mathematics and Computer Science*, vol. 20, no. 4, pp. 619–635, 2010.

[9] K. Kemir, F. Ben Hmida, J. Ragot, and M. Gossa, "Novel optimal recursive filter for state and fault estimation of linear systems with unknown disturbances." *International Journal of Applied Mathematics and Computer Science*, vol. 21, no. 4, pp. 629–638, 2011.

[10] M. Luzar, A. Czajkowski, M. Witczak, and M. Mrugalski, "Actuators and sensors fault diagnosis with dynamic, state-space neural networks," in *Proc. Methods and Models in Automation and Robotics, MMAR*, Międzyzdroje, Poland, 2012.

[11] P. M. Frank and T. Marcu, "Diagnosis strategies and systems. principles, fuzzy and neural approaches." in *Intelligent Systems and Interfaces (Teodorescu H. N., Mlynek D., Kandel A. and Zimmermann H. J. (Eds.)*. Boston: Kluwer Academic Publishers, 2000.

[12] S. H. Wang, E. J. Davison, and P. Dorato, "Observing the states of systems with unmeasurable disturbances," *IEEE Transactions on Automatic Control*, vol. 20, no. 5, pp. 716–717, 1975.

[13] D. Koenig and S. Mammar, "Design of a class of reduced unknown inputs non-linear observer for fault diagnosis," in *Proc. American Control Conference, ACC*, Arlington, USA, 2002.

[14] A. M. Pertew, H. J. Marquez, and Q. Zhao, "$\mathcal{H}_\infty$ synthesis of unknown input observers for non-linear lipschitz systems," *International Journal of Control*, vol. 78, no. 15, pp. 1155–1165, 2005.

[15] A. Zemouche and M. Boutayeb, "Observer design for Lipschitz nonlinear systems: the discrete time case," *IEEE Trans. Circuits and Systems - II:Express Briefs*, vol. 53, no. 8, pp. 777–781, 2006.

[16] H. Li and M. Fu, "A linear matrix inequality approach to robust h_∞ filtering," *IEEE Trans. Signal Processing*, vol. 45, no. 9, pp. 2338–2350, 1997.

[17] D. Stipanovic and D. Siljak, "Robust stability and stabilization of discrete-time non-linear: the lmi approach," *International Journal of Control*, vol. 74, no. 5, pp. 873–879, 2001.

[18] A. Zemouche, M. Boutayeb, and G. Iulia Bara, "Observer for a class of Lipschitz systems with extension to $\mathcal{H}_\infty$ performance analysis," *Systems and Control Letters*, vol. 57, no. 1, pp. 18–27, 2008.

[19] S. Boyd, E. Feron, L. E. Ghaoui, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory.* Philadelphia: Siam, 1994.