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On a Time-Varying Stochastic Small Gain Theorem

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***Abstract*— In this note, a *small gain* result is presented for a class of discrete-time time-varying Markovian jump systems with state-dependent noise. Using the stabilizing solutions of some generalized Riccati equations, the result is obtained based on a Lyapunov type argument.**

I. INTRODUCTION

The class of Markovian jump linear systems is very appropriate to model plant whose structure is subject to random abrupt changes. The theory related to this class of systems is now well established. Problems such as stability, optimal and robust control, as well as important applications of such systems, can be found in several references in the current literature, for instance in [1], [2], [5], [6] and the reference therein.

A well known result in linear control theory is the so called *small gain* theorem which in resume allows to obtain informations concerning the Lyapunov stability of systems by using input-output properties of the systems. Such a result has been established for the deterministic case for time invariant systems in [7] and for the time-varying case in [3]. In [4], the authors give a corresponding result for continuous-time stochastic systems with state-dependent white noise.

In the present note, our aim is to give a small gain result for the class of discrete-time time-varying stochastic systems. More specifically, the stochastic nature of the considered class of systems results from the presence of state-dependent noise and Markovian jumping parameters. Using the stabilizing solutions of some generalized Riccati equations, the result is obtained based on a Lyapunov type argument.

This paper is organized as follows: Section 2 gives some preliminary definitions and results. The small gain theorem is given in Section 3 which also includes an application to stability radius estimation. Section 4 ends this note.

II. PRELIMINARIES

A. Dynamical model

Consider the system $\mathbb{G}$ having the state space representation described by:

$$\mathbb{G}:\begin{cases} x(t+1)=\left(A_0(t,\theta_t)+\sum_{k=1}^{r} w_k(t)A_k(t,\theta_t)\right)x(t) \\ \qquad\qquad +B(t,\theta_t)v(t) \\ z(t)=C(t,\theta_t)x(t)+D(t,\theta_t)v(t) \end{cases} \tag{II.1}$$

where $x(t)\in\mathbb{R}^n$ is the system state vector, $v(t)\in\mathbb{R}^{m_v}$ is the external input, $z(t)\in\mathbb{R}^{n_z}$ is the output, $\left(w(t)=(w_1(t),\cdots,w_r(t))^T\right)$ is a sequence of independent random vectors and the triple $(\{\theta_t\}_{t\geqslant 0},\{P_t\}_{t\geqslant 0},\Theta)$ is a time-inhomogeneous Markov chain defined on a given probability space $(\Omega,\mathcal{F},\mathrm{P})$ with the finite states set $\Theta=\{1,\cdots,N\}$ and the sequence of transition probability matrices $\{P_t\}_{t\geqslant 0}$.

Concerning the processes $\{\theta_t\}_{t\geqslant 0}$ and $\{w_t\}_{t\geqslant 0}$, the following assumptions are made:

H1) $\{w(t)\}_{t\geqslant 0}$ is a sequence of independent random vectors with the following properties:

$$E\left[w(t)\right]=0,\quad E\left[w(t)w^T(t)\right]=I_r,\quad t\geqslant 0$$

I_r being the identity matrix of size r. As usual, throughout the paper $E\left[\cdot\right]$ stands for the mathematical expectation and the superscript T denotes the transposition of a vector or a matrix.

H2) For each $t\geqslant 0$ the σ-algebra $\mathcal{F}_t$ is independent of the σ-algebra $\mathcal{G}_t$, where $\mathcal{F}_t=\sigma(w(s);0\leqslant s\leqslant t)$ and $\mathcal{G}_t=\sigma(\theta_s;0\leqslant s\leqslant t)$.

H3) i) The transition probability matrices $P_t,\ t\geqslant 0$, are nondegenerate stochastic matrices.

ii) $\pi_0(i)=\mathrm{P}\{\theta_0=i\}>0,\ 1\leqslant i\leqslant N$.

Note that in the developments of this note, the Markov chain is not prefixed, but it is assumed that the initial distributions $\pi_0=(\pi_0(1),\cdots,\pi_0(N))$ lie in the subset

$$\mathcal{M}_N=\left\{\pi_0|\pi_0(i)>0,1\leqslant i\leqslant N,\sum_{i=1}^{N}\pi_0(i)=1\right\}$$

H4) There exists an initial distribution $\tilde{\pi}_0\in\mathcal{M}_N$ such that for the Markov chain $(\{\tilde{\theta}_t\}_{t\geqslant 0},\{P_t\}_{t\geqslant 0},\Theta)$ (with initial distribution $\tilde{\pi}_0$), there exists $\delta>0$ such that

$$\tilde{\pi}_t(i)=\mathrm{P}\{\tilde{\theta}_t=i\}\geqslant\delta,\ t\geqslant 0,\ i\in\Theta \tag{II.2}$$

Regarding the coefficients of the system (II.1) we make the assumption:

H5) $\{A_k(t,i)\}_{t\geqslant 0},\ 0\leqslant k\leqslant r,\ \{B(t,i)\}_{t\geqslant 0},\ \{C(t,i)\}_{t\geqslant 0}$ and $\{D(t,i)\}_{t\geqslant 0}$, $i\in\Theta$, are bounded matrix sequences.

B. Lyapunov type operators

Let $\mathcal{S}_n\in\mathbb{R}^{n\times n}$ be the linear subspace of real symmetric matrices. Set $\mathcal{S}_n^N=\mathcal{S}_n\oplus\mathcal{S}_n\oplus\cdots\oplus\mathcal{S}_n$. $\mathcal{S}_n^N$ is a real ordered Hilbert space. The usual inner product on $\mathcal{S}_n^N$ is:

$$\langle X,Y\rangle=\sum_{i=1}^{N}\mathrm{Tr}(X(i)Y(i)) \tag{II.3}$$

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for all $X = (X(1), \cdots, X(N))$ and $Y = (Y(1), \cdots, Y(N))$ in $\mathcal{S}_n^N$.

Based on the sequences $\{A_k(t,i)\}_{t\geqslant 0}$, $0 \leqslant k \leqslant r$, $i \in \Theta$, and $\{P_t\}_{t\geqslant 0}$ introduced previously, we define the following linear operators:

$$\begin{aligned}\mathcal{L}_t : \mathcal{S}_n^N &\longrightarrow \mathcal{S}_n^N \\ S &\longmapsto \mathcal{L}_t S = (\mathcal{L}_t S(1), \cdots, \mathcal{L}_t S(N))\end{aligned} \tag{II.4}$$

where

$$\mathcal{L}_t S(i) = \sum_{k=0}^{r} \sum_{j=1}^{N} p_t(j,i) A_k(t,j) S(j) A_k^T(t,j) \tag{II.5}$$

The operators $\mathcal{L}_t$ are called the Lyapunov type operators associated to the system (II.1). The adjoint operator of $\mathcal{L}_t$ with respect to the inner product (II.3) is noted $\mathcal{L}_t^*$.

Let $\mathcal{R}(t,s)$ be the linear evolution operator defined on $\mathcal{S}_n^N$ by the sequence $\{\mathcal{L}_t\}_{t\geqslant 1}$ as follows:

$$\mathcal{R}(t,s) = \begin{cases} \mathcal{L}_{t-1}\mathcal{L}_{t-2}\cdots\mathcal{L}_s & \text{if } t > s \geqslant 0 \\ I_{\mathcal{S}_n^N} & \text{if } t = s \end{cases} \tag{II.6}$$

where $I_{\mathcal{S}_n^N}$ is the identity operator on $\mathcal{S}_n^N$.

We are now in position to introduce the following definitions:

Definition 1. We say that the sequence $\{\mathcal{L}_t\}_{t\geqslant 0}$ generates an exponentially stable evolution if there exist $\beta \geq 1,\ q \in (0,1)$ such that

$$\|\mathcal{R}(t,s)\| \leq \beta q^{t-s}$$

$\forall t \geqslant s \geqslant 0$.

Definition 2. We say that the zero state equilibrium of the system (II.1) with $v_k \equiv 0$, is

- Strongly exponentially stable in mean square (SESMS) (or equivalently $(\mathbb{A}, \mathbb{P})$ is SESMS) if there exist $\beta \geq 1,\ q \in (0,1)$ such that
$$\|\mathcal{R}(t,s)\| \leq \beta q^{t-s}$$
$\forall t \geqslant s \geqslant 0$, where $\mathbb{P} := \{P_t\}_{t\geq 0}$, $\mathbb{A} = \{A_t\}_{t\geq 0}$, $A_t = (A_0(t), A_1(t), \cdots, A_r(t))$ and $A_k(t) = (A_k(t,1), A_k(t,2), \cdots, A_k(t,N))$, $0 \leqslant k \leqslant r$.
- Exponentially stable in the mean square sense with conditioning of type I (ESMSCI) (or equivalently $(\mathbb{A}, \mathbb{P})$ is ESMSCI) if there exist $\beta \geq 1,\ q \in (0,1)$ such that for any Markov chain $(\{\theta_t\}_{t\geq 0},\ \{P_t\}_{t\geq 0},\ \Theta)$ we have
$$E\left[|\Phi(t,s)x_s|^2 \middle| \theta_s = i\right] \leq \beta |x_s|^2 q^{t-s},\ \forall\, t \geq s \geq 0$$
$i \in \Theta_s,\ x_s \in \mathbb{R}^n$, where $\Phi(t,s)$ is the fundamental random matrix solution of system (II.1), and $\Theta_s = \{i \in \Theta | \mathrm{P}\{\theta_s = i\} > 0\}$.
- Exponentially stable in the mean square sense (ESMS) (or equivalently $(\mathbb{A}, \mathbb{P})$ is ESMS), if there exist $\beta \geq 1,\ q \in (0,1)$ such that for any Markov chain $(\{\theta_t\}_{t\geq 0},\ \{P_t\}_{t\geq 0},\ \Theta)$ we have
$$E\left[|\Phi(t,s)x_s|^2\right] \leq \beta |x_s|^2 q^{t-s},\ \forall\, t \geq s \geq 0,\ x_s \in \mathbb{R}^n$$

Remark 1. The stability notions defined above are not equivalent in the general case. SESMS implies the two other stability notions. Note however that under assumption (**H4**), the three stability notions are equivalent. Hence, in the rest of the paper we will only use SESMS definition. For more details on this topic, one can refer to [6].

Definition 3. We say that system (II.1) is stochastically detectable if there exist sequences $\{K(t,i)\}_{t\geqslant 0}$, $i \in \Theta$ such that the zero state equilibrium of the following system:

$$\begin{aligned} x(t+1) = \Big(&A_0(t,\theta_t) + K(t,\theta_t)C(t,\theta_t) \\ &+ \sum_{k=1}^{r} w_k(t) A_k(t,\theta_t)\Big) x(t) \end{aligned} \tag{II.7}$$

is SESMS.

Let $\mathbb{C} = \{C_t\}_{t\geq 0},\ C_t = (C(t,1), C(t,2), \cdots, C(t,N))$. We end this subsection by the following definition.

Definition 4. We say that the pair $(C_t, \mathcal{L}_t)$ is detectable, or equivalently the triple $(\mathbb{C}, \mathbb{A}, \mathbb{P})$ is detectable if there exists a bounded sequence $\{H_t\}_{t\geqslant 0}$, where $H_t = (H(t,1), H(t,2), \cdots, H(t,N))$, such that the sequence $\{\mathcal{L}_t^H\}_{t\geqslant 0}$ generates an exponentially stable evolution, where $\mathcal{L}_t^H$ is defined by $\mathcal{L}_t^H X = (\mathcal{L}_t^H X(1), \cdots, \mathcal{L}_t^H X(N))$, $X \in S_n^N$, with

$$\begin{aligned} \mathcal{L}_t X(i) = &\sum_{j=1}^{N} p_t(j,i) \left(A_0(t,j) + H(t,j)C(t,j)\right) X(j) \\ &\times \left(A_0(t,j) + H(t,j)C(t,j)\right)^T \\ &+ \sum_{k=1}^{r} \sum_{j=1}^{N} p_t(j,i) A_k(t,j) X(j) A_k^T(t,j) \end{aligned} \tag{II.8}$$

C. Input-output operators

For each $t \geqslant 1$ we denote $\tilde{\mathcal{H}}_t = \mathcal{F}_{t-1} \vee \mathcal{G}_t$ and $\tilde{\mathcal{H}}_0 = \sigma(\theta_0)$. In the following $l^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbb{R}^{m_v}\}$ is the real Hilbert space of all sequences $\{v_t\}_{t\geqslant 0}$ of m_v-dimensional random vectors with the properties that $\forall t \geqslant 0$, v_t is $\tilde{\mathcal{H}}_t$-measurable and $\sum_{t=0}^{\infty} E\left[|v_t|^2\right] < \infty$. In this paper, the inputs $v = \{v_t\}_{t\geqslant 0}$ are stochastic processes in $l^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbb{R}^{m_v}\}$. The norm induced by the usual inner product of the Hilbert space $l^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbb{R}^{m_v}\}$ is:

$$\|v\|_{l^2_{\tilde{\mathcal{H}}}} = \left(\sum_{t=0}^{\infty} E\left[|v_t|^2\right]\right)^{\frac{1}{2}}$$

for all $v \in l^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbb{R}^{m_v}\}$. Let $x(t,0,v)$ be the solution of (II.1) corresponding to the input $v = \{v_t\}_{t\geqslant 0}$ with the initial condition $x(0,0,v) = 0$, and let

$$z(t,0,v) = C(t,\theta_t)x(t,0,v) + D(t,\theta_t)v(t) \tag{II.9}$$

be the corresponding output. It follows from Corollary 3.9 (i) in [6] that if (II.1) is SESMS then $z = \{z(t,0,v)\}_{t\geqslant 0} \in l^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbb{R}^{m_v}\}$ for any $v \in l^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbb{R}^{m_v}\}$. Since $v \longrightarrow z(t,0,v)$ is a linear dependence, if system (II.1) is SESMS, one can define a linear operator $\mathcal{T} : l^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbb{R}^{m_v}\} \longrightarrow l^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbb{R}^{n_z}\}$ by:

$$(\mathcal{T}v)(t) = z(t,0,v) = C(t,\theta_t)x(t,0,v) + D(t,\theta_t)v(t) \tag{II.10}$$

for every $v \in l^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbb{R}^{m_v}\}$. The linear operator $\mathcal{T}$ will be called input-output operator defined by the system (II.1) while system (II.1) is known as a state space representation of the operator $\mathcal{T}$.

Suppose that (II.1) is SESMS and $D(t,\theta_t)$ is square and boundedly invertible, if $\{\hat{\mathcal{L}}_t\}_{t\geqslant 0}$ generates an exponentially stable evolution where the operator $\hat{\mathcal{L}}_t$ is defined by:

$$\begin{aligned} \hat{\mathcal{L}}_t : \mathcal{S}_n^N &\longrightarrow \mathcal{S}_n^N \\ S &\longmapsto \hat{\mathcal{L}}_t S = \left(\tilde{\mathcal{L}}_t S(1),\cdots,\tilde{\mathcal{L}}_t S(N)\right) \end{aligned}$$

$$\begin{aligned} \hat{\mathcal{L}}_t S(i) = &\sum_{j=1}^{N} p_t(j,i)\hat{A}_0(t,j)S(j)\hat{A}_0^T(t,j) \\ &+ \sum_{k=1}^{r}\sum_{j=1}^{N} p_t(j,i)A_k(t,j)S(j)A_k^T(t,j) \end{aligned}$$

and

$$\hat{A}_0(t,\theta_t) = A_0(t,\theta_t) - B(t,\theta_t)D^{-1}(t,\theta_t)C(t,\theta_t)$$

then one may define the inverse of the input-output operator $\mathcal{T}$ associated to (II.1), namely $\mathcal{T}^{-1} : l^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbb{R}^{m_v}\} \longrightarrow l^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbb{R}^{m_v}\}$ which is the input-output operator coresponding to the system:

$$\begin{cases} x(t+1) = \left(\hat{A}_0(t,\theta_t) + \sum_{k=1}^{r} w_k(t)A_k(t,\theta_t)\right)x(t) \\ \qquad\qquad +B(t,\theta_t)D^{-1}(t,\theta_t)v(t) \\ z(t) = -D^{-1}(t,\theta_t)C(t,\theta_t)x(t) + D^{-1}(t,\theta_t)v(t) \end{cases}$$

We end this section by giving the following result (stochastic time-varying bounded real lemma) which will be useful in the development of our main results:

Proposition 1. Under the considered assumptions, the following assertions are equivalent:

i) The zero state equilibrium of system (II.1) is SESMS and $\|\mathcal{T}\| < \gamma$.

ii) The generalized discrete-time Riccati equation (II.11) has a bounded and stabilizing solution $\tilde{X}(t) = (\tilde{X}(t,1),\tilde{X}(t,2),\cdots,\tilde{X}(t,N)) \in \mathcal{S}_n^{N_+}$, $\forall t \geqslant 0$, that verifies (II.12) (for $t \geqslant 0, i \in \Theta$):

$$\begin{aligned} X(t,i) = &\Pi_{i1}(t,X(t+1)) + C^T(t,i)C(t,i) \\ &- \Pi_{i2}(t,X(t+1))\left(\Pi_{i3}^{\gamma}(t,X(t+1))\right)^{-1}\star \end{aligned} \tag{II.11}$$

$$\Pi_{i3}^{\gamma}(t,X(t+1) \leqslant -\epsilon I_{m_v},\ \epsilon \in (0,\gamma^2 - \|\mathcal{T}\|^2) \tag{II.12}$$

where

$$\begin{cases} \mathcal{E}_i(t,X(t+1)) = \sum_{j=1}^{N} p_t(i,j)X(t+1,j) \\ \Pi_{i1}(t,X(t+1)) = \sum_{k=0}^{r} A_k^T(t,i)\mathcal{E}_i(t,X(t+1)) \\ \qquad\qquad \times A_k(t,i) \\ \Pi_{i2}(t,X(t+1)) = A_0^T(t,i)\mathcal{E}_i(t,X(t+1))B(t,i) \\ \qquad\qquad +C^T(t,i)D(t,i) \\ \Pi_{i3}^{\gamma}(t,X(t+1)) = B^T(t,i)\mathcal{E}_i(t,X(t+1))B(t,i) \\ \qquad\qquad +D^T(t,i)D(t,i) - \gamma^2 I_{m_v} \end{cases}$$

Recall that $\tilde{X}(t)$ is called stabilizing solution if the resulting bounded gain sequence $\{F_t^{\gamma}\}_{t\geqslant 0}$, $F_t^{\gamma} = (F^{\gamma}(t,1),F^{\gamma}(t,2),\cdots,F^{\gamma}(t,N))$, $F^{\gamma}(t,i) = -\left(\Pi_{i3}^{\gamma}(t,X(t+1))\right)^{-1}\Pi_{i2}^T(t,X(t+1))$ ensures that the closed loop system resulting from (II.1) whith $v(t) = F^{\gamma}(t,\theta_t)x(t)$ is SESMS.

iii) There exists $\mathcal{P}(t) = (\mathcal{P}(t,1),\mathcal{P}(t,2),\cdots,\mathcal{P}(t,N)) \in \mathcal{S}_n^N$ and a positive scalar ξ such that (for $t \geqslant 0, i \in \Theta$):

$$\begin{bmatrix} \tilde{\Pi}_{i1}(t,\mathcal{P}(t+1)) & \Pi_{i2}(t,\mathcal{P}(t+1)) \\ \star & \Pi_{i3}^{\gamma}(t,\mathcal{P}(t+1)) \end{bmatrix} \leqslant -\xi I \tag{II.13}$$

where:

$$\begin{aligned} \tilde{\Pi}_{i1}(t,\mathcal{P}(t+1)) = &\Pi_{i1}(t,\mathcal{P}(t+1)) + C^T(t,i)C(t,i) \\ &- \mathcal{P}(t,i) \end{aligned}$$

and

$$0 < \underline{\eta} \leqslant \lambda_{\min}[\mathcal{P}(t,i)] \leqslant \lambda_{\max}[\mathcal{P}(t,i)] \leqslant \bar{\eta} < +\infty \tag{II.14}$$

Proof. The equivalence ii) $\Longleftrightarrow$ iii) can be deduced from the general result given in Theorem 5.12 from [6]. A sketch of the proof of the equivalence i) $\Longleftrightarrow$ iii), which is based on the properties of some input-output operators, is given in the Appendix. ■

Remark 2. A similar result as in Proposition 1 (equivalence i) $\Longleftrightarrow$ ii)) has been recently reported in [8]. Note however that the authors used a stronger assumption than **H4**. Note that this assumption is used to prove the boundedness of the solutions of some generalized discrete-time Riccati equation. Indeed, it follows from [8] that the authors require the satisfaction of relation (II.2) for all initial distributions while in our case we require the existence of at least one initial distribution such that (II.2) is verified. This is due to the fact that the solution of the generalized Riccati equation does not depend upon the initial distribution of the Markov chain. Hence, if the boundedness is obtained for an initial distribution of the Markov chain, it will be true for all other initial distributions.

III. Main results

In what follows, we will first introduce an auxiliary result which will be used in the proof of the proposed time-varying stochastic small gain theorem.

Theorem 1. Consider system (II.1), with $m_v = n_z$, and assume that it is SESMS. Denote by $\mathcal{T}$ the corresponding input-output operator from $l^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbb{R}^{m_v}\} \longrightarrow l^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbb{R}^{n_z}\}$. Under the considered assumptions **H1-H5**, If $\|\mathcal{T}\| < 1$ then the input-output operator $I-\mathcal{T}$ is invertible and its inverse has a realization which is SESMS.

Proof. First, note that under assumptions **H1-H3** and **H5**, and following similar arguments as in the proof of Proposition 8.4 in [6], one can shows that if $\|\mathcal{T}\| < 1$ then $\left(I - D^T(t,i)D(t,i)\right) \geqslant \epsilon_0 I$ for some positive scalar $\epsilon_0 < 1$, $\forall t \geqslant 0$, $i \in \Theta$. Therefore for each $i \in \Theta$, the eigenvalues of the matrix $D(t,i)$ are located inside the disc $|\lambda| \leqslant \sqrt{(1-\epsilon_0)}$, $\forall t \geqslant 0$. Hence we obtain that $(I - D(t,i))$ is boundedly invertible $\forall t \geqslant 0$, $i \in \Theta$.
One can deduce from Proposition 1 that the generalized discrete-time Riccati equation:

$$\begin{aligned} X(t,i) &= \Pi_{i1}(t, X(t+1)) + C^T(t,i)C(t,i) \\ &\quad - \Pi_{i2}(t, X(t+1)) \left(\Pi^1_{i3}(t, X(t+1))\right)^{-1} \star \end{aligned} \quad \text{(III.1)}$$

where $t \geqslant 0, i \in \Theta$, admits a bounded and stabilizing solution $X(t) = (X(t,1), X(t,2), \cdots, X(t,N)) \in \mathcal{S}_n^{N+}$, $\forall t \geqslant 0$. Using some matrix algebra, one can show that (III.1) can be rewritten ($\forall t \geqslant 0, i \in \Theta$) as:

$$\begin{aligned} X(t,i) &= \tilde{A}_0^T(t,i)\mathcal{E}_i(t, X(t+1))\tilde{A}_0(t,i) + \bar{\Pi}_{i1}(t, X(t+1)) \\ &\quad + (\bar{C}(t,i) - F^1(t,i))^T \left(-\Pi^1_{i3}(t, X(t+1))\right) \star \end{aligned} \quad \text{(III.2)}$$

where

$$\begin{cases} \bar{C}(t,i) = (I - D(t,i))^{-1}C(t,i) \\ \tilde{A}_0(t,i) = A_0(t,i) + B(t,i)\bar{C}(t,i) \\ \bar{\Pi}_{i1}(t, X(t+1)) = \sum_{k=1}^r A_k^T(t,i)\mathcal{E}_i(t, X(t+1))A_k(t,i) \\ F^1(t,i) = -\left(\Pi^1_{i3}(t, X(t+1))\right)^{-1} \Pi^T_{i2}(t, X(t+1)) \end{cases}$$

Denote $\tilde{C}(t,i) = \left(-\Pi^1_{i3}(t, X(t+1))\right)^{\frac{1}{2}} (\bar{C}(t,i) - F^1(t,i))$. System (II.1), with $A_0(t,i)$ replaced by $\tilde{A}_0(t,i)$ and $C(t,i)$ by $\tilde{C}(t,i)$, $\forall t \geqslant 0, i \in \Theta$, is stochastically detectable. Indeed, by choosing $K(t,i) = -B(t,i)\left(-\Pi^1_{i3}(t, X(t+1))\right)^{-\frac{1}{2}}$, $\forall t \geqslant 0$, $i \in \Theta$ one has $\tilde{A}_0(t,i) + K(t,i)\tilde{C}(t,i) = A_0(t,i) + B(t,i)F^1(t,i)$ and one knows that:

$$x(t+1) = \Big(A_0(t,\theta_t) + B(t,\theta_t)F^1(t,\theta_t) + \sum_{k=1}^r w_k(t)A_k(t,\theta_t)\Big) x(t)$$

is SESMS since $X(t) = (X(t,1), X(t,2), \cdots, X(t,N)) \in \mathcal{S}_n^{N+}$, $\forall t \geqslant 0$ is a bounded and stabilizing solution of (III.1). Hence, from Corollary 4.1 in [6], it follows that the pair $(\tilde{C}_t, \tilde{\mathcal{L}}_t)$ is detectable where $\tilde{C}_t = (\tilde{C}(t,1), \tilde{C}(t,2), \cdots, \tilde{C}(t,N))$ and the operator $\tilde{\mathcal{L}}_t$ is defined by:

$$\begin{aligned} \tilde{\mathcal{L}}_t : \mathcal{S}_n^N &\longrightarrow \mathcal{S}_n^N \\ S &\longmapsto \tilde{\mathcal{L}}_t S = \left(\tilde{\mathcal{L}}_t S(1), \cdots, \tilde{\mathcal{L}}_t S(N)\right) \end{aligned}$$

and

$$\begin{aligned} \tilde{\mathcal{L}}_t S(i) &= \sum_{j=1}^N p_t(j,i)\tilde{A}_0(t,j)S(j)\tilde{A}_0^T(t,j) \\ &\quad + \sum_{k=1}^r \sum_{j=1}^N p_t(j,i)A_k(t,j)S(j)A_k^T(t,j) \end{aligned}$$

Note also that (III.2) can be rewritten in the form of discrete-time backward affine equation:

$$X_t = \tilde{\mathcal{L}}_t^* X_{t+1} + \breve{C}_t \quad \text{(III.3)}$$

where $\breve{C}_t = (\breve{C}(t,1), \breve{C}(t,2), \cdots, \breve{C}(t,N))$, $\forall t \geqslant 0$, and $\breve{C}(t,i) = \tilde{C}^T(t,i)\tilde{C}(t,i)$, $\forall t \geqslant 0$, $i \in \Theta$. Hence, it follows from Theorem 4.1 in [6] that the sequence $\{\tilde{\mathcal{L}}_t\}_{t\geqslant 0}$ generates an exponentially stable evolution. Finally, using Theorem 2.12 and Theorem 3.7 from [6] one concludes that $(I - \mathcal{T})^{-1}$ is SESMS. Hence the proof is complete. ∎

Consider the systems:

$$\mathbb{G}_1 : \begin{cases} x_1(t+1) = \left(A_0^1(t,\theta_t) + \sum_{k=1}^r w_k(t)A_k^1(t,\theta_t)\right) x_1(t) \\ \qquad\qquad +B_1(t,\theta_t)v_1(t) \\ z_1(t) = C_1(t,\theta_t)x_1(t) + D_1(t,\theta_t)v_1(t) \end{cases} \quad \text{(III.4)}$$

$$\mathbb{G}_2 : \begin{cases} x_2(t+1) = \left(A_0^2(t,\theta_t) + \sum_{k=1}^r w_k(t)A_k^2(t,\theta_t)\right) x_2(t) \\ \qquad\qquad +B_2(t,\theta_t)v_2(t) \\ z_2(t) = C_2(t,\theta_t)x_2(t) + D_2(t,\theta_t)v_2(t) \end{cases} \quad \text{(III.5)}$$

where $x_i(t) \in \mathbb{R}^{n_i}$, $i \in \{1,2\}$, $z_1(t)$, $v_2(t) \in \mathbb{R}^{n_z}$, $z_2(t)$, $v_1(t) \in \mathbb{R}^{m_v}$. Let $\mathcal{T}_1$: $l^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbb{R}^{m_v}\} \longrightarrow l^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbb{R}^{n_z}\}$, $\mathcal{T}_2$: $l^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbb{R}^{n_z}\} \longrightarrow l^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbb{R}^{m_v}\}$, their corresponding input-output operators respectively.

Theorem 2. (The small gain theorem) Assume:

a) Assumptions **H1-H5** are verified.
b) The zero state equilibria of (III.4) and (III.5) are SESMS.
c) $\|\mathcal{T}_1\| < \gamma$ and $\|\mathcal{T}_2\| < \frac{1}{\gamma}$ for some $\gamma > 0$.

Under these conditions, the input-output operator $I - \mathcal{T}_1\mathcal{T}_2$ is invertible and its inverse has a realization which is SESMS.

Proof. Set $x(t) = \begin{pmatrix} x_1^T(t) & x_2^T(t) \end{pmatrix}^T$. A state space representation of the product operator $\mathcal{T}_1\mathcal{T}_2$ is given by:

$$\mathbb{G}_{12} : \begin{cases} x(t+1) = (A_0(t,\theta_t) + \sum_{k=1}^r w_k(t)A_k(t,\theta_t))\, x(t) \\ \qquad\qquad +B(t,\theta_t)v_2(t) \\ z_1(t) = C(t,\theta_t)x(t) + D(t,\theta_t)v_2(t) \end{cases} \quad \text{(III.6)}$$

where:

$$A_0(t,i) = \begin{pmatrix} A_0^1(t,i) & B_1(t,i)C_2(t,i) \\ 0 & A_0^2(t,i) \end{pmatrix},$$

$$A_k(t,i) = \begin{pmatrix} A_k^1(t,i) & 0 \\ 0 & A_k^2(t,i) \end{pmatrix}, 1 \leqslant k \leqslant r$$

$$B(t,i) = \begin{pmatrix} B_1(t,i)D_2(t,i) \\ B_2(t,i) \end{pmatrix},$$

$$C(t,i) = \begin{pmatrix} C_1(t,i) & D_1(t,i)C_2(t,i) \end{pmatrix},$$
$$D(t,i) = D_1(t,i)D_2(t,i)$$

The conclusions of Theorem 2 can be obtained directly using Theorem 1. To this end, one has to first verify that (III.6) verifies the assumptions of Theorem 1. That is:

i) $\|\mathcal{T}_1\mathcal{T}_2\| < 1$: directly follows from the inequality $\|\mathcal{T}_1\mathcal{T}_2\| \leqslant \|\mathcal{T}_1\| \cdot \|\mathcal{T}_2\| < 1$.

ii) **The zero state equilibrium of (III.6) is SESMS**: we have to show that

$$\begin{cases} x_1(t+1) = \left(A_0^1(t,\theta_t) + \sum_{k=1}^r w_k(t)A_k^1(t,\theta_t)\right)x_1(t) \\ \qquad\qquad +B_1(t,\theta_t)C_2(t,i)x_2(t) \\ x_2(t+1) = \left(A_0^2(t,\theta_t) + \sum_{k=1}^r w_k(t)A_k^2(t,\theta_t)\right)x_2(t) \end{cases}$$

is SESMS. One sees that the second equation is SESMS by assumption. Using the estimates in Theorem 3.18 from [6] with $g_0(t) = B_1(t,\eta_t)C_2(t,\eta_t)x_2(t)$ and $g_k(t) = 0$ for $k \geq 1$, it follows that the first component $x_1(t)$ is SESMS.

The assumptions in Theorem 1 being verified, the conclusion follows directly. ∎

To end this section, we will now give an application of the proposed small gain theorem. Consider the linear system subject to parametric uncertainties:

$$x(t+1) = \Bigg(A_0(t,\theta_t) + B(t,\theta_t)\Delta(t,\theta_t)C(t,\theta_t) + \sum_{k=1}^r w_k(t)A_k(t,\theta_t)\Bigg)x(t) \qquad \text{(III.7)}$$

where $\{A_k(t,i)\}_{t\geqslant 0}$, $0 \leqslant k \leqslant r$, $\{B(t,i)\}_{t\geqslant 0}$, $\{C(t,i)\}_{t\geqslant 0}$, $i \in \Theta$, are assumed to be known matrices and $\{\Delta(t,i)\}_{t\geqslant 0}$ are unknown matrices. System (III.7) is the perturbed model of the nominal system:

$$x(t+1) = \left(A_0(t,\theta_t) + \sum_{k=1}^r w_k(t)A_k(t,\theta_t)\right)x(t) \quad \text{(III.8)}$$

If the zero state equilibrium of the nominal system is SESMS, we ask the question whether the zero state equilibrium of the perturbed model (III.8) remains SESMS for $\Delta(t,i) \neq 0$. This is the concept of stability robustness. A classical measure of stability robustness is the concept of stability radius. Before giving the definition of the stability radius, we first introduce a norm in the set of uncertainties. Let $\mathbf{\Delta} = \{\Delta_t\}_{t\geqslant 0}$, $\Delta_t = (\Delta(t,1), \Delta(t,2), \cdots, \Delta(t,N))$. We set:

$$|\mathbf{\Delta}| = \sup_{t\geqslant 0}\left(\max_{i\in\Theta}\left(\lambda_{\max}\left(\Delta^T(t,i)\Delta(t,i)\right)\right)^{\frac{1}{2}}\right) \qquad \text{(III.9)}$$

Definition 5. The stability radius of the nominal system (III.7), or equivalently, the stability radius of the pair $(\mathbb{A}, \mathbb{P})$ with respect to the structured uncertainties described by the pair $(\mathbb{B}, \mathbb{C})$ where $\mathbb{B} = \{B_t\}_{t\geq 0}$, $B_t = (B(t,1), B(t,2), \cdots, B(t,N))$, is the number $\rho_L[\mathbb{A}, \mathbb{P}|\mathbb{B}, \mathbb{C}] = \inf\{\rho > 0 | \exists\mathbf{\Delta}$ with $|\mathbf{\Delta}| \leqslant \rho$ such that the zeros state equilibrium of the corresponding system (III.7) is not SESMS$\}$.

A lower bound of the stability radius introduced above is given by the following result:

Theorem 3. Assume that the zero state equilibrium of the nominal system (III.7) is SESMS. Let $\mathcal{T}$: $l^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbb{R}^m\} \longrightarrow l^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbb{R}^p\}$ be the input-output operator of the fictitious system defined by:

$$\mathbb{G}_f : \begin{cases} x(t+1) = \left(A_0(t,\theta_t) + \sum_{k=1}^r w_k(t)A_k(t,\theta_t)\right)x(t) \\ \qquad\qquad +B(t,\theta_t)v(t) \\ z(t) = C(t,\theta_t)x(t) \end{cases} \qquad \text{(III.10)}$$

then we have

$$\rho_L[\mathbb{A}, \mathbb{P}|\mathbb{B}, \mathbb{C}] \geqslant \|\mathcal{T}\|^{-1} \qquad \text{(III.11)}$$

Proof. The result follows from Theorem 2 where $\mathbb{G}_1$ corresponds to $\mathbb{G}_f$ and $\mathbb{G}_2$ is given by:

$$\mathbb{G}_2 : \Big\{ z_2(t) = \Delta(t,\theta_t)v_2(t)$$

∎

IV. Conclusion

In this note, a small gain result has been proposed for a class of discrete-time time-varying Markovian jump systems with state-dependent noise. An application of such a result to the estimation of the stability radius of this class of systems has been also considered.

Appendix

A sketch of the proof of the equivalence i)$\Longleftrightarrow$iii) from Proposition 1.

i) $\Longrightarrow$ iii) : By assumption, one knows that the zero state equilibrium of (II.1) is SESMS and $\|\mathcal{T}\| < \gamma$. Fix $0 < \rho_1^2 < \gamma^2 - \|\mathcal{T}\|^2$ and set $\hat{\gamma} = (\gamma^2 - \rho_1^2)^{\frac{1}{2}}$. We have $\|\mathcal{T}\| < \hat{\gamma}$. Define the perturbed plant $\mathbb{G}_{\epsilon_0}$

$$\mathbb{G}_{\epsilon_0} : \begin{cases} x(t+1) = \left(A_0(t,\theta_t) + \sum_{k=1}^r w_k(t)A_k(t,\theta_t)\right)x(t) \\ \qquad\qquad +B(t,\theta_t)v(t) \\ z^{\epsilon_0}(t) = C^{\epsilon_0}(t,\theta_t)x(t) + D^{\epsilon_0}(t,\theta_t)v(t) \end{cases} \qquad \text{(IV.1)}$$

where

$$C^{\epsilon_0}(t,\theta_t) = \begin{bmatrix} C(t,\theta_t) \\ \epsilon_0\mathbb{I} \end{bmatrix}, \quad D^{\epsilon_0}(t,\theta_t) = \begin{bmatrix} D(t,\theta_t) \\ \mathbf{0} \end{bmatrix}.$$

Obviously, the zero state equilibrium of $\mathbb{G}_{\epsilon_0}$ is SESMS. Hence one can shows, using the boundedness property of the matrix sequences $\{A_k(t,i)\}_{t\geqslant 0}$, $0 \leq k \leq r$ and $\{B(t,i)\}_{t\geqslant 0}$ for all $t \geq 0$, $i \in \Theta$, that there exists a positive constant α such that $\|x\|^2_{l^2_{\tilde{\mathcal{H}}}} \leqslant \alpha\left(|x_0|^2 + \|v\|^2_{l^2_{\tilde{\mathcal{H}}}}\right)$, $\forall x_0 \in \mathbb{R}^n$, for all $v \in l^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbb{R}^{m_v}\}$. Such an inequality may be derived applying Corollary 3.9 (i) in [2]. Then, for the particular case $x_0 = 0$,

one gets $\|x\|^2_{l^2_{\tilde{\mathcal{H}}}} \leqslant \alpha \|v\|^2_{l^2_{\tilde{\mathcal{H}}}}$, for all $v \in l^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbb{R}^{m_v}\}$.
Now, note that

$$\begin{aligned}\|z^{\epsilon_0}\|^2_{l^2_{\tilde{\mathcal{H}}}} &= \|z\|^2_{l^2_{\tilde{\mathcal{H}}}} + {\epsilon_0}^2 \|x\|^2_{l^2_{\tilde{\mathcal{H}}}} \leqslant \|z\|^2_{l^2_{\tilde{\mathcal{H}}}} + {\epsilon_0}^2 \alpha \|v\|^2_{l^2_{\tilde{\mathcal{H}}}} \\ &\leqslant \left[\|\mathcal{T}\|^2 + {\epsilon_0}^2 \alpha\right] \|v\|^2_{l^2_{\tilde{\mathcal{H}}}}\end{aligned} \tag{IV.2}$$

for all $v \in l^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbb{R}^{m_v}\}$. Choose $\beta = \left[\frac{\hat{\gamma}^2 - \|\mathcal{T}\|^2}{2\alpha}\right]^{\frac{1}{2}}$. Fix $0 < \epsilon_0 \leq \beta$, then we have $\|\mathcal{T}\|^2 + {\epsilon_0}^2\alpha \leq \|\mathcal{T}\|^2_\infty + \frac{1}{2}(\hat{\gamma}^2 - \|\mathcal{T}\|^2) = \frac{1}{2}(\hat{\gamma}^2 + \|\mathcal{T}\|^2)$. So we obtain

$$\|\mathcal{T}\|^2 + {\epsilon_0}^2 \alpha < \hat{\gamma}^2 \tag{IV.3}$$

Plugging (IV.3) in (IV.2) we obtain $\|z^{\epsilon_0}\|^2_{l^2_{\tilde{\mathcal{H}}}} < \hat{\gamma}^2 \|v\|^2_{l^2_{\tilde{\mathcal{H}}}}$ for all $v \in l^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbb{R}^{m_v}\}$, and hence $\|\mathcal{T}_{\epsilon_0}\| < \hat{\gamma}$. Now define the system $\mathbb{G}_\epsilon$

$$\mathbb{G}_\epsilon : \begin{cases} x(t+1) = \left(A_0(t, \theta_t) + \sum_{k=1}^r w_k(t) A_k(t, \theta_t)\right) x(t) \\ \qquad\qquad +B(t, \theta_t) v(t) \\ z^\epsilon(t) = C^\epsilon(t, \theta_t) x(t) + D^\epsilon(t, \theta_t) v(t) \end{cases} \tag{IV.4}$$

where

$$C^\epsilon(t, \theta_t) = \begin{bmatrix} C^{\epsilon_0}(t, \theta_t) \\ \rho_2 \mathbb{I} \end{bmatrix}, \quad D^\epsilon(t, \theta_t) = \begin{bmatrix} D^{\epsilon_0}(t, \theta_t) \\ \mathbf{0} \end{bmatrix}.$$

Obviously, the zero state equilibrium of $\mathbb{G}_\epsilon$ is SESMS and $\|x\|^2_{l^2_{\tilde{\mathcal{H}}}} \leqslant \alpha \|v\|^2_{l^2_{\tilde{\mathcal{H}}}}$, for all $v \in l^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbb{R}^{m_v}\}$.
Now, note that

$$\begin{aligned}\|z^\epsilon\|^2_{l^2_{\tilde{\mathcal{H}}}} &= \|z\|^2_{l^2_{\tilde{\mathcal{H}}}} + ({\rho_2}^2 + {\epsilon_0}^2) \|x\|^2_{l^2_{\tilde{\mathcal{H}}}} \\ &= \|z^{\epsilon_0}\|^2_{l^2_{\tilde{\mathcal{H}}}} + {\rho_2}^2 \|x\|^2_{l^2_{\tilde{\mathcal{H}}}} \\ &\leqslant \|z^{\epsilon_0}\|^2_{l^2_{\tilde{\mathcal{H}}}} + {\rho_2}^2 \alpha \|v\|^2_{l^2_{\tilde{\mathcal{H}}}} \leqslant \left[\|\mathcal{T}_{\epsilon_0}\|^2 + {\rho_2}^2 \alpha\right] \|v\|^2_{l^2_{\tilde{\mathcal{H}}}}\end{aligned} \tag{IV.5}$$

for all $v \in l^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbb{R}^{m_v}\}$. Choose $\beta' = \min\left\{\rho_1, \left[\frac{\hat{\gamma}^2 - \|\mathcal{T}_{\epsilon_0}\|^2}{2\alpha}\right]^{\frac{1}{2}}\right\}$. Fix $0 < \rho_2 \leqslant \beta'$, then we have $\|\mathcal{T}_{\epsilon_0}\|^2 + {\rho_2}^2\alpha \leq \|\mathcal{T}_{\epsilon_0}\|^2 + \frac{1}{2}(\hat{\gamma}^2 - \|\mathcal{T}_{\epsilon_0}\|^2) = \frac{1}{2}(\hat{\gamma}^2 + \|\mathcal{T}_{\epsilon_0}\|^2)$. So we obtain

$$\|\mathcal{T}_{\epsilon_0}\|^2 + {\rho_2}^2 \alpha < \hat{\gamma}^2 \leqslant \breve{\gamma}^2 \tag{IV.6}$$

where $\breve{\gamma} = (\gamma^2 - \rho_2^2)^{\frac{1}{2}}$. Hence $\|z^\epsilon\|^2_{l^2_{\tilde{\mathcal{H}}}} < \breve{\gamma}^2 \|v\|^2_{l^2_{\tilde{\mathcal{H}}}}$ for all $v \in l^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbb{R}^{m_v}\}$, and $\|\mathcal{T}_\epsilon\| < \breve{\gamma}$.
Following similar arguments as in [6] (Chapter 8), with some modifications related to the time varying nature of the problem considered here, one can show that under assumptions **H**1-**H**5, the generalized discrete-time Riccati equation (IV.7) has a bounded solution
$\tilde{X}_\epsilon(t) = (\tilde{X}_\epsilon(t, 1), \tilde{X}_\epsilon(t, 2), \cdots, \tilde{X}_\epsilon(t, N)) \in \mathcal{S}_n^{N+}$, $\forall t \geqslant 0$, that verifies (IV.8) (for $t \geqslant 0, i \in \Theta$):

$$\begin{aligned}X(t, i) &= \Pi_{i1}(t, X(t+1)) + (C^\epsilon(t, i))^T C^\epsilon(t, i) \\ &\quad - \Pi^\epsilon_{i2}(t, X(t+1)) \left({}^\epsilon\Pi^{\breve{\gamma}}_{i3}(t, X(t+1))\right)^{-1} \star\end{aligned} \tag{IV.7}$$

$${}^\epsilon\Pi^{\breve{\gamma}}_{i3}(t, X(t+1) \leqslant -\nu I_{m_v}, \ \nu \in (0, \breve{\gamma}^2 - \|\mathcal{T}_\epsilon\|^2) \tag{IV.8}$$

where

$$\begin{cases} \Pi^\epsilon_{i2}(t, X(t+1)) = A_0^T(t, i) \mathcal{E}_i(t, X(t+1)) B(t, i) \\ +(C^\epsilon(t, i))^T D^\epsilon(t, i) \\ {}^\epsilon\Pi^{\breve{\gamma}}_{i3}(t, X(t+1)) = B^T(t, i) \mathcal{E}_i(t, X(t+1)) B(t, i) \\ +(D^\epsilon(t, i))^T D^\epsilon(t, i) - \breve{\gamma}^2 I_{m_v} \end{cases}$$

After multiplying out all the matrices, one obtains

$$\begin{aligned}&\Pi_{i1}(t, \tilde{X}^\epsilon(t+1)) + C^T(t, i) C(t, i) - \tilde{X}_\epsilon(t, i) \\ &- \Pi_{i2}(t, \tilde{X}_\epsilon(t+1)) \left(\Pi^{\breve{\gamma}}_{i3}(t, \tilde{X}_\epsilon(t+1))\right)^{-1} \star + \rho_2^2 I \\ &= -\epsilon_0^2 I < 0\end{aligned} \tag{IV.9}$$

where $\Pi^{\breve{\gamma}}_{i3}(t, \tilde{X}_\epsilon(t+1)) = B^T(t, i) \mathcal{E}_i(t, X(t+1)) B(t, i) + D^T(t, i) D(t, i) - \breve{\gamma}^2 I_{m_v}$. Clearly, $\tilde{X}_\epsilon(t, i) \geq \epsilon_0^2 \mathbb{I} > 0$, $i \in \Theta$, $\forall t \geq 0$. Finally, apply the Schur complement property to show that (II.13) is verified. Hence, the proof is complete.

iii) $\Longrightarrow$ i) The proof is quite similar to the time-invariant case. The details are omitted here. ■

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