2013 European Control Conference (ECC)
July 17-19, 2013, Zürich, Switzerland.

On Kinematic Control Extremals*

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***Abstract*— This paper suggests the possible means for optimal kinematic control problems classification. The classification is partially illustrated by two examples with state-constraints taken from the mechanics of controlled systems. The first of these is connected with the application of kinematic control problems to the inference of the general mathematical models of mechanical systems while the second is a problem of the time-optimal motion of a point on a flat surface that represents the movement of a vehicle on a multilane highway.**

I. INTRODUCTION

Extremal problems with kinematic control arise quite often, not only in the mechanics of control systems, but also in such areas as management science and economics. It's important to indicate that we consider a specific problem to be the kinematic control problem when the function of control has a dimension of speed, acceleration, or their combination. It would be useful to create a classification of mentioned problems in order to shape some patterns that might help in finding their extremals. In this paper we will discuss only Pontryagin extremals that satisfy the maximum principle for optimal control problems with state constraints by R.V. Gamkrelidze [1] which was proven by A. V. Arutyunov, D. Y. Karamzin and F. L. Pereira and first presented in Russian Academy of Sciences in 2011.

II. CLASSIFICATION

Let us introduce the possible classification for problems with kinematic control.

1. Problems connected with relative motion's influence on transportation motion (gyrostats, dynamic oscillation dampers).

2. Problems connected with transportation motion's influence on relative motion (stabilization of inverted pendulum applied to robotics, segways, etc.).

3. Problems aimed at finding motions of dynamic simulation stands (for example, kinematic simulations of flights of aircraft and spacecraft on aeromechanical dynamic stands [2]).

4. Problems of optimal control with known Lagrangian.

5. Problems of optimal control with known kinematically admissible motions.

Examples of 4th and 5th categories will be presented in this paper.

III. THE PROBLEMS WITH KNOWN LAGRANGIAN

A.

Let us consider a situation when the only information about the system that we have is its Lagrangian $L = T - P$, where $T = T(q, \dot{q}, t)$ - kinetic energy, $P = P(q, \dot{q}, t)$ - generalized potential, $q = (q_1, ...q_N)$ - generalized coordinates. We will take a look at a problem of minimizing the Hamilton action functional

$$J = \int_{t_0}^{t_1} L(q, \dot{q})dt$$

between two specified states $q^0 = q(t_0), q^1 = q(t_1)$ at two specified times t_0, t_1 regarding $\dot{q} = v$ as a kinematic control using the maximum principle. It can be directly inferred that

$$p(t) = \frac{\partial L(q,\dot{q})}{\partial \dot{q}},$$
$$\dot{p}(t) = \frac{\partial L(q,\dot{q})}{\partial q},$$

and, therefore, that

$$\frac{d}{dt}\left(\frac{\partial L(q,\dot{q})}{\partial \dot{q}}\right) - \frac{\partial L(q,\dot{q})}{\partial q} = 0 \quad (1)$$

and $H = pv - L(q, v) = p\dot{q} - L(q, \dot{q}) \equiv const$, which are Lagrange equations and first integral respectively. In other words, in this case Pontryagin extremals are the same as Hamilton extremals, and the solution of equations (1) describes true evolution (or true motion) of this system. However, it's known that such a solution doesn't necessarily minimize functional in question [3].

B.

Now let us try to answer whether it is possible to get a model of a mechanical system using Pontryagin extremals of kinematic control in more difficult situations where the system has a scleronomous state constraint $g_1(q) \leq 0$ in a special case: $g_1 = b^T q, \ b \neq 0$. The problem can be formulated as follows:

$$\begin{cases} \dot{q} = v \\ g_1(q) \leq 0 \\ J = \int_{t_0}^{t_1} L(q, v)dt \to \min\limits_{v(\cdot)\in L_\infty^n} \end{cases}$$
$$q(t_0) = a^0, \ q(t_1) = a^1,$$
$$t_0, t_1, a^0, a^1 \text{ - fixed}$$
$$L(q, v) = T(q, v) - P(q);$$

*This work was supported by the grant of the Government of the Russian Federation designed to support scientific research projects implemented under the supervision of leading scientists at Russian institutions of higher learning. Institution: MSU, the contract number: 11.G34.31.0054

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Having introduced Lagrange functional:

$$\mathcal{L}(q,v,p,\nu_1)=\lambda_0 J+\int\limits_{t_0}^{t_1} p(\dot{q}-v)dt+\int\limits_{t_0}^{t_1} g_1(q)d\nu_1$$

where $p(\cdot)=(p_1(\cdot)..p_n(\cdot))$ - absolutely continuous function and ν_1 - nonnegative Radon measure, which act as Lagrange multipliers (it can be shown that $\lambda_0=1$), we can apply the maximum principle. Extended Pontryagin function for this problem has the following form:

$$\tilde{H}=H-\mu_1(t)s_1(q,v)=pv-L(q,v)-\mu_1(t)s_1(q,v),$$
where $s_1=\frac{\partial g_1}{\partial q}v=b^T v.$

Let upper index "0" denote Pontryagin extremal. Then:

- $\dot{p}=-\frac{\partial H^0}{\partial q}+\mu_1(t)\frac{\partial s_1^0}{\partial q}=-\frac{\partial H^0}{\partial q}$
- $\frac{\partial \tilde{H}^0}{\partial v}=0|\Rightarrow p-\frac{\partial L^0}{\partial v}-\mu_1\frac{\partial s_1^0}{\partial v}=0|\Rightarrow$
  $p=\frac{\partial L^0}{\partial v}-\mu_1\frac{\partial s_1^0}{\partial v}$ - generalized impulse, $\mu_1(t)=\int\limits_t^T d\nu_1.$

Here μ_1 is a nonincreasing left continuous on $(t_0;\ t_1)$ function, which equals to a constant on every segment of time, where the optimal path lies in the interior of the set defined by state constraint; in addition $\mu_1(t_1)=0$.

As a result, we have a mathematical model of mechanical system with one scleronomous constraint:

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}=\lambda_1(t)\frac{\partial g_1(q)}{\partial q}, & q(t_0)=a^0,\\ \frac{d\mu_1}{dt}=\lambda_1(t), & q(t_1)=a^1,\\ \lambda_1(t)g_1(q(t))=0, & \mu_1(t_1)=0;\\ g_1(q(t))\le 0; & \end{cases} \qquad (2)$$

If $\mu_1(t)$ is an absolutely continuous function, then mathematical model (2) is correct (the derivative $\frac{d\mu_1}{dt}$ exists).

If $\mu_1(t)$ is a saltus function $h_1(t)$, then the mathematical model can be modified using the results of V. F. Zhuravlev for systems with unilateral constraints [4]. Let t_k be a moment of saltus (impact), $k=1,..N;\ 1\le N\le\infty$. Then the model (2) can be written in the following form:

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}=0,\\ \frac{d\mu_1}{dt}=0,\\ g_1(q(t))<0; \end{cases} \quad \text{if } t\ne t_k,$$

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}=\lambda_1(t_k)b^T,\\ g_1(q(t_k))=0; \end{cases} \quad \text{if } t=t_k,$$

$$\dot{q}(t_k+0)-\dot{q}(t_k)=-2b^T\dot{q}A^{-1}b\cdot\frac{1}{b^TA^{-1}b},\ k=1,..N; \qquad (3)$$

Here we assume that $T=\frac{1}{2}q^TAq$. Formula (3) is the solution of the problem of determining the after-impact state using known before-impact state in case of ideal impact, i.e. ideal constraint: $T_-=T_+$ [4]. Thus, using the maximum principle we managed to get a mathematical model of mechanical system which conforms to classical results.

IV. THE PROBLEMS WITH KNOWN KINEMATICAL ADMISSIBLE MOTIONS

A.

Let us consider a set of the following problems. We will suppose that the only information about the system we have is the description of its admissible motions from the kinematic point of view. For example, the motion of an object on a flat surface can be described by the following system of differential equations and constraints:

$$\begin{cases} \dot{x}=\mathrm{v}\cos\theta, & 0\le \mathrm{v}(t)\le \mathrm{v}_m,\\ \dot{y}=\mathrm{v}\sin\theta, & \\ \dot{\theta}=\omega, & |\omega(t)|\le\omega_m,\\ \dot{\omega}=u, & |u(t)|\le\nu,\\ \dot{\mathrm{v}}=\mathrm{w}, & |\mathrm{w}(t)|\le\mu; \end{cases} \qquad (4)$$

where x and y are the coordinates of a point on a surface, θ is the angle between the object's velocity vector and the positive direction of the x-axis, ω is the angular velocity, v is the scalar velocity, u and w are kinematic controls. This model also has control constraints and pure state constraints. The problem is to minimize the time required to move from known initial manifold to the terminal manifold. It is worth mentioning that the presented system can be viewed as a generalization of such well-known models as Dubins car and Markov car [5],[6],[7],[8]. Solutions and extremals of such problems can later be used as a program - desirable - motions of significantly more complex dynamic models of different vehicles such as segways, robots, bycicles, cars and so on.

B.

Let us consider a problem of the fastest turn for a vehicle in model (4) with the following initial and terminal conditions provided its speed remains constant $\mathrm{v}(t)\equiv 1$:

$$(x(0),\,y(0),\,\theta(0),\,\omega(0))=(0,0,0,0),$$
$$(x(t_k),\,y(t_k),\,\theta(t_k),\,\omega(t_k))=(\gamma_x,\,\gamma_y,\,\vartheta,0),$$

where t_k is the finishing time, γ_x, γ_y are arbitrary constants, ϑ is fixed. The problem can be written in a more accurate form:

$$\begin{cases} \dot{x}=\cos\theta, & \\ \dot{y}=\sin\theta, & z(0)=0,\\ \dot{\theta}=\omega, & z(t_k)\to\min\limits_{u\in U};\\ \dot{\omega}=u, & q=(x,y,\theta,\omega,z)^T \text{ is smooth},\\ \dot{z}=1 & \end{cases} \qquad (5)$$

Control constraints:

$$\begin{aligned} &u(\cdot)\in U=\{u(\cdot)\in L_\infty: u(t)\in R\}\\ &R=\{u|r_1(u)\le 0,\ r_2(u)\le 0\}\\ &r_1(u)=u-1;\\ &r_2(u)=-u-1; \end{aligned} \qquad (6)$$

State constraints:

$$\begin{aligned} &\omega(\cdot)\in\Omega=\{\omega(\cdot)\in C:\omega(t)\in G\}\\ &G=\{\omega|g_1(\omega)\le 0,\ g_2(\omega)\le 0\}\\ &g_1(\omega)=\omega-\omega_m;\\ &g_2(\omega)=-\omega-\omega_m; \end{aligned} \qquad (7)$$

These equations are considered to be nondimensionalized. In order to apply the maximum principle [1] for the problem with state-constraints mentioned above, let us define all the necessary functions.

– *characteristic function of state-constraints:*

$$s_i(q,u) = \frac{\partial g_i(q)}{\partial q} \cdot f(q,u) \,|\Rightarrow\; s_1(u) = u,\;\; s_2(u) = -u;$$

– *extended Pontryagin function*:

$$\bar{H}(p,q,u,\mu,\lambda_0) = <p, f(q,u)> - <\mu, S(u)> = \\ p_x \cos\theta + p_y \sin\theta + p_\theta \omega + p_\omega u + p_z - \mu_1 u + \mu_2 u;$$

– *small Lagrange function*:

$$l(q(t_k),\lambda) = \lambda_0 z(t_k) + \lambda_\theta[\theta(t_k) - \vartheta] + \lambda_\omega[\omega(t_k)].$$

Admissible controlled process $\{q^0(t),\;\; u^0(t),\;\; [0;\,t_k]\}$ satisfies the Pontryagin maximum principle, if the following Lagrange multipliers:

- vector $\lambda = (\lambda_0, \lambda_x, \lambda_y, \lambda_\theta, \lambda_\omega)^T : \lambda_0 \geq 0$;
- absolutely continuous function $p = (p_x, p_y, p_\theta, p_\omega, p_z) : [0; t_k] \to R^5$;
- function $\mu = (\mu_1, \mu_2)^T : [0; t_k] \to R^2$;
- function $\eta = (\eta_1, \eta_2)^T : [0; t_k] \to R^2, \eta_i \in L_\infty$;

exist, λ_0, p, μ are not equal to zero simultaneously and conditions a) - e) are satisfied:

a) <u>*Maximum condition.*</u>

$$\max_{|u|\leq 1} \bar{H}(p(t), q^0(t), u) = \bar{H}(p(t), q^0(t), u^0)$$
a.e. $t \in [0; t_k]$

Therefore $u = \operatorname{sign}(p_\omega - \mu_1 + \mu_2)$.

b) <u>*Adjoint system and transversability condition.*</u>

$$\dot{p} = -\frac{\partial \bar{H}}{\partial q} \;\text{ a.e. } t \in [0; t_k], \\ p(t_k) = -\frac{\partial l(q(t_k),\lambda)}{\partial q(t_k)}.$$

Therefore $p_x \equiv 0$, $p_y \equiv 0$, $p_\theta \equiv -\lambda_\theta$, $p_\omega = \lambda_\theta t + \alpha$, $p_z \equiv -\lambda_0$, where $\alpha = \text{const}$.

c) <u>*Hamiltonian stationarity condition.*</u>

$$\bar{H}(p(t), q^0(t), u^0) \equiv 0 \;\text{ a.e. } t \in [0;\; t_k].$$

d) <u>*Complementary slackness condition:*</u>

$$\eta_i \geq 0 \;\text{ a.e. } t \in [0; t_k], \\ <\eta(t), R(t)> = 0 \;\text{ a.e. } t \in [0; t_k].$$

Therefore $\eta_1(t) \cdot [u(t) - 1] = 0$; $\eta_2(t) \cdot [-u(t) - 1] = 0$ a.e. $t \in [0; t_k]$.

e) <u>*Additional conditions*</u>

$$\frac{\partial \bar{H}(p,\mu,q,u)}{\partial u} = \eta_1(t) \cdot \frac{\partial r_1(u)}{\partial u} + \eta_2(t) \cdot \frac{\partial r_2(u)}{\partial u}$$
a.e. $t \in [0; t_k^0]$

Therefore $p_\omega(t) - \mu_1(t) + \mu_2(t) = \eta_1(t) - \eta_2(t)$.
Properties of μ:
– vector-function μ is left continuous on $(0; t_k)$ and $\mu(t_k) = 0$;
– every μ_i is nonincreasing;
– every μ_i remains constant on every segment of time, where optimal trajectory lies in the interior of the set defined by i state constraint.

Let us take a look at two principally different cases:
1) The extremal path doesn't have a boundary arc of non-zero length, i.e. $g_1(\omega(t)) = \omega(t) - \omega_m < 0$ and $g_2(\omega(t)) = -\omega(t) - \omega_m < 0\; \forall t \in [0, t_k]$.
2) The extremal path has a boundary arc of non-zero length, i.e. $\exists[\tau_1, \tau_2] \subset [0, t_k]$, $[\tau_1, \tau_2] \neq \emptyset$ and $\exists i \in 1, 2 :\;\; \forall t \in [\tau_1, \tau_2] \;\;\; g_i(\omega(t)) = \omega(t) - \omega_m = 0$

For both of these cases the extremal paths do not include any singular arc [9], which therefore indicates, that they do not include chattering arcs. We are not including the proof of this fact because it can be relatively easily inferred from the maximum principle stated above. It's worth mentioning that, according to M.I. Zelikin [10], for the same model without state constraints i.e. (5),(6) and for certain terminal manifolds not only does chattering extremals exist, but they are also required for an optimal junction between the regular and singular arcs as can be concluded from the generalized Kelly-Clebsch condition. Also, he managed to find necessary conditions for optimality of such chattering extremals. Due to the fact that chattering extremals are difficult to realize in practice, their absence makes the solution more applicable to real vehicles after taking into account their dynamics.

<u>First case</u>. From the properties of function μ stated in e) condition it follows that $\mu_{1,2} \equiv 0$. Consequently, according to a) and b): $\bar{H}(p,q,u,\mu,\lambda_0) = p_\theta \omega + p_\omega u + p_z$, $\;u = \operatorname{sign}[p_\omega - \mu_1 + \mu_2] = \operatorname{sign}(p_\omega)$. All the Lagrange multipliers can be proven to be unique up to normalization and found w.r.t. (5)-(7) and the necessary conditions for optimality a)-e); the corresponding control function can be found as well:

$$u|_{[0;\sqrt{\vartheta})} = 1,\; u|_{(\sqrt{\vartheta};2\sqrt{\vartheta}]} = -1; \\ \lambda_0 = \sqrt{\vartheta},\; \lambda_\theta = -1,\; p_x \equiv 0,\; p_y \equiv 0, \\ p_\theta \equiv 1,\; p_\omega(t) = -t + \sqrt{\vartheta},\; p_z \equiv -\sqrt{\vartheta}; \\ \eta_1 = p_\omega \cdot I|_{p_\omega > 0},\; \eta_2 = -p_\omega \cdot I|_{p_\omega < 0},\; \mu_i \equiv 0.$$

Here and in the next case we consider ϑ to be positive. Therefore, the uniqueness of the Lagrange multipliers leads to the conclusion that this Pontryagin extremal is the optimal solution of the problem. Having integrated the differential equations (5), we can conclude that the first case takes place when $\sqrt{\vartheta} < \omega_m$.

<u>Second case</u>. This problem is significantly more difficult due to the boundary arc, that is why to save space we will just state the results which were inferred from (5)-(7) and a)-e). All the Lagrange multipliers can be proven to be unique up to normalization and found w.r.t. (5) and the necessary conditions for optimality a)-e); the corresponding control

function can be found as well:

$$u|_{[0;\omega_m)} = 1,\ u|_{(\omega_m;\,\vartheta/\omega_m)} = 0,\ u|_{[\vartheta/\omega_m;\omega_m+\vartheta/\omega_m]} = -1.$$

$$\begin{array}{l} \lambda_0 = 1,\ \lambda_\theta = -1/\omega_m,\ p_x \equiv 0,\ p_y \equiv 0, \\ p_\theta \equiv 1/\omega_m,\ p_\omega(t) = -t/\omega_m + \vartheta/\omega_m^2,\ p_z \equiv -1, \\ \mu_2 \equiv 0. \end{array}$$

$$\mu_1(t) = \begin{cases} \vartheta/\omega_m^2 - 1, & t \in [0;\omega_m] \\ \vartheta/\omega_m^2 - t/\omega_m, & t \in (\omega_m;\vartheta/\omega_m] \\ 0, & t \in (\vartheta/\omega_m;\omega_m + \vartheta/\omega_m] \end{cases}$$

Therefore, the uniqueness of the Lagrange multipliers leads to the conclusion that this Pontryagin extremal is the optimal solution of the problem. Having integrated the differential equations (5), we can conclude that the second case takes place when $\sqrt{\vartheta} \geq \omega_m$. To sum up, we found the only optimal path for problem B and now can apply this solution to another problem.

C.

Let us consider a problem of the fastest change of lane for a vehicle in model (4) provided its speed remains constant: $\mathrm{v}(t) \equiv 1$. All the equations and constraints as well as initial manifold will remain the same as for the problem (5)-(7), whilst the terminal manifold will change:

$$(x(t_k), y(t_k), \theta(t_k), \omega(t_k), z(t_k)) = (\gamma_x, \eta, 0, 0, \gamma_z),$$

where $\gamma_x, \gamma_z -$ arbitary constants, $\eta -$ fixed.

Let η be positive (in case it's negative the process will be symmetrical). This time we are not going to apply the maximum principle, because it leads to the necessity to deal with Freshnel integrals which aren't easy to analyze. Instead we will try to use just found optimal paths from problem B and watch if this approach reduces complexity. Let us break the problem down into two stages:

I. The fastest ϑ angle turn. The initial and terminal conditions:

$$\begin{array}{l} (x(0),\ y(0),\ \theta(0),\ \omega(0), z(0)) = (0,0,0,0,0), \\ (x(t_k^1),\ y(t_k^1),\ \theta(t_k^1,\ \omega(t_k^1),\ ,z(t_k^1)) = (\gamma_x^1,\ \gamma_y^1,\ \vartheta, 0, \gamma_z^1), \end{array}$$

Due to the fact that we already know the analytical solution, the precise terminal point can be found as soon as ϑ and ω_m are known. It would be appropriate to constrain possible paths of a vehicle and forbid driving perpendicularly to the road during a change of lane. In addition, if $\eta > 0$ then, apparently, ϑ has to be positive as well. In other words, we suppose $0 < \vartheta < \frac{\pi}{2}$.

II. The fastest $(-\vartheta)$ angle turn. The initial condition for this stage is the terminal condition of the previous one, which is considered to be a known function of ϑ and ω_m. The terminal condition of the second stage:

$$(x(t_k^2), y(t_k^2), \theta(t_k^2), \omega(t_k^2,\ z(t_k^2)) = (\gamma_x^2, \gamma_y^2, 0, 0,\ \gamma_z^2)$$

This problem is symmetrical to the I; therefore, we can find the precise terminal point knowing ϑ and ω_m. Consequently, we can find the connection between η which determines the desirable change in y-coordinate and ϑ:

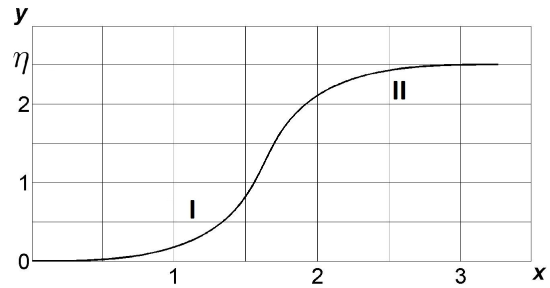

Fig. 1. The trajectory of the system on xy plane

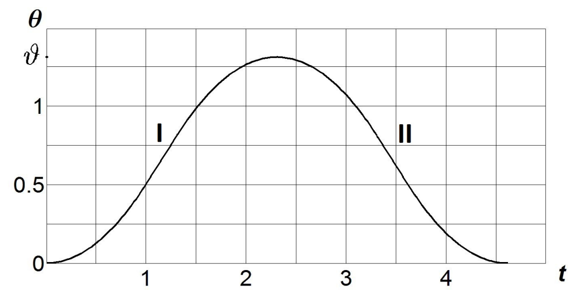

Fig. 2. The dependence of angle θ on time

$$y(t_k^2) = \int_0^{2\left(\omega_m + \vartheta/\omega_m\right)} \sin\theta(t,\vartheta)\cdot dt = \eta.$$

It is worth mentioning that $\theta(\vartheta, t)$ and ϑ have the same sign $\forall t \in (0;t_k)$; in addition $0 \leq \theta(\vartheta,t) \leq \vartheta \leq \pi/2$, thus, $\sin\theta(\vartheta,t) \in [0;1]\ \forall t \in [0;t_k]$. Therefore, knowing η we can find out if corresponding ϑ exists – apparently it does provided η isn't too great – and make a junction between stages I and II. This controlled process will satisfy initial and terminal conditions as well as all the constraints.

It's a well known fact that a junction between two optimal paths isn't necessarily optimal. However, in this particular case, with help of numerical modulation, it can be shown that the resulting process satisfies the maximum principle as long as $\vartheta < \frac{\pi}{2}$ and therefore at least it is a Pontryagin extremal.

V. CONCLUSIONS

In the first example, it has been shown that there is an alternative method of inferring complex mathematical models of mechanical systems applying the maximum principle to specially formulated problems with kinematic control. Acquired results concur with classical ones.

The second example implies that in cases where the theorem produces results that are not informative enough for finding analytical solutions, it might be consistent to apply the maximum principle to less complex problems. Found optimal paths can be later used for analyzing the full size problem. Kinematic control problems for systems with known kinematical admissible motions and their extremals can be used as program trajectories for more complex dynamical systems or for testing the quality of personal control on virtual reality training stands.

As it follows from both examples, the Pontryagin maximum principle in a form of Gamkreligze [1] appears to be a very helpful tool for problems with kinematic control. It happens largely because this maximum principle is obtained in Hamiltonian form which is quite natural for kinematic problems.

The presented classification is far from being complete, and some more though is required in order to finish and enhance it. In the future, such classification may help in developing special techniques for finding Pontryagin extremals.

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