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Solution of a Singular H_∞ Control Problem for Linear Systems with State Delays

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***Abstract*— An H_∞ control problem for linear systems with point-wise and distributed state delays is considered. The case, where the output equation does not contain the control variable, is treated. In this case, the problem is singular, i.e., the game-theoretic Riccati equation approach is not applicable to its solution. A method of solution of this problem, based on its regularization and asymptotic solution of the regularized problem, is proposed.**

I. INTRODUCTION

Controlled systems with disturbances (uncertainties) in dynamics are extensively studied in the literature (see e.g. [1], [2], [3], [4] and references therein). Two classes of disturbances are usually distinguished: (1) disturbances belonging to a known bounded set of Euclidean space; (2) quadratically integrable disturbances. For controlled systems with quadratically integrable disturbances, the H_∞ control problem is frequently considered. The H_∞ control problem has been studied for systems without and with delays in the state variables (see e.g. [2], [3], [5], [6], [7]).

If the rank of the matrix of coefficients for the control variable in the output equation equals to the dimension of the control, then the solution of the H_∞ control problem can be reduced to a solution of a game-theoretic Riccati equation. In the case of un-delayed systems, the Riccati equation is finite dimensional, while in the case of delayed systems it is infinite dimensional. The infinite dimensional Riccati equation can be reduced to a hybrid set of three matrix equations of Riccati type: algebraic, ordinary differential and first-order partial differential ones.

If the rank of the matrix of coefficients for the control in the output is smaller then the dimension of the control, then the mentioned above game-theoretic Riccati equation approach to the solution of H_∞ problem is failed, because in this case the corresponding Riccati equation does not exists. Such H_∞ control problems are called singular or nonstandard. For un-delayed dynamics, singular H_∞ control problems were studied in the literature (see e.g [8], [9], [10] for linear systems, [11] for nonlinear systems, and references therein).

In the present paper, we consider a singular H_∞ control problem for a class of systems with delays in the state variables. A regularization of this problem is proposed leading to a new - H_∞ cheap control problem. The latter is solved by adapting a singular perturbation technique. Then, it is shown on how accurately the controller, solving the H_∞ cheap control problem, solves the original singular H_∞ control problem.

The following main notations are used in the paper: (1) E^n is the n-dimensional real Euclidean space, $\|\cdot\|$ is the norm in this space; (2) $L^2[0,+\infty;,E^n]$ is the space of n-dimensional vector-functions quadratically integrable on the interval $[0,+\infty)$, $\|\cdot\|_{L^2}$ is the norm in this space; (3) $\mathrm{col}(x,y)$, where $x \in E^n$, $y \in E^m$, denotes the column block-vector with upper block x and lower block y; (4) I_n is the n-dimensional identity matrix; (5) the superscript prime denotes the transposition of a matrix A, $(A^{'})$ and a vector x, $(x^{'})$.

II. PROBLEM STATEMENT

The following controlled system with point-wise and distributed time delays in the state variable is considered:

$$\frac{dZ(t)}{dt} = \mathcal{A}Z(t) + \mathcal{H}Z(t-h) + \int_{-h}^{0} \mathcal{G}(\tau)Z(t+\tau)d\tau + \mathcal{B}u(t) + \mathcal{F}w(t), \tag{1}$$

$$\zeta(t) = \mathrm{col}\{\mathcal{C}Z(t),\ \mathcal{M}u(t)\}, \tag{2}$$

where $t \geq 0$, $Z(t) \in E^n$, $u(t) \in E^r$, $(n \geq r)$, (u is a control);, $w(t) \in E^q$, ($w(t)$ is a disturbance); $\zeta(t) \in E^p$, ($\zeta(t)$ is an output); $h > 0$ is a given constant time delay; $\mathcal{A}$, $\mathcal{H}$, $\mathcal{B}$, $\mathcal{F}$, $\mathcal{C}$, $\mathcal{M}$ are given constant matrices and $\mathcal{G}(\tau)$ is a given matrix-valued function of corresponding dimensions.

Assuming that $w(t) \in L^2[0,+\infty;E^q]$, let us consider the following functional:

$$\mathcal{J}(u,w) = \Big(\|\zeta(t)\|_{L^2}\Big)^2 - \gamma^2\Big(\|w(t)\|_{L^2}\Big)^2, \tag{3}$$

where $\gamma > 0$ is a given constant.

The H_∞ control problem with a performance level γ for the system (1)-(2) is to find a controller $u^*[Z(\cdot)](t)$ that ensures the inequality $\mathcal{J}(u^*,w) \leq 0$ along trajectories of (1) for all $w(t) \in L^2[0,+\infty;E^q]$ and for $Z(t) = 0,\ t \leq 0$.

Consider the matrices

$$\mathcal{D} = \mathcal{C}^{'}\mathcal{C}, \quad \mathcal{N} = \mathcal{M}^{'}\mathcal{M}. \tag{4}$$

It is seen that the matrices $\mathcal{D}$ and $\mathcal{N}$ are symmetric and at least positive semi-definite. Moreover, if rank$\mathcal{M} = r$, then the matrix $\mathcal{N}$ is positive definite. In such a case, we can write down the following hybrid set of Riccati-type matrix algebraic, ordinary and partial differential equations with deviating arguments for the matrices $\mathcal{P}$, $\mathcal{Q}(\tau)$, $\mathcal{R}(\tau,\rho)$ in the domain $\Omega = \{(\tau,\rho) : \tau \in [-h,0],\ \rho \in [-h,0]\}$:

$$\mathcal{P}\mathcal{A} + \mathcal{A}^{'}\mathcal{P} + \mathcal{P}\mathcal{S}\mathcal{P} + \mathcal{Q}(0) + \mathcal{Q}^{'}(0) + \mathcal{D} = 0, \tag{5}$$

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$$\frac{d\mathcal{Q}(\tau)}{d\tau} = \Big(\mathcal{A} + \mathcal{S}\mathcal{P}\Big)^{'}\mathcal{Q}(\tau) + \mathcal{P}\mathcal{G}(\tau) + \mathcal{R}(0,\tau), \qquad (6)$$

$$\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\rho}\right)\mathcal{R}(\tau,\rho) = \mathcal{G}^{'}(\tau)\mathcal{Q}(\rho) + \mathcal{Q}^{'}(\tau)\mathcal{G}(\rho) + \mathcal{Q}^{'}(\tau)\mathcal{S}\mathcal{Q}(\rho), \qquad (7)$$

where $\mathcal{S} = \gamma^{-2}\mathcal{F}\mathcal{F}^{'} - \mathcal{B}\mathcal{N}^{-1}\mathcal{B}^{'}$.

The matrices $\mathcal{Q}(\tau)$ and $\mathcal{R}(\tau,\rho)$ are subject to the boundary conditions

$$\mathcal{Q}(-h) = \mathcal{P}\mathcal{H}, \qquad (8)$$

$$\mathcal{R}(-h,\tau) = \mathcal{H}^{'}\mathcal{Q}(\tau), \quad \mathcal{R}(\tau,-h) = \mathcal{Q}^{'}(\tau)\mathcal{H}. \qquad (9)$$

The triple $\{\mathcal{P}, \mathcal{Q}(\tau), \mathcal{R}(\tau,\rho), \ (\tau,\rho) \in \Omega\}$ is called a solution of the set (5)-(9) if: (a) it satisfies this set; (b) the matrix-valued function $\mathcal{Q}(\tau)$ is continuously differentiable for $\tau \in [-h,0]$; (c) the matrix-valued function $\mathcal{R}(\tau,\rho)$ is continuous for $(\tau,\rho) \in \Omega$; (d) the partial derivatives $\partial\mathcal{R}(\tau,\rho)/\partial\tau$ and $\partial\mathcal{R}(\tau,\rho)/\partial\rho$ are piece-wise continuous and bounded, while their sum is continuous for $(\tau,\rho) \in \Omega$.

Along with the set (5)-(9), let us consider the linear system

$$dZ(t)/dt = [\mathcal{A} - \mathcal{B}\mathcal{N}^{-1}\mathcal{B}^{'}\mathcal{P}]Z(t) + \mathcal{H}Z(t-h) + \int_{-h}^{0}[\mathcal{G}(\tau) - \mathcal{B}\mathcal{N}^{-1}\mathcal{B}^{'}\mathcal{Q}(\tau)]Z(t+\tau)d\tau, \quad t \geq 0. \qquad (10)$$

Lemma 1 [12]. *Let there exist a solution $\{\mathcal{P}, \mathcal{Q}(\tau), \mathcal{R}(\tau,\rho)\}$ of (5)-(9) such that*

$$\mathcal{P}^{'} = \mathcal{P}, \quad \mathcal{R}^{'}(\tau,\rho) = \mathcal{R}(\rho,\tau), \qquad (11)$$

and the system (10) is exponentially stable. Then, the controller

$$u^{*}[Z(\cdot)](t) = -\mathcal{N}^{-1}\mathcal{B}^{'}\left[\mathcal{P}Z(t) + \int_{-h}^{0}\mathcal{Q}(\tau)Z(t+\tau)d\tau\right] \qquad (12)$$

solves the H_∞ control problem (1)-(3).

Remark 1. *Note that for the particular case of the problem (1)-(3) ($\mathcal{G}(\tau) \equiv 0$, $\mathcal{M} = I_r$), the result similar to Lemma 1 was obtained in [13].*

Remark 2. *Lemma 1 presents solvability conditions of the H_∞ control problem (1)-(3) and a controller solving this problem. Due to the expression for the matrix $\mathcal{S}$ and (12), one can use this lemma only if the matrix $\mathcal{N}$ is invertible, and consequently, the rank of the matrix $\mathcal{M}$ equals r (the dimension of the control vector). Otherwise, Lemma 1 is not applicable.*

The objective of this paper is to develop a method of solution of the H_∞ control problem (1)-(3) in the case rank$\mathcal{M} < r$. More precisely, in what follows we consider such a problem for the case

$$\mathcal{M} = 0. \qquad (13)$$

The H_∞ control problem (1)-(3) subject to the condition (13) is a singular (nonstandard) H_∞ control problem.

III. TRANSFORMATION OF THE H_∞ CONTROL PROBLEM (1)-(3),(13)

In what follows, we assume:

(A1) the matrix $\mathcal{B}$ has full rank r;

(A2) the matrix-valued function $\mathcal{G}(\tau)$ is piece-wise continuous for $\tau \in [-h,0]$;

(A3) the matrix $\mathcal{B}^{'}\mathcal{D}\mathcal{B}$ is positive definite;

(A4) $\mathcal{H}\mathcal{B} = 0$;

(A5) $\mathcal{G}(\tau)\mathcal{B} = 0$, $\tau \in [-h,0]$.

By $\mathcal{B}_c$ we denote a complement matrix to the matrix $\mathcal{B}$, i.e., the dimension of $\mathcal{B}_c$ is $n \times (n-r)$, and the block matrix $(\mathcal{B}_c, \mathcal{B})$ is nonsingular. Based on the assumption A3, consider the following matrix:

$$\mathcal{L} = \mathcal{B}_c - \mathcal{B}(\mathcal{B}^{'}\mathcal{D}\mathcal{B})^{-1}\mathcal{B}^{'}\mathcal{D}\mathcal{B}_c. \qquad (14)$$

By using the matrix $\mathcal{L}$, we transform (similarly to [14]) the state in the H_∞ control problem (1)-(3),(13) as follows:

$$Z(t) = \Big(\mathcal{L}, \mathcal{B}\Big)z(t), \qquad (15)$$

where $z(t)$ is a new state.

Due to results of [14], the transformation (15) is nonsingular, and by using this transformation and the assumptions (A1),(A3)-(A5), the H_∞ control problem (1)-(3),(13) becomes

$$\frac{dz(t)}{dt} = Az(t) + Hz(t-h) + \int_{-h}^{0}G(\tau)z(t+\tau)d\tau + Bu(t) + Fw(t), \quad t \geq 0, \qquad (16)$$

$$\zeta(t) = \mathrm{col}\{\zeta_z(t), 0\}, \quad \zeta_z(t) = Cz(t), \qquad (17)$$

$$J(u,w) = \Big(\|\zeta_z(t)\|_{L^2}\Big)^2 - \gamma^2\Big(\|w(t)\|_{L^2}\Big)^2. \qquad (18)$$

where

$$A = \Big(\mathcal{L}, \mathcal{B}\Big)^{-1}\mathcal{A}\Big(\mathcal{L}, \mathcal{B}\Big), \qquad (19)$$

$$H = \Big(\mathcal{L}, \mathcal{B}\Big)^{-1}\mathcal{H}\Big(\mathcal{L}, \mathcal{B}\Big) = \begin{pmatrix} H_1 & 0 \\ H_3 & 0 \end{pmatrix}, \qquad (20)$$

$$G(\tau) = \Big(\mathcal{L}, \mathcal{B}\Big)^{-1}\mathcal{G}(\tau)\Big(\mathcal{L}, \mathcal{B}\Big) = \begin{pmatrix} G_1(\tau) & 0 \\ G_3(\tau) & 0 \end{pmatrix}, \qquad (21)$$

$$B = \Big(\mathcal{L}, \mathcal{B}\Big)^{-1}\mathcal{B} = \begin{pmatrix} 0 \\ I_r \end{pmatrix}, \quad F = \Big(\mathcal{L}, \mathcal{B}\Big)^{-1}\mathcal{F}, \qquad (22)$$

$$C = \mathcal{C}\Big(\mathcal{L}, \mathcal{B}\Big), \qquad (23)$$

the matrices H_1 and $G_1(\tau)$ are of the dimension $(n-r) \times (n-r)$, while the matrices H_3 and $G_3(\tau)$ are of the dimension $r \times (n-r)$.

Note that, due to (4),(23) and results of [14], the matrix $D = C^{'}C$ can be represented as follows:

$$D = \Big(\mathcal{L}, \mathcal{B}\Big)^{'}\mathcal{D}\Big(\mathcal{L}, \mathcal{B}\Big) = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \qquad (24)$$

where

$$D_1 = \mathcal{B}_c^{'}\mathcal{D}\mathcal{L}, \quad D_2 = \mathcal{B}^{'}\mathcal{D}\mathcal{B}, \qquad (25)$$

and the $(n-r) \times (n-r)$-matrix D_1 is symmetric positive semi-definite, while the $r \times r$-matrix D_2 is symmetric positive definite.

Let us partition the state vector $z(t)$, the matrix A and the matrix F into blocks as follows:

$$z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \quad F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \tag{26}$$

where the vector $x(t)$ is of the dimension $(n-r)$; the vector $y(t)$ is of the dimension r; the matrices A_1, A_2, A_3 and A_4 are of the dimensions $(n-r)\times(n-r)$, $(n-r)\times r$, $r\times(n-r)$ and $r \times r$, respectively; the matrices F_1 and F_2 are of the dimensions $(n-r) \times q$ and $r \times q$, respectively.

Then, the system (16) and the functional (18) can be rewritten as

$$\frac{dx(t)}{dt} = A_1 x(t) + A_2 y(t)$$

$$+H_1 x(t-h) + \int_{-h}^{0} G_1(\tau) x(t+\tau) d\tau + F_1 w(t), \tag{27}$$

$$\frac{dy(t)}{dt} = A_3 x(t) + A_4 y(t)$$

$$+H_3 x(t-h) + \int_{-h}^{0} G_3(\tau) x(t+\tau) d\tau + u(t) + F_2 w(t), \tag{28}$$

$$J(u,w) = \int_{0}^{+\infty} \Big[x^{'}(t) D_1 x(t) + y^{'}(t) D_2 y(t)$$

$$-\gamma^2 w^{'}(t) w(t) \Big] dt. \tag{29}$$

Remark 3. *It is seen that in the system (27)-(28), the state y is controlled directly, while the state x is controlled through the state y. Moreover, the directly controlled state y is delay-free.*

The H_∞ control problem for the system (27)-(28) and the functional (29) is to find a controller $u^*[x(\cdot), y(\cdot)](t)$ that ensures the inequality $J(u^*, w) \le 0$ along trajectories of (27)-(28) for all $w(t) \in L^2[0, +\infty; E^q]$ and for $x(t) = 0,\ t \le 0,\ y(0) = 0$. Since the transformation (15) is invertible, this problem is equivalent to the H_∞ control problem (1)-(3),(13). Also, it should be noted that since the functional (29) does not contain a control cost, a set of Riccati-type equations, similar to (5)-(7), cannot be written down for the H_∞ control problem (27)-(28),(29) meaning that this problem is singular. In what follows, we concentrate on the solution of this problem, and we call it the Singular H_∞ Control Problem (SHICP). A result, similar to Lemma 1, cannot be obtained for the SHICP. In order to solve the SHICP, we propose below a regularization of this problem and an asymptotic analysis of the regularized problem.

IV. REGULARIZATION OF THE SHICP

In order to study the SHICP, we replace it by a regular H_∞ control problem, which is close in a proper sense to the SHICP. This new H_∞ control problem has the same equations of dynamics (27)-(28) as the SHICP has. However, the functional in the new problem differs from the one in the SHICP. This functional has the "regular" form, i.e., it contains a quadratic control cost, and it is close to the one in the SHICP. Namely, this new functional has the form

$$J_\varepsilon(u,w) = \int_{0}^{+\infty} \Big[x^{'}(t) D_1 x(t) + y^{'}(t) D_2 y(t)$$

$$+\varepsilon^2 u^{'}(t) u(t) - \gamma^2 w^{'}(t) w(t) \Big] dt, \tag{30}$$

where $\varepsilon > 0$ is a small parameter.

The new H_∞ control problem is to find a controller $u_\varepsilon^*[x(\cdot), y(\cdot)](t)$ that ensures the inequality $J_\varepsilon(u_\varepsilon^*, w) \le 0$ along trajectories of (27)-(28) for all $w(t) \in L^2[0, +\infty; E^q]$ and for $x(t) = 0,\ t \le 0,\ y(0) = 0$. Since the parameter ε is small, this new problem is a cheap control problem [12]. We call this problem the H_∞ Cheap Control Problem (HICCP). The HICCP is the regular H_∞ control problem, and it can be solved in the way, similar to Lemma 1. However, such a way is rather complicated, because the corresponding set of Riccati-type equations is, in general, of a high dimension and it is ill posed for $\varepsilon \to +0$. Below, we construct a simplified controller of the HICCP and apply it to the SHICP.

V. CONSTRUCTING A SIMPLIFIED CONTROLLER FOR THE HICCP

In this section, using results of [12], we construct a simplified controller for the HICCP. This construction is based on a converting the HICCP to an H_∞ control problem for a singularly perturbed system, and an asymptotic decomposition of the latter into two much simpler ε-free subproblems, the slow and fast ones.

A. Transformation of the HICCP

By the control transformation $u(t) = (1/\varepsilon) v(t)$, where v is a new control, the HICCP becomes

$$\frac{dx(t)}{dt} = A_1 x(t) + A_2 y(t)$$

$$+H_1 x(t-h) + \int_{-h}^{0} G_1(\tau) x(t+\tau) d\tau + F_1 w(t), \tag{31}$$

$$\varepsilon \frac{dy(t)}{dt} = \varepsilon \Big\{ A_3 x(t) + A_4 y(t)$$

$$+H_3 x(t-h) + \int_{-h}^{0} G_3(\tau) x(t+\tau) d\tau \Big\} + B v(t) + \varepsilon F_2 w(t), \tag{32}$$

$$J_{\text{reg}}(v,w) = \int_{0}^{+\infty} \Big[x^{'}(t) D_1 x(t) + y^{'}(t) D_2 y(t)$$

$$+v^{'}(t) v(t) - \gamma^2 w^{'}(t) w(t) \Big] dt. \tag{33}$$

Note that the system (31)-(32) is singularly perturbed [15]. The state variables $x(\cdot)$ and $y(\cdot)$ are the slow and fast ones, respectively. It is seen that in this system, the slow state variable is with a delay, while the fast state variable is delay free.

B. Slow Subproblem

The dynamics equation and the functional of the slow subproblem are obtained from (31)-(33) by setting there formally $\varepsilon = 0$ and re-denoting x, y, v and J_{reg} by x_s, y_s, v_s and J^s_{reg}, respectively. Thus, we have

$$\frac{dx_s(t)}{dt} = A_1 x_s(t) + A_2 y_s(t)$$

$$+H_1 x_s(t-h) + \int_{-h}^{0} G_1(\tau) x_s(t+\tau) d\tau + F_1 w(t), \quad t \geq 0, \tag{34}$$

$$v_s(t) = 0, \quad t \geq 0, \tag{35}$$

$$J^s_{\text{reg}} = \int_0^{+\infty} \Big[x_s^{'}(t) D_1 x_s(t) + y_s^{'}(t) D_2 y_s(t)$$

$$+ v_s^{'}(t) v_s(t) - \gamma^2 w^{'}(t) w(t) \Big] dt. \tag{36}$$

The problem (34)-(36) is an H_∞ control problem for a specific kind of differential-algebraic systems with state delays. The H_∞ control problem for another type of time delay differential-algebraic systems was considered in [16].

By substituting (35) into (36), we obtain

$$J^s_{\text{reg}} = \int_0^{T} \Big[x_s^{'}(t) D_1 x_s(t) + y_s^{'}(t) D_2 y_s(t)$$

$$- \gamma^2 w_s^{'}(t) w_s(t) \Big] dt. \tag{37}$$

Since the variable $y_s(t)$ does not satisfy any equation for $t \in [0, +\infty)$, one can choose it to satisfy a desirable property of the system (34). This means that the variable $y_s(t)$ can be considered as a control variable in the system (34). Thus, the functional (37), calculated along trajectories of this system, depends on the control variable $y_s(t)$ and the disturbance $w(t) \in L^2[0, +\infty; E^q]$, i.e., $J^s_{\text{reg}} = J^s_{\text{reg}}(y_s, w)$. For the system (34), we can formulate the following H_∞ control problem with a performance level γ: to find a controller $y^*_s[x_s(\cdot)](t)$ that ensures the inequality $J^s_{\text{reg}}(y^*_s, w) \leq 0$ along trajectories of (34) for all $w(t) \in L^2[0, +\infty; E^q]$ and $x_s(t) = 0, \ t \ \leq 0$. This H_∞ control problem is called the Slow H_∞ Control Subproblem (SHICS) associated with the HICCP.

Consider the following hybrid set of Riccati-type matrix algebraic, ordinary differential and first-order partial differential equations with deviating arguments for the unknown matrices P_s, $Q_s(\tau)$, $R_s(\tau, \rho)$:

$$P_s A_1 + A_1^{'} P_s + P_s S_s P_s + Q_s(0) + Q_s^{'}(0) + D_1 = 0, \tag{38}$$

$$\frac{dQ_s(\tau)}{d\tau} = \Big(A_1^{'} + P_s S_s \Big) Q_s(\tau) + P_s G_1(\tau) + R_s(0, \tau), \tag{39}$$

$$\left(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial \rho} \right) R_s(\tau, \rho) = G_1^{'}(\tau) Q_s(\rho) + Q_s^{'}(\tau) G_1(\rho)$$

$$+ Q_s^{'}(\tau) S_s Q_s(\rho)], \tag{40}$$

where $S_s = \gamma^{-2} F_1 F_1^{'} - A_2 D_2^{-1} A_2^{'}$.

The set of equations (38)-(40) is subject to the boundary conditions

$$Q_s(-h) = P_s H_1, \tag{41}$$

$$R_s(-h, \tau) = H_1^{'} Q_s(\tau), \quad R_s(\tau, -h) = Q_s^{'}(\tau) H_1. \tag{42}$$

Along with the set (38)-(42), consider the system

$$\frac{dx_s(t)}{dt} = (A_1 - A_2 D_2^{-1} A_2^{'} P_s) x_s(t) + H_1 x_s(t-h)$$

$$+ \int_{-h}^{0} [G_1(\tau) - A_2 D_2^{-1} A_2^{'} Q_s(\tau)] x_s(t+\tau) d\tau, \quad t \geq 0. \tag{43}$$

Similarly to Lemma 1, one has the following lemma.

Lemma 2. *Let there exist a solution* $\{P_s, Q_s(\tau), R_s(\tau, \rho)\}$ *of (38)-(42) in the domain* Ω*, such that*

$$P_s^{'} = P_s, \qquad R_s^{'}(\tau, \rho) = R_s(\rho, \tau), \tag{44}$$

and the system (43) is exponentially stable. Then, the controller

$$y^*_s[x_s(\cdot)](t) = -D_2^{-1} A_2^{'} \Big[P_s x_s(t)$$

$$+ \int_{-h}^{0} Q_s(\tau) x_s(t+\tau) d\tau \Big], \tag{45}$$

solves the SHICS.

C. Fast Subproblem

The fast subproblem is obtained in the following three stages. First, the slow variable $x(\cdot)$ is removed from the equation (32) and the functional (33). Second, the following transformation of variables is made in the resulting equation and functional:

$$t = \varepsilon \xi, \quad y(\varepsilon \xi) = y_f(\xi), \quad v(\varepsilon \xi) = v_f(\xi), \quad w(\varepsilon \xi) = w_f(\xi),$$

$$J_{\text{reg}}(v(\varepsilon \xi), w(\varepsilon \xi)) = \varepsilon J^f_{\text{reg}}(v_f(\xi), w_f(\xi)), \tag{46}$$

where ξ, y_f, v_f, w_f and J^f_{reg} are new independent variable, state, control, disturbance and functional, respectively.

Thus, we obtain the system and the functional

$$\frac{dy_f(\xi)}{d\xi} = \varepsilon A_4 y_f(\xi) + v_f(\xi) + \varepsilon F_2 w_f(\xi), \quad \xi \geq 0, \tag{47}$$

$$J^f_{\text{reg}}(v_f, w_f) = \int_0^{+\infty} \Big[y_f^{'}(\xi) D_2 y_f(\xi) + v_f^{'}(\xi) v_f(\xi)$$

$$- \gamma^2 w_f^{'}(\xi) w_f(\xi) \Big] d\xi. \tag{48}$$

Finally, neglecting formally the terms with the multiplier ε in (47) yields the system

$$\frac{dy_f(\xi)}{d\xi} = v_f(\xi), \qquad \xi \geq 0. \tag{49}$$

For the system (49) and the functional (48), the H_∞ control problem with a performance level γ can be formulated as follows. To find a controller $v^*_f[y_f(\xi)]$ that stabilizes (49) and ensures the inequality $J^f_{\text{reg}}(v^*_f, w_f) \leq 0$ along its trajectories for all $w_f(\xi) \in L^2[0, +\infty; E^q]$ and for $y_f(0) = 0$. This H_∞ control problem is called the Fast H_∞ Control Subproblem (FHICS) associated with the HICCP.

Let K be any Hurwitz matrix of the dimension $r \times r$. Then, the controller

$$v_f^*[y_f(\xi)] = K y_f(\xi) \tag{50}$$

solves the FHICS.

Note, that the FHICS is a particular case of the infinite horizon H_∞ control problem, considered in [2]. Due to results of this book, if there exists a solution P_f of the algebraic matrix Riccati-type equation

$$-(P_f)^2 + D_2 = 0, \tag{51}$$

such that $-P_f$ is a Hurwitz matrix, then the matrix gain K in (50) can be chosen as

$$K = -P_f. \tag{52}$$

Since the matrix D_2 is positive definite, then there exist the unique positive definite solution of (51)

$$P_f = D_2^{1/2}, \tag{53}$$

where the superscript "1/2" denotes the unique symmetric positive definite square root of respective symmetric positive definite matrix.

Now, using (52) and (53), we have $K = -D_2^{1/2}$. Hence, the controller

$$v_f^*[y_f(\xi)] = -D_2^{1/2} y_f(\xi) \tag{54}$$

solves the FHICS.

D. Composite Controller for the HICCP

In this subsection, based on the control $v_s(t)$, given by (35), the controller $y_s^*[x_s(\cdot)](t)$, solving the SHICS, and the controller $v_f^*[y_f(\xi)]$, solving the FHICS, we construct a so called composite controller $v_c[x(\cdot), y(\cdot)](t)$. This controller, being multiplied by $1/\varepsilon$, constitutes a controller $u_c[x(\cdot), y(\cdot)](t) = (1/\varepsilon)v_c[x(\cdot), y(\cdot)](t)$, which solves the HICCP for all sufficiently small $\varepsilon > 0$.

The composite controller $v_c[x(\cdot), y(\cdot)](t)$ is constructed in the form

$$v_c[x(\cdot), y(\cdot)](t) = v_s(t) + v_f^*[\tilde{y}(t/\varepsilon)], \tag{55}$$

where $\tilde{y}(t/\varepsilon)$ is defined as follows

$$\tilde{y}(t/\varepsilon) \stackrel{\triangle}{=} y(t) - y_s^*[x(\cdot)](t). \tag{56}$$

Substituting (35) and (54) into (55), and using (45), (53) and (56) yield after some rearrangement

$$v_c[x(\cdot), y(\cdot)](t) = -D_2^{-1/2}\Big(A_2' P_s x(t) + D_2 y(t) + \int_{-h}^{0} A_2' Q_s(\tau) x(t+\tau) d\tau\Big), \tag{57}$$

where $D_2^{-1/2}$ is the inverse matrix to $D_2^{1/2}$.

Theorem 1 [12]. *Let there exist a solution $\{P_s(t), Q_s(t,\tau), R_s(t,\tau,\rho)\}$ of (38)-(42) in the domain Ω, satisfying the conditions (44) and providing the system (43) to be exponentially stable. Then, there exists a positive number ε^*, such that the controller $u_c[x(\cdot), y(\cdot)](t) = (1/\varepsilon)v_c[x(\cdot), y(\cdot)](t)$, where $v_c[x(\cdot), y(\cdot)](t)$ is given by (57), solves the HICCP for all $\varepsilon \in (0, \varepsilon^*]$.*

Remark 4. *Theorem 1 presents ε-free solvability conditions for the HICCP valid for all sufficiently small values of $\varepsilon > 0$. Moreover, these conditions are of a lower dimension than the dimension of the HICCP. The gain matrices in the controller $u_c[x(\cdot), y(\cdot)](t)$, solving the HICCP, also are ε-free.*

VI. SOLUTION OF THE SHICP

Due to Theorem 1, the employing the controller $u(t) = u_c[(x(\cdot), y(\cdot)](t)$ in the system (27)-(28) subject to the initial conditions $x(t) = 0,\ t \le 0,\ y(0) = 0$, yields the following inequality for all $w(t) \in L^2[0, +\infty; E^q]$ and all $\varepsilon \in (0, \varepsilon^*]$:

$$J(u_c, w) + \int_0^{+\infty} v_c'[x(\cdot), y(\cdot)](t) v_c[x(\cdot), y(\cdot)](t) dt \le 0, \tag{58}$$

where the arguments $x(\cdot)$ and $y(\cdot)$ for v_c constitute the solution $\mathrm{col}\Big(x(t,\varepsilon), y(t,\varepsilon)\Big)$ of the system (27)-(28) with the control $u(t) = u_c[(x(\cdot), y(\cdot)](t)$ and the initial conditions $x(t) = 0,\ t \le 0,\ y(0) = 0$.

From the inequality (58), one directly has $J(u_c, w) \le 0$, which implies that the controller $u(t) = u_c[(x(\cdot), y(\cdot)](t)$ solves the SHICP for all $\varepsilon \in (0, \varepsilon^*]$ if there exist a solution $\{P_s(t), Q_s(t,\tau), R_s(t,\tau,\rho)\}$ of (38)-(42) in the domain Ω, satisfying the conditions of Theorem 1.

Remark 5. *The inequality (58) yields a stronger inequality than $J(u_c, w) \le 0$. Namely,*

$$J(u_c, w) \le - \int_0^{+\infty} v_c'[x(\cdot,\varepsilon), y(\cdot,\varepsilon)](t) v_c[x(\cdot,\varepsilon), y(\cdot,\varepsilon)](t) dt. \tag{59}$$

The integral in the right-hand side of this inequality depends on $\varepsilon \in (0, \varepsilon^*]$. The following theorem gives an estimate of this integral for small enough $\varepsilon > 0$.

Theorem 2. *Let there exist a solution $\{P_s(t), Q_s(t,\tau), R_s(t,\tau,\rho)\}$ of (38)-(42) in the domain Ω, satisfying the conditions of Theorem 1. Then there exists a positive number ε_1^*, ($\varepsilon_1^* \le \varepsilon^*$), such that, for all $w(t) \in L^2[0, +\infty; E^q]$ and all $\varepsilon \in (0, \varepsilon_1^*]$, the following inequality is satisfied:*

$$0 \le \int_0^{T} v_c'[x(\cdot,\varepsilon), y(\cdot,\varepsilon)](t) v_c[x(\cdot,\varepsilon), y(\cdot,\varepsilon)](t) dt \le a\varepsilon\Big(\|w(t)\|_{L^2}\Big)^2, \tag{60}$$

where $a > 0$ is some constant independent of $w(t)$ and ε.

Proof. In order to save the space, we present here a sketch of the proof. The left-hand inequality in (60) is obvious. Proceed to the proof of the right-hand one. Asymptotic analysis of the system (27)-(28) with the control $u(t) = u_c[(x(\cdot), y(\cdot)](t)$ and the initial conditions $x(t) = 0,\ t \le 0,\ y(0) = 0$ leads to the existence of a constant $0 < \varepsilon_1^* \le \varepsilon^*$ such that, for all $w(t) \in L^2[0, +\infty; E^q]$ and all $\varepsilon \in (0, \varepsilon_1^*]$, the following inequalities are valid:

$$0 \le \sup_{t\in[0,+\infty)} \Big\|x(t,\varepsilon) - \bar{x}(t)\Big\| \le a_1 \varepsilon^{1/2} \Big\|w(t)\Big\|_{L^2}, \tag{61}$$

$$0 \leq \sup_{t\in[0,+\infty)} \Big\| y(t,\varepsilon) - \bar{y}(t) \Big\| \leq a_1 \varepsilon^{1/2} \Big\| w(t) \Big\|_{L^2}, \quad (62)$$

where $a_1 > 0$ is some constant independent of $w(t)$ and ε,

$$\bar{x}(t) = \int_0^t \bar{\Phi}_x(t,s) F_1 w(s) ds, \quad (63)$$

$$\bar{y}(t) = \int_0^t \bar{\Phi}_y(t,s) F_1 w(s) ds, \quad (64)$$

the $(n-r)\times(n-r)$-matrix-valued function $\bar{\Phi}_x(t,s)$ is the solution of the problem

$$\frac{d\bar{\Phi}_x(t,s)}{dt} = \Big(A_1 - A_2 D_2^{-1} A_2^{'} P_s\Big)\bar{\Phi}_x(t,s) + H_1 \bar{\Phi}(t-h,s)$$
$$+ \int_{-h}^{0} \Big(G_1(\tau) - A_2 D_2^{-1} A_2^{'} Q_s(\tau)\Big)\bar{\Phi}_x(t+\tau,s) ds, \quad (65)$$
$$0 \leq s \leq t < +\infty,$$
$$\bar{\Phi}_x(t,s) = 0, \quad t < s; \quad \bar{\Phi}_x(s,s) = I_{n-r}, \quad (66)$$

and the $r \times (n-r)$-matrix-valued function $\bar{\Phi}_y(t,s)$ has the form

$$\bar{\Phi}_y(t,s) = -D_2^{-1} A_2^{'} P_s \bar{\Phi}_x(t,s)$$
$$- \int_{-h}^{0} D_2^{-1} A_2^{'} Q_s(\tau) \bar{\Phi}_x(t+\tau,s) d\tau. \quad (67)$$

Now, by using the inequalities (61)-(62) and the equations (63)-(67), one obtains after some rearrangement the inequality

$$\sup_{t\in[0,+\infty)} \Big\| v_c[x(\cdot,\varepsilon), y(\cdot,\varepsilon)](t) \Big\| \leq a_2 \varepsilon^{1/2} \Big\| w(t) \Big\|_{L^2}, \quad (68)$$

where $a_2 > 0$ is some constant independent of $w(t)$ and ε.

The inequality (68) directly yields the right-hand inequality in (60).

As a direct consequence of Theorem 2, we obtain the following corollary.

Corollary 1. *Let the conditions of Theorem 2 be satisfied. Then, there exists a positive constant ε_2^*, ($\varepsilon_2^* \leq \varepsilon_1^*$), and a function $g(\varepsilon)$, ($0 \leq g(\varepsilon) \leq a\varepsilon$, $\varepsilon \in (0, \varepsilon_2^*]$, the constant $a > 0$ is defined in Theorem 2), such that for all $\varepsilon \in (0, \varepsilon_2^*]$ the controller $u_c[x(\cdot,\varepsilon), y(\cdot,\varepsilon)](t)$ solves the singular H_∞ control problem for the system (27)-(28) with the following functional:*

$$J_g(u,w) = \int_0^{+\infty} \Big[x^{'}(t) D_1 x(t) + y^{'}(t) D_2 y(t)$$
$$-(\gamma_g(\varepsilon))^2 w^{'}(t) w(t)\Big] dt, \quad (69)$$

where the performance level $\gamma_g(\varepsilon)$ has the form $\gamma_g(\varepsilon) = \sqrt{\gamma^2 - g(\varepsilon)} > 0$.

Remark 6. *Due to Corollary 1, the controller $u_c[x(\cdot,\varepsilon), y(\cdot,\varepsilon)](t)$ solves not only the original SHICP (27)-(28),(29), but also the singular H_∞ control problem (27)-(28),(69) with a slightly smaller performance level. This smaller performance level is close to the original one γ and it satisfies the limit equality $\lim_{\varepsilon\to+0}\gamma_g(\varepsilon) = \gamma$.*

VII. CONCLUSIONS

A singular H_∞ control problem for a linear system with point-wise and distributed time delays in the state was considered. Under proper assumptions, this system was transformed equivalently to the system consisting of two modes, one of which is controlled directly, while the other is controlled through the first one. The state of the directly controlled mode is delay-free, and its dimension coincides with the control's dimension. Due to this transformation, the initially formulated H_∞ control problem was converted to a new singular H_∞ control problem. The latter was regularized by adding to the functional a quadratic control cost with a small weight ε^2, ($\varepsilon > 0$), which yields the H_∞ cheap control problem.This problem was solved by its asymptotic partitioning into two much simpler subproblems (the slow and fast ones), and constructing a composite controller based on the controllers of these subproblems. It was shown that the composite controller, solving the H_∞ cheap control problem, also solves the singular H_∞ control problem. Moreover, it was shown that this controller also solves a singular H_∞ control problem with a smaller performance level, depending on ε. This smaller performance level tends to the original one for $\varepsilon \to +0$.

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