

Distributed bounded-error state estimation for partitioned systems based on practical robust positive invariance

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Abstract—In this paper we propose a novel partition-based state estimator for linear discrete-time systems composed of physically coupled subsystems affected by bounded disturbances. The proposed scheme is distributed in the sense that each local state estimator exploits suitable pieces of information from parent subsystems. Moreover, differently from schemes based on moving horizon estimation, it does not require the on-line solution to optimization problems. Our method also guarantees the satisfaction of constraints on local estimation errors. We achieve our aims exploiting the notion of practical robust positive invariance developed in [1]. As an example, we illustrate the use of the distributed state estimator for reconstructing the states of a power network system.

I. INTRODUCTION

In modern engineering there are several examples of applications composed of a large number of subsystems and for which centralized operations can be very expensive. For instance, the use of centralized controllers and state estimators can be hampered by the complexity of the design stage or the demanding computational and communication requirements for on-line operations. An alternative approach is to decompose the plant into physically coupled subsystems and design local controllers and state estimators associated to each subsystem. The main advantage is that local units can operate in parallel using computational resources available at subsystem locations. Approaches with these features have been studied since the 1970's under the banner of decentralized and distributed control.

Available distributed state estimation schemes can be classified according to different criteria. First, the goal of a local state estimator can be either to reconstruct the state of the overall plant [2], [3] or a subset of it [4], [5], [6], [7], [8]. In this paper we will consider the latter goal and focus on partition-based estimators, meaning that a local estimator must reconstruct only the state of the corresponding subsystem and different subsystems have non-overlapping states. Second, the topology of the communication network connecting local estimators can be different, ranging from all-to-all communication [4] to transmission of information only between physically coupled subsystems [5], [6], [7], [8] arranged in a parent-child relation. Third, local estimators

can be based on unconstrained models [4], [5], [6] or can cope with constraints on system variables such as disturbances and states [7] or estimation errors [8].

In this paper we propose a novel partition-based state estimator for linear discrete-time subsystems affected by bounded disturbances. Similarly to the method proposed in [7] and [8], our scheme is distributed in the sense that computation of local state estimates can be performed in parallel but only after each estimator has received suitable pieces of information from parent subsystems. Moreover, as in [8], state estimators account for constraints on subsystem disturbances and guarantee the fulfillment of *a priori* specified constraints on local estimation errors. Differently from the scheme in [7], that is based on moving horizon estimation, and similarly to [8], local estimators have a Luenberger structure and therefore do not require the on-line solution to optimization problems. Furthermore, most operations required for the design of a local estimator can be done using computational resources collocated with the corresponding subsystem and centralized steps require only the analysis of a system whose order is equal to the number of subsystems.

In order to guarantee convergence of state estimates in absence of disturbances and fulfillment of constraints on the estimation error, we rely on the notion of practical robust positive invariance developed in [1] that is applied to the error dynamics. We also highlight that most of the appealing computational features of our method directly follow from results reported in [1] for the case of polytopic constraints. Since practical robust positive invariance implies worst-case robustness against the propagation of errors between subsystems, our design method involves some degree of conservatism and can not be always applied. Therefore, in the attempt of maximizing chances of successful design, we provide guidelines on the choice of local estimator parameters.

Compared to the distributed state estimator proposed in [8], our scheme has several distinctive features. First, the use of the notion of practical robust positive invariance instead of the more standard concept of robust positive invariance, allows us to achieve a less conservative design procedure (see [9] for a discussion on the degree of conservativeness of various invariance concepts). Second, our local estimators can take advantage of the knowledge of parents' outputs and this can be fundamental for successful estimator design, as demonstrated in Section V through an example. Third, the method in [8] requires to analyze in a centralized fashion the

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stability of a system whose order is equal to the sum of the orders of all subsystems.

The paper is structured as follows. Local state estimators are described in Section II. In Section III we introduce practical robust decentralized invariance and show how it can be applied for guaranteeing convergence of estimators and constraint satisfaction. In Section IV we detail the design of local estimators. In Section V we illustrate the use of the distributed state estimator for reconstructing the states of a power network system. Section VI is devoted to conclusions.

Notation. We use $a : b$ for the set of integers $\{a, a + 1, \dots, b\}$. The symbol \mathbb{R}_+^n stands for the vectors in \mathbb{R}^n with nonnegative elements. The column vector with s components v_1, \dots, v_s is $\mathbf{v} = (v_1, \dots, v_s)$. The symbol \oplus denotes the Minkowski sum, i.e. $A = B \oplus C$ if and only if $A = \{a : a = b + c, b \in B, c \in C\}$. Moreover, $\bigoplus_{i=1}^s G_i = G_1 \oplus \dots \oplus G_s$. The symbol $\mathbf{1}$ (resp. $\mathbf{0}$) denotes a matrix or a column vector with all elements equal to 1 (resp. 0). Given a matrix $A \in \mathbb{R}^{n \times n}$, $\|A\|_1 = \sum_{i=1}^n \sum_{j=1}^n |A_{ij}|$. Given a vector $x \in \mathbb{R}^n$ and a set $\mathbb{S} \subseteq \mathbb{R}^n$, $\text{dist}(x, \mathbb{S}) = \inf_{s \in \mathbb{S}} \|x - s\|$.

The set $\mathbb{X} \subseteq \mathbb{R}^n$ is Robust Positively Invariant (RPI) [10] for $x(t+1) = f(x(t), w(t))$, $w(t) \in \mathbb{W} \subseteq \mathbb{R}^m$ if $x(t) \in \mathbb{X} \Rightarrow f(x(t), w(t)) \in \mathbb{X}, \forall w(t) \in \mathbb{W}$. The RPI set \mathbb{X} is maximal if it includes every other RPI set. The set $\mathbb{X} \subseteq \mathbb{R}^n$ is positively invariant set for $x(t+1) = f(x(t))$ if $x(t) \in \mathbb{X} \Rightarrow f(x(t)) \in \mathbb{X}$.

The set $\mathbb{X} \subseteq \mathbb{R}^n$ is a λ -contractive RPI set, with $\lambda \in [0, 1)$ for $x(t+1) = f(x(t))$ if $x(t) \in \mathbb{X} \Rightarrow f(x(t)) \in \lambda \mathbb{X}$.

A polyhedron \mathbb{X} is the intersection of finitely many half spaces and a polytope is a bounded polyhedron. A \mathcal{C} -set is a set that is compact, convex and contains the origin.

II. DISTRIBUTED STATE ESTIMATOR (DSE)

We consider a discrete-time Linear Time Invariant (LTI) system

$$\begin{aligned} \mathbf{x}^+ &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{D}\mathbf{w} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned} \quad (1)$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^p$ and $\mathbf{w} \in \mathbb{R}^r$ are the state, the input, the output and the disturbance, respectively, at time t and \mathbf{x}^+ stands for \mathbf{x} at time $t+1$. The state is partitioned into M state vectors $x_{[i]} \in \mathbb{R}^{n_i}$, $i \in \mathcal{M} = 1 : M$ such that $\mathbf{x} = (x_{[1]}, \dots, x_{[M]})$ and $n = \sum_{i \in \mathcal{M}} n_i$. Similarly, the input, the output and the disturbance are partitioned into M vectors $u_{[i]} \in \mathbb{R}^{m_i}$, $y_{[i]} \in \mathbb{R}^{p_i}$, $w_{[i]} \in \mathbb{R}^{r_i}$, $i \in \mathcal{M}$ such that $\mathbf{u} = (u_{[1]}, \dots, u_{[M]})$, $m = \sum_{i \in \mathcal{M}} m_i$, $\mathbf{y} = (y_{[1]}, \dots, y_{[M]})$, $p = \sum_{i \in \mathcal{M}} p_i$, $\mathbf{w} = (w_{[1]}, \dots, w_{[M]})$ and $r = \sum_{i \in \mathcal{M}} r_i$.

We assume the dynamics of the i -th subsystem is

$$\begin{aligned} \Sigma_{[i]} : \quad x_{[i]}^+ &= A_{ii}x_{[i]} + B_i u_{[i]} + \sum_{j \in \mathcal{N}_i} A_{ij}x_{[j]} + D_i w_{[i]} \\ y_{[i]} &= C_i x_{[i]} \end{aligned} \quad (2)$$

where $A_{ij} \in \mathbb{R}^{n_i \times n_j}$, $i, j \in \mathcal{M}$, $B_i \in \mathbb{R}^{n_i \times m_i}$, $D_i \in \mathbb{R}^{n_i \times r_i}$, $C_i \in \mathbb{R}^{p_i \times n_i}$ and \mathcal{N}_i is the set of parents of subsystem i defined as $\mathcal{N}_i = \{j \in \mathcal{M} : A_{ij} \neq 0, i \neq j\}$. Since $y_{[i]}$ depends on the local state $x_{[i]}$ only, subsystems $\Sigma_{[i]}$

are output-decoupled and then $\mathbf{C} = \text{diag}(C_1, \dots, C_M)$. Similarly, subsystems $\Sigma_{[i]}$ are input- and disturbance-decoupled, i.e. $\mathbf{B} = \text{diag}(B_1, \dots, B_M)$ and $\mathbf{D} = \text{diag}(D_1, \dots, D_M)$. We also assume

$$w_{[i]} \in \mathbb{W}_i \quad (3)$$

where $\mathbb{W}_i \subset \mathbb{R}^{r_i}$ is a bounded set.

In this section we propose a DSE for (1). We define for $i \in \mathcal{M}$ the local state estimator

$$\begin{aligned} \tilde{\Sigma}_{[i]} : \quad \tilde{x}_{[i]}^+ &= A_{ii}\tilde{x}_{[i]} + B_i u_{[i]} - L_{ii}(y_{[i]} - C_i \tilde{x}_{[i]}) + \\ &\quad \sum_{j \in \mathcal{N}_i} A_{ij}\tilde{x}_{[j]} - \sum_{j \in \mathcal{N}_i} \delta_{ij} L_{ij}(y_{[j]} - C_j \tilde{x}_{[j]}) \end{aligned} \quad (4)$$

where $\tilde{x}_{[i]} \in \mathbb{R}^{n_i}$ is the state estimate, $L_{ij} \in \mathbb{R}^{n_i \times p_j}$ are gain matrices and $\delta_{ij} \in \{0, 1\}$. Hereafter we assume $\delta_{ij} = 0$ and $L_{ij} = 0$ if $j \notin \mathcal{N}_i$. This implies that $\tilde{\Sigma}_{[i]}$ depends only on local variables ($\tilde{x}_{[i]}$, $u_{[i]}$ and $y_{[i]}$) and parents' variables ($\tilde{x}_{[j]}$ and $y_{[j]}$, $j \in \mathcal{N}_i$). Binary parameters δ_{ij} , $j \in \mathcal{N}_i$ can be chosen to take advantage of the knowledge of parents' outputs ($\delta_{ij} = 1$) or to reduce the amount of information received from parents ($\delta_{ij} = 0$).

Defining the state estimation error as

$$e_{[i]} = x_{[i]} - \tilde{x}_{[i]}, \quad (5)$$

from (2), (4) and (5), we obtain the local error dynamics

$$e_{[i]}^+ = \bar{A}_{ii}e_{[i]} + \sum_{j \in \mathcal{N}_i} \bar{A}_{ij}e_{[j]} + D_i w_{[i]} \quad (6)$$

where $\bar{A}_{ii} = A_{ii} + L_{ii}C_i$ and $\bar{A}_{ij} = A_{ij} + \delta_{ij}L_{ij}C_j$, $i \neq j$. Our main goal is to solve the following problem.

Problem 1: Design local state estimators $\tilde{\Sigma}_{[i]}$, $i \in \mathcal{M}$ that

- (a) are nominally convergent, i.e. when $\mathbb{W} = \{0\}$ it holds

$$\|e_{[i]}(t)\| \rightarrow 0 \text{ as } t \rightarrow \infty \quad (7)$$

- (b) guarantee

$$e_{[i]}(t) \in \mathbb{E}_i, \forall t \geq 0 \quad (8)$$

where $\mathbb{E}_i \subseteq \mathbb{R}^{n_i}$ are prescribed sets containing the origin in their interior. \square

Defining the collective variable $\mathbf{e} = (e_{[1]}, \dots, e_{[M]}) \in \mathbb{R}^n$, from (6) one obtains the collective dynamics of the estimation error

$$\mathbf{e}^+ = \bar{\mathbf{A}}\mathbf{e} + \mathbf{D}\mathbf{w} \quad (9)$$

where the matrix $\bar{\mathbf{A}}$ is composed of blocks \bar{A}_{ij} , $i, j \in \mathcal{M}$.

We equip system (9) with constraints $\mathbf{e} \in \mathbb{E} = \prod_{i \in \mathcal{M}} \mathbb{E}_i$ and $\mathbf{w} \in \mathbb{W} = \prod_{i \in \mathcal{M}} \mathbb{W}_i$. In Section III we address Problem 1 under the following assumptions

Assumption 1: The matrices \bar{A}_{ii} , $i \in \mathcal{M}$, are Schur.

Assumption 2: The sets \mathbb{E}_i and \mathbb{W}_i , $i \in \mathcal{M}$ are \mathcal{C} -sets.

Let \mathbf{L} be the matrix composed of blocks L_{ij} , $i, j \in \mathcal{M}$, we highlight that if \mathbf{L} is such that $\bar{\mathbf{A}}$ is Schur, then property (7) holds. If, in addition, Assumption 2 holds, by choosing properly the set \mathbb{E} , then there exists an RPI set $\Omega \subset \mathbb{E}$ for the constrained system (9) and $\mathbf{e}(0) \in \Omega$ guarantees property (8). Remarkably, when sets \mathbb{E}_i and \mathbb{W}_i are polytopes, an RPI set Ω can be found solving a Linear Programming (LP)

problem [10], [11]. However the LP problem includes the collective model (1) in the constraints and computations become prohibitive for large n .

In absence of coupling between subsystems (i.e. $A_{ij} = 0$, $i \neq j$) the estimator dynamics (4) and error dynamics (6) are decoupled as well. Therefore, under Assumptions 1 and 2, properties (7) and (8) can be guaranteed computing RPI sets $\Omega_i \subseteq \mathbb{E}_i$ for each local error dynamics and requiring $e_{[i]}(0) \in \Omega_i$. Furthermore, if \mathbb{E}_i and \mathbb{W}_i are polytopes, the computation of sets Ω_i , $i \in \mathcal{M}$ requires the solution of M LP problems that can be solved in parallel using computational resources collocated with subsystems. In order to propose a partially decentralized design procedure in presence of coupling between subsystems one has to take into account how coupling propagates errors between subsystems. As we will show in the next section, the notion of practical robust positive invariance, proposed in [1] allows one to study precisely this issue and offers a computationally feasible, yet conservative, procedure for solving Problem 1.

III. PRACTICAL ROBUST POSITIVE INVARIANCE FOR STATE ESTIMATION

In this section, we show how the main results of [1], applied to the error dynamics (6) equipped with constraints (3) and (8), allow one to guarantee properties (7) and (8).

Given a collection of sets $\mathbb{S} = \{\mathbb{S}_i, i \in \mathcal{M}\}$, $\mathbb{S}_i \subset \mathbb{R}^{n_i}$ and a set $\Theta \subset \mathbb{R}_+^M$, we define a parameterized family of sets $\mathcal{S}(\mathbb{S}, \Theta) = \{(\theta_1 \mathbb{S}_1, \dots, \theta_M \mathbb{S}_M) : \theta \in \Theta\}$, where $\theta = (\theta_1, \dots, \theta_M)$. Intuitively, scalars θ_i can be interpreted as scaling factors.

Definition 1: The family of sets $\mathcal{S}(\mathbb{S}, \Theta)$ is practical Robust Positive Invariant (pRPI) for the constrained local error dynamics given by (6), (3) and (8), if, for all $i \in \mathcal{M}$ and all $(\theta_1 \mathbb{S}_1, \dots, \theta_M \mathbb{S}_M) \in \mathcal{S}(\mathbb{S}, \Theta)$, one has

$$\theta_i \mathbb{S}_i \subseteq \mathbb{E}_i \quad (10a)$$

$$\bar{A}_{ii} \theta_i \mathbb{S}_i \oplus \bigoplus_{j \in \mathcal{N}_i} \bar{A}_{ij} \theta_j \mathbb{S}_j \oplus D_i \mathbb{W}_i \subseteq \theta_i^+ \mathbb{S}_i \quad (10b)$$

$$(\theta_1^+ \mathbb{S}_1, \dots, \theta_M^+ \mathbb{S}_M) \in \mathcal{S}(\mathbb{S}, \Theta) \quad (10c)$$

Assumption 3: The sets \mathbb{S}_i , $i \in \mathcal{M}$ are \mathcal{C} -sets containing the origin in their interior.

The main issue we will address in the sequel is the following: given \mathbb{S} is there any nonempty set $\Theta \subset \mathbb{R}_+^M$ such that the family $\mathcal{S}(\mathbb{S}, \Theta)$ is pRPI? In order to provide an answer, in [1] it is proposed to first derive the dynamics of the scaling factors θ_i . More precisely, for all $i, j \in \mathcal{M}$ we set

$$\mu_{ij} = \begin{cases} \min_{\mu \geq 0} \{\mu : \bar{A}_{ij} \mathbb{S}_j \subseteq \mu \mathbb{S}_i\} & \text{if } i = j \text{ or } j \in \mathcal{N}_i \\ 0 & \text{otherwise} \end{cases} \quad (11)$$

$$\alpha_i = \min_{\beta \geq 0} \{\beta : D_i \mathbb{W}_i \subseteq \beta \mathbb{S}_i\}. \quad (12)$$

and define the collective dynamics of the scaling factors

$$\theta^+ = T\theta + \alpha \quad (13)$$

where the entries of $T \in \mathbb{R}^{M \times M}$ are $T_{ij} = \mu_{ij}$ and $\alpha = (\alpha_1, \dots, \alpha_M)$. It is easy to show that (13) guarantees

$$e_{[i]} \in \theta_i \mathbb{S}_i \Rightarrow e_{[i]}^+ \in \theta_i^+ \mathbb{S}_i. \quad (14)$$

For fulfilling (10a), let us define

$$\Theta_0 = \{\theta \in \mathbb{R}^M : \forall i \in \mathcal{M}, \theta_i \mathbb{S}_i \subseteq \mathbb{E}_i\} \quad (15)$$

The key assumption used in [1] for providing a set Θ that makes $\mathcal{S}(\mathbb{S}, \Theta)$ a pRPI family is the following one.

Assumption 4: (i) T is Schur.

(ii) The unique equilibrium point $\bar{\theta}$ of system (13) is such that $\bar{\theta} \in \Theta_0$.

(iii) The set Θ is an invariant set for system (13) and constraint set Θ_0 , i.e. $\forall \theta \in \Theta \subseteq \Theta_0$, $\theta^+ \in \Theta$.

Lemma 1 ([1]): Let Assumptions 1-4 hold. Then,

(i) there is a non-trivial convex and compact positively invariant set Θ for system (13) equipped with constraints $\theta \in \Theta_0$;

(ii) $\mathcal{S}(\mathbb{S}, \Theta)$ is pRPI for (6) with constraints (3) and (8). \square

Lemma 1 guarantees that

$$\theta(0) \in \Theta \text{ and } e_{[i]}(0) \in \theta_i(0) \mathbb{S}_i, \forall i \in \mathcal{M} \Rightarrow e_{[i]}(t) \in \theta_i(t) \mathbb{S}_i, \forall i \in \mathcal{M}, \forall t \geq 0 \quad (16)$$

Furthermore, as shown in [1], $\text{dist}(e_{[i]}(t), \bar{\theta}_i \mathbb{S}_i) \rightarrow 0$ as $t \rightarrow \infty$. In the nominal case, i.e. $\mathbb{W} = \{0\}$, one has $\alpha = 0$ in (13). Then $\bar{\theta} = 0$ and property (7) is guaranteed. Also (8) holds since, from (16) and (10a) one has $e_{[i]}(t) \in \theta_i(t) \mathbb{S}_i \subseteq \mathbb{E}_i$. Therefore, Problem 1 is solved if we can design local state estimators fulfilling the assumptions of Lemma 1. A design procedure to achieve this goal is proposed in Section IV.

Remark 1: Note that, according to (16), the initialization of the local estimators requires to find a suitable initial state $\theta(0) \in \Theta$ for system (13) and this is a centralized operation. In order to allow each estimator to locally compute its initial state, one can build offline a hyper-rectangle $\bar{\Theta} = \prod_{i=1}^M [\underline{\theta}_i, \bar{\theta}_i]$ contained in Θ and choose $\tilde{x}_{[i]}(0)$ such that $x_{[i]}(0) - \tilde{x}_{[i]}(0) \in [\underline{\theta}_i, \bar{\theta}_i]$.

IV. DESIGN OF LOCAL ESTIMATORS

In this section, we propose a method to design the distributed state estimator presented in Sections II and III. The key issue is how to compute suitable gains L_{ij} and binary variables δ_{ij} such that Assumption 4 holds. From now on we consider polytopic sets \mathbb{E}_i , \mathbb{W}_i and \mathbb{S}_i , $i \in \mathcal{M}$ verifying Assumptions 2 and 3. Without loss of generality we can write

$$\mathbb{E}_i = \{h_{i,\tau}^T e_{[i]} \leq 1, \forall \tau \in 1 : \bar{\tau}_i\} = \{\mathcal{H}_i e_{[i]} \leq \mathbf{1}\} \quad (17a)$$

$$\mathbb{W}_i = \{f_{i,v}^T w_{[i]} \leq 1, \forall v \in 1 : \bar{v}_i\} = \{\mathcal{F}_i w_{[i]} \leq \mathbf{1}\} \quad (17b)$$

$$\mathbb{S}_i = \{g_{i,\psi}^T s_{[i]} \leq 1, \forall \psi \in 1 : \bar{\psi}_i\} = \{\mathcal{G}_i s_{[i]} \leq \mathbf{1}\} \quad (17c)$$

where $\mathcal{H}_i = (h_{i,1}^T, \dots, h_{i,\bar{\tau}_i}^T) \in \mathbb{R}^{\bar{\tau}_i \times n_i}$, $\mathcal{F}_i = (f_{i,1}^T, \dots, f_{i,\bar{v}_i}^T) \in \mathbb{R}^{\bar{v}_i \times r_i}$ and $\mathcal{G}_i = (g_{i,1}^T, \dots, g_{i,\bar{\psi}_i}^T) \in \mathbb{R}^{\bar{\psi}_i \times n_i}$. The design procedure is summarized in Algorithm 1 that is composed of three parts. Operations in part (A) can be

Algorithm 1

Input: polytopic sets $\mathbb{E}_i, \mathbb{W}_i, i \in \mathcal{M}$ verifying Assumption 2.

Output: A pRPI family of sets $\mathcal{S}(\mathcal{S}, \Theta)$.

(A) *Decentralized steps.* For all $i \in \mathcal{M}$,

- (I) compute the matrix L_{ii} such that \bar{A}_{ii} is Schur and has as many zero eigenvalues as possible;
- (II) compute a λ_i -contractive set \mathbb{S}_i for

$$e_{[i]}^+ = \bar{A}_{ii} e_{[i]} \quad (18)$$

verifying $\mathbb{S}_i \subseteq \mathbb{E}_i$ and set $\mu_{ii} = \lambda_i$;

- (III) compute α_i as in (12).

(B) *Distributed steps.* For all $i \in \mathcal{M}$,

- (I) if $\delta_{ij} = 1$, compute the matrix $L_{ij}, \forall j \in \mathcal{N}_i$ solving

$$\min_{L_{ij}} \|\bar{A}_{ij}\|_1 \quad (19)$$

- (II) compute μ_{ij} as in (11).

(C) *Centralized steps*

- (I) if matrix T is not Schur **stop**;
 - (II) compute set Θ_0 as in (15) and the equilibrium point $\bar{\theta}$ of system (13). If $\bar{\theta} \notin \Theta_0$ **stop**;
 - (III) compute the maximal invariant set Θ_∞ of system (13) equipped with constraint Θ_0 ;
 - (IV) compute an inner box approximation $\bar{\Theta}$ of Θ_∞ .
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executed in parallel using computational resources associated with subsystems, i.e. in a decentralized fashion. Steps in part (B) have a distributed nature, meaning that computations are decentralized but they can be performed only after each system has received suitable pieces of information from its parents. Finally, design steps in part (C) require centralized computations involving only the M -th order system (13). Next, we comment each step of Algorithm 1 in details.

A. Part (A)

Step (AI) is the easiest one and it can be performed only if pairs $(A_{ii}, C_i), i \in \mathcal{M}$, are detectable that is guaranteed by Assumption 1. The requirement of placing eigenvalues of \bar{A}_{ii} in zero is motivated by step (AII).

The computation of sets \mathbb{S}_i as in step (AII) has been suggested in [1] and it is based on the argument that sets $(1 - \lambda_i)\mathbb{S}_i$ can be used for compensating coupling terms in the error dynamics. Remarkably, using the efficient procedures proposed in [11], the computation of a set \mathbb{S}_i amounts to solving the optimization problem

$$\mathcal{P}_i(\mathbb{S}_i^0, k_i) : \min_{\alpha_i, \beta_i, \{\mathbb{S}_i^s\}_{s=1}^{k_i}} \alpha_i \quad (20a)$$

$$\alpha_i \in [0, 1), \quad \mathbb{S}_i^{k_i} \subseteq \alpha \mathbb{S}_i^0 \quad (20b)$$

$$\beta_i \in \mathbb{R}_+, \quad \bigoplus_{s=0}^{k_i-1} \mathbb{S}_i^s \subseteq \beta_i \mathbb{E}_i \quad (20c)$$

$$\mathbb{S}_i^s = \bar{A}_{ii}^s \mathbb{S}_i^0, \quad \forall s = 1, \dots, k_i \quad (20d)$$

where $k_i \in \mathbb{N}$ and the set $\mathbb{S}_i^0 \subset \mathbb{R}^{n_i}$ are provided as inputs. In particular, (20) is an LP problem and the set \mathbb{S}_i can be computed as $\mathbb{S}_i = \beta_i^{-1} \bigoplus_{s=0}^{k_i-1} \mathbb{S}_i^s$. Furthermore, the contractivity parameter is $\lambda_i = \frac{\delta_i + \alpha_i - 1}{\delta_i}$, where $\delta_i = \min_{\tilde{\delta}} \{ \tilde{\delta} : \bigoplus_{s=0}^{k_i-1} \mathbb{S}_i^s \subseteq \tilde{\delta} \mathbb{S}_i^0, \tilde{\delta} \geq 1 \}$. Note that also δ_i can be computed solving an LP problem. As shown in [11], since the matrix \bar{A}_{ii} is Schur, then, given a \mathcal{C} -set \mathbb{S}_i^0 , there exists a sufficiently large k_i such that problem (20) is feasible. Moreover, if all eigenvalues of \bar{A}_{ii} are zero, feasibility of (20) can be guaranteed setting $k_i = n_i$. Indeed since $\bar{A}_{ii}^{n_i} = \mathbf{0}$ we have $\mathbb{S}_i^{n_i} = \{0\}$ and hence, irrespectively of \mathbb{S}_i^0 , constraints (20b) hold with $\alpha_i = 0$. In addition, since from (20d) sets $\{\mathbb{S}_i^s\}_{s=1}^{k_i-1}$ are polytopes containing the origin, then there exists β_i such that constraints (20c) hold. We highlight that the scalar μ_{ii} computed as in (11) is equal to the contractivity parameter λ_i . Step (AIII) focuses on the computation of scalars α_i . From (12) and (17b), using procedures proposed in [12], we have $\alpha_i = \max_{\psi \in 1:\bar{\psi}_i} \{z_i\}$ where

$$z_i = \max_{w_{[i]}} g_{i,\psi} D_i w_{[i]}, \quad \mathcal{F}_i w_{[i]} \leq \mathbf{1}. \quad (21)$$

Therefore, step (AIII) requires the solution to the ψ_i LP problems (21).

B. Part (B)

For the computation of matrices L_{ij} and parameters μ_{ij} , each system $\Sigma_{[i]}$ needs to receive the matrix C_j and the set \mathbb{S}_j from parents $j \in \mathcal{N}_i$ such that $\delta_{ij} = 1$.

In step (BI), if $\delta_{ij} = 1$, the computation of matrices $L_{ij}, j \in \mathcal{N}_i$ is required. Since the choice of L_{ij} affects the coupling term \bar{A}_{ij} and hence the Schurness of matrix T , we propose to reduce the magnitude of coupling by minimizing the magnitude of \bar{A}_{ij} as in (19). In particular, the minimization of $\|\bar{A}_{ij}\|_1$ in (19) amounts to an LP problem. So far the parameters δ_{ij} have been considered fixed. However, if in step (BI) one obtains $L_{ij} = 0$ for some $j \in \mathcal{N}_i$, it is impossible to reduce the magnitude of the coupling term \bar{A}_{ij} and, from (4), the knowledge of $y_{[j]}$ is useless. This suggests to revise the choice of δ_{ij} and set $\delta_{ij} = 0$. In step (BII), since \mathbb{S}_i are polytopes, using procedures proposed in [12] we can compute scalars μ_{ij} as

$$\mu_{ij} = \max_{\psi \in 1:\bar{\psi}_i} \{ \max_{s_{[j]}} g_{i,\psi} \bar{A}_{ij} s_{[j]} : \mathcal{G}_j s_{[j]} \leq \mathbf{1} \}. \quad (22)$$

that requires the solution of $\bar{\psi}_i$ LP problems.

C. Part (C)

In step (CI) we check the Schurness of matrix T . If the test fails one cannot fulfill Assumption 4-(i) and the only possibility is to restart the algorithm after increasing the number of variables δ_{ij} that are equal to one. In step (CII), since the sets \mathbb{S}_i and \mathbb{E}_i are polytopes, using results from [12] the computation of the set Θ_0 can be done as follows

$$\Theta_0 = \prod_{i=1}^M [0, \bar{\theta}_i], \quad \bar{\theta}_i = \left(\max_{\tau \in 1:\bar{\tau}_i} \{ \sup_{s_{[i]}} h_{i,\tau} s_{[i]} : \mathcal{G}_i s_{[i]} \leq \mathbf{1} \} \right)^{-1}. \quad (23)$$

Moreover, in step (CII) we compute the equilibrium point $\bar{\theta}$ of system (13). If $\bar{\theta} \notin \Theta_0$ we can not guarantee property (8) and therefore the algorithm stops. Note that if $\mathbb{W}_i = \{0\}$, $\forall i \in \mathcal{M}$, the equilibrium point $\bar{\theta}$ is the origin and hence $\bar{\theta} \in \Theta_0$ by construction.

According to Assumption 4-(iii), the set Θ of all feasible contractions θ is computed as an RPI set for system (13) and constraints $\theta \in \Theta_0$. In particular, since T is Schur and Θ_0 is a polytope, using results from [13] we can compute the maximal RPI set Θ_∞ by solving a suitable LP problem.

As discussed in Remark 1, a decentralized initialization of state estimators is possible computing an hyperrectangle $\bar{\Theta}$ contained in Θ_∞ . This is done in step (CIV). More precisely, using results from [14], we can set $\bar{\Theta} = \prod_{i=1}^M [0, \theta_i]$ where

$$\bar{\theta}_i = \max_{\theta \in \Theta_\infty} \gamma^T \tilde{\theta}, \quad (24)$$

$$\gamma = (\gamma_1, \dots, \gamma_M), \quad \gamma_i = (\max_{\theta} \theta_i : \theta \in \Theta_\infty)^{-1}. \quad (25)$$

As described in [14], the vector γ is used for maximizing the volume of $\bar{\Theta}$. From (24) and (25) the computation of the hyper-rectangle $\bar{\Theta}$ requires the solution of $M + 1$ LP optimization problems.

V. EXAMPLES

In this section, we apply the proposed distributed state estimator to a power network system composed of several power generation areas coupled through tie-lines. The dynamics of an area equipped with primary control and linearized around equilibrium value for all variables is described by the following continuous-time LTI model [15]

$$\Sigma_{[i]}^C : \quad \dot{x}_{[i]} = A_{ii}^C x_{[i]} + \bar{B}_i^C \bar{u}_{[i]} + \sum_{j \in \mathcal{N}_i} A_{ij}^C x_{[j]} + w_{[i]} \quad (26)$$

where $x_{[i]} = (\Delta\theta_i, \Delta\omega_i, \Delta P_{m_i}, \Delta P_{v_i})$ is the state, $\bar{u}_{[i]} = (\Delta P_{ref,i}, \Delta P_{L,i})$ is composed of the control input of each area and the local power load and \mathcal{N}_i is the sets of parent areas, i.e. areas directly connected to $\Sigma_{[i]}^C$ through tie-lines. In (26), $w_{[i]} \in \mathbb{R}^{n_i}$ is the disturbance term for the i -th area and it is bounded in the polytopic set $\mathbb{W}_i \subset \mathbb{R}^{n_i}$. The matrices of system (26) are defined in [16]. For the meaning of constants as well as parameter values we defer the reader to Section 1 of [16]. We obtain models $\Sigma_{[i]}^C$ by discretizing models $\Sigma_{[i]}^C$ with 1 sec sampling time, using zero-order-hold and treating $\bar{u}_{[i]}$, $x_{[j]}$, $j \in \mathcal{N}_i$ and $w_{[i]}$ as exogenous signals. We note that using the proposed discretization scheme the set of parents \mathcal{N}_i does not change. In the following, we present different designs of the distributed state estimator for the power network composed of four areas shown in Figure 1 (Scenario 1 of [16]¹). In Example 1 and 2, for each area, we consider the following bounds on the state estimation error

$$\mathbb{E}_i = \{e_{[i]} \in \mathbb{R}^{n_i} : \|e_{[i,1]}\|_\infty \leq 0.005, \\ \|e_{[i,k]}\|_\infty \leq 0.01, \quad k \in 2 : 4\}. \quad (27)$$

¹For the simulations, we use the load power steps given in Section 1.1 of [16] and the control inputs computed using MPC controllers as in Section 2 of [16].

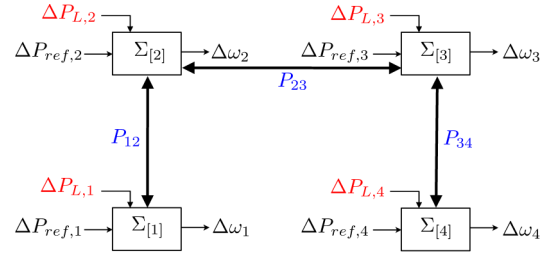


Fig. 1. Power network system composed of four areas

We highlight that constraints (27) amounts to tolerating state estimation errors less than 10% of the maximum value assumed by the state variables. In Example 3, we consider $\mathbb{E}_i = 2\mathbb{E}_i$, $\forall i \in \mathcal{M}$.

A. Example 1

As first example, we consider $\delta_{ij} = 1$, $\forall i \in \mathcal{M}$, $\forall j \in \mathcal{N}_i$, $\mathbb{W}_i = \{0\}$, $\forall i \in \mathcal{M}$, i.e. no disturbances act on the system, and we assume to measure only the angular speed deviation $\Delta\omega_{[i]}$ of each area. Therefore the outputs are given by

$$y_{[i]} = C_i x_{[i]}, \quad C_i = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}. \quad (28)$$

In this case, Algorithm 1 stops in Step (CI): because the computed sets \mathbb{S}_i are such that T is not Schur. We highlight that from the results of Step (BI), all parameters δ_{ij} could be reset to zero. Indeed, for matrices C_i in (28), it is impossible to reduce the magnitude of the coupling terms $\bar{A}_{ij} = A_{ij} + L_{ij}C_j$ by solving optimization problems (19).

B. Example 2

We consider $\mathbb{W}_i = \{0\}$, $\forall i \in \mathcal{M}$, i.e. no disturbances act on the system, and we assume to measure both $\Delta\theta_{[i]}$ and $\Delta\omega_{[i]}$ for each area. Therefore the outputs are given by

$$y_{[i]} = C_i x_{[i]}, \quad C_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \quad (29)$$

First we consider $\delta_{ij} = 0$, $\forall i \in \mathcal{M}$, $\forall j \in \mathcal{N}_i$. In this case, as in the first example, since we cannot take advantage of parents' outputs, Algorithm 1 stops before its conclusion. Indeed, it is impossible to find sets \mathbb{S}_i such that T is Schur. This example shows that if we also consider more output variables for each subsystem, Algorithm 1 can stop in Step (CI) due the magnitude of the coupling terms \bar{A}_{ij} . Now we consider $\delta_{ij} = 1$, $\forall i \in \mathcal{M}$, $\forall j \in \mathcal{N}_i$. In this case we can reduce the magnitude of the coupling terms. Solving optimization problems (19), we can compute matrices L_{ij} such that $\bar{A}_{ij} = \mathbf{0}$, hence the Schurness of matrix T is guaranteed since sets \mathbb{S}_i are λ_i -contractive. In this case, $T = \text{diag}(0.932, 0.843, 0.711, 0.889)$ and $\bar{\Theta} = \{\theta \in \mathbb{R}^4 : 0 \leq \theta_i \leq 1, \forall i = 1 : 4\}$. We note that if matrix T is diagonal, Step (CIV) of Algorithm 1 can be skipped since $\Theta_\infty = \bar{\Theta}$. We performed an estimation experiment initializing the local state estimators $\tilde{\Sigma}_{[i]}$, $i \in \mathcal{M}$ with $\tilde{x}_{[i]}(0) = x_{[i]}(0) - e_{[i]}(0)$, where $e_{[i]}(0)$ is a vertex of the set \mathbb{S}_i . Since no disturbances act on the system, the state estimation error converges to zero as $t \rightarrow \infty$, i.e. (7) is verified.

C. Example 3

We consider $\mathbb{W}_i = \{w_{[i]} \in \mathbb{R}^{n_i} : \|w_{[i,k]}\|_\infty \leq 10^{-5}, k = 1 : 4\}$, $\forall i \in \mathcal{M}$ and output variables given in (29). As in Example 2, by considering $\delta_{ij} = 1$, $\forall i \in \mathcal{M}$, $\forall j \in \mathcal{N}_i$, Algorithm 1 does not stop at any intermediate step. We have performed a similar experiment as in Example 2, but generating random values $w_{[i]}(t)$ from the uniform distribution on \mathbb{W}_i and keeping $w_{[i]}(t)$ constant between sample instants. In Figure 2 and 3, we show the maximum estimation error² at the beginning of the experiment (Figure 2) and for $t \geq 10$ (Figure 3). In particular, even in presence of disturbances on the system the state estimation error $e_{[i]}(t)$ lies in the set \mathbb{E}_i , $\forall i \in \mathcal{M}$.

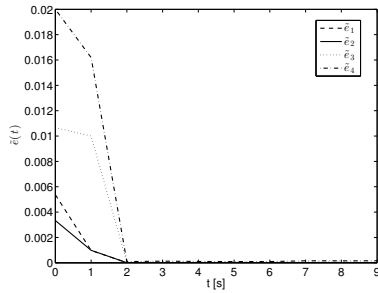


Fig. 2. Maximum estimation error $\tilde{e}_{[j]}(t)$, $t = 0 : 9$ defined as in (30), for simulation in Example 3.

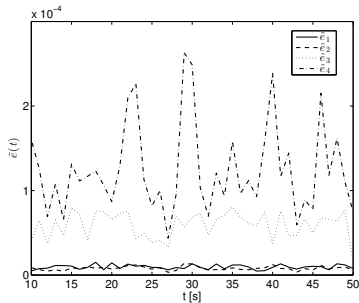


Fig. 3. Maximum estimation error $\tilde{e}_{[j]}(t)$, $t = 10 : 50$ defined as in (30), for simulation in Example 3.

VI. CONCLUSIONS

In this paper, we proposed a novel partition-based state estimator for linear discrete-time subsystems affected by bounded disturbances. The proposed distributed state estimator guarantees the convergence of the overall state estimation and also boundedness on the state estimation error. Since our design method involves some degree of conservatism, we provided an algorithm for the choice of local estimator parameters. Moreover, many steps of the design procedure involve decentralized computations. Similarly to [8] our

²The maximum state estimation error is defined as

$$\tilde{e}_{[j]}(t) = \max_{i \in \mathcal{M}} |x_{[i,j]}(t) - \tilde{x}_{[i,j]}(t)| \quad (30)$$

where $x_{[i,j]}$ and $\tilde{x}_{[i,j]}$ are, respectively, the real and estimated state trajectory of the j -th state of the i -th subsystem.

state estimation algorithm can be directly used together with state-feedback distributed control schemes such as [17] although further research is needed for assessing the stability properties of the closed-loop system. In the future, we will also consider the problem of decentralizing completely computations required in the design process. This would lead to state-estimators that can be designed using local computational resources only, so coping with the plug-and-play design requirements of the model predictive control proposed in [18].

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