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Mean Square Stability of Non-homogeneous Markov Jump Linear Systems using Interval Analysis

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Abstract— **This paper deals with a discrete-time Markov jump linear system with a non-homogeneous Markov Chain. In particular, we consider the situation when the transition probability matrix of the non-homogeneous Markov Chain is varying in an interval, and obtained a sufficient condition for the mean square stability of the proposed system.**

I. INTRODUCTION

Markov jump linear system (MJLS) is an important class of stochastic switched system in which the switched law is governed by a finite state Markov chain (or a finite state Markov process). The motivation on the study of the class of systems is the fact that many dynamical systems subject to random abrupt changes can be modeled by MJLS such as manufacturing system, networked control system, etc. The theory of stability, optimal and robust control, as well as important applications of such systems, can be found in several references in the current literature, for instance in [5], [8], [9], [11] and the references therein.

For a MJLS, most of the works in the literature assume that the underlying Markov chain (or process) is time-homogeneous, which implies that the transition probability matrix (TPM) is time-invariant (or sojourn time follows an exponential distribution). However, this assumption is not verified when considering failure prone systems for example. Usually, in this class of systems, a time-homogeneous Markov chain (or process) is used to model the random failures, which implies a time-invariant TPM (or a constant failure rate). In reality however, this assumption is often violated because the failure probabilities (or rate) of a component usually depends on many factors, for example, its age, the degree of usage, etc.

For a non-homogeneous MJLS, the investigation of exponential mean square stability using Lyapunov's second method is carried out in [12]. A sufficient condition for the mean square stability using Kronecker products is obtained in [8]. When the variation of TPM (or switching rate matrix) is confined to vary in a convex hull with known vertices, second moment stability is addressed in [1], [6] and [7]. The other kind of description is in an element-wise way. In this situation, the second moment stability is carried out with a given switching rate matrix along with its error bounds ([3], [18]).

In this paper, we consider the mean square stability of a discrete-time MJLS with a non-homogeneous Markov chain. Our contribution in this article is to consider the variation of TPM in an interval, that means the time-varying nature of TPM is captured as an interval TPM. By using the results of interval analysis, we provide a sufficient condition for the mean square stability of a non-homogeneous discrete-time MJLS.

The rest of the article is organized as follows: Section II gives the description of a discrete-time MJLS with interval TPM. Mathematical preliminaries are presented in Section III. In section IV, a sufficient condition for the mean square stability of a discrete-time MJLS is obtained. In Section V, a numerical example is given to illustrate the proposed results, and the concluding remarks are addressed in Section VI.

Notation : Let $\mathbb{R}^n$ be the n-dimensional real Euclidean space. A^T is the transpose of a matrix A. $L \geq_e M$ (or $L \leq_e M$) denotes the element-wise inequalities of matrices L, M. $\rho(A)$ represents the spectral radius of a matrix A. $tr(A)$ represents the trace of a matrix A. The standard vector norm in $\mathbb{R}^n$ is indicated by $||.||$ the corresponding induced norm of a matrix A by $||A||$. Let $\mathbb{O}$ denote a zero matrix or a zero vector of appropriate dimension. $\mathbb{I}_n$ denotes the identity matrix of dimension $n \times n$. The empty set is represented by $\emptyset$.

II. MODEL DESCRIPTION

Consider the following discrete-time MJLS

$$x_{k+1} = H(\sigma_k)x_k, \quad H(\sigma_k) \in \mathbb{R}^{n\times n} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state of the system. Let $(\Omega, \mathcal{F}, Pr)$ be the underlying probability space, where Ω is the sample space, $\mathcal{F}$ is the sigma algebra of events and Pr is the probability measure. Let the expectation of a random variable be denoted by $E\{.\}$. The form process $\{\sigma_k\}_{k\geq 0}$ is described by a discrete-time Markov chain with state space $\underline{N} = \{1, 2, ...N\}$, and TPM $\Pi_k = (\pi_{ij}(k))_{N\times N}$ where

$$Pr\{\sigma_{k+1} = j|\sigma_k = i\} = \pi_{ij}(k)$$

with $\pi_{ij}(k) \geq 0$ and $\sum_{j=1}^{N} \pi_{ij}(k) = 1$, $\forall k \geq 0$, which is a property of the stochastic matrix. Let $\pi_0 = (\pi_i(0))_{1\times N}$ be the initial probability mass function of $\{\sigma_k\}_{k\geq 0}$ defined by

$$\pi_0 = [Pr\{\sigma_0 = 1\}\ Pr\{\sigma_0 = 2\}\ \cdots\ Pr\{\sigma_0 = N\}]_{1\times N}$$

The notation $H(\sigma_k)$ denote the dependence of matrix H on the Markovian form process $\{\sigma_k\}_{k\geq 0}$.

If $\Pi_k = \Pi$, $\forall k \geq 0$, then $\{\sigma_k\}_{k\geq 0}$ is regarded as a time-homogeneous Markov chain. If Π_k is dependent on k, then $\{\sigma_k\}_{k\geq 0}$ is known as a non-homogeneous Markov chain or a

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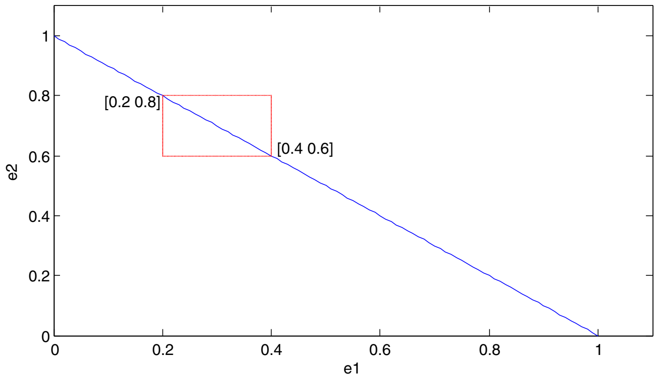

Fig. 1. Variation of first row ($[e_1\ e_2]$) of Π_k. Note that $e_1 + e_2 = 1$ since Π_k is stochastic.

time-inhomogeneous Markov chain. In the sequel, we present the time-varying TPM Π_k as an interval matrix.

Characterization of TPM as an interval matrix:

Let $\{\sigma_k\}_{k\geq 0}$ be a finite state non-homogeneous Markov chain with TPM $\Pi_k = (\pi_{ij}(k))_{N\times N}$ assumed to be varying in an interval. Such an interval matrix is defined by

Definition 1: Let $P \in \mathbb{R}^{N\times N}$ and $Q \in \mathbb{R}^{N\times N}$ be non-negative matrices with $P \leq_e Q$. We represent Π_k which is varying in an interval $[P, Q]$ as

$$\Pi_k = [P, Q] \in \mathbb{IR}^{NN}$$

where Π_k is a $N \times N$ stochastic matrix with $P \leq_e \Pi_k \leq_e Q$, for all k. Assume that P and Q are such that $[P, Q] \neq \emptyset$.

Remark 1: Given $\Pi_k = [P, Q]$, a convex polytope in which Π_k is varying can be obtained. Consider the following examples

Example 1: Let

$$\Pi_k = \begin{bmatrix} [0.2, 0.4] & [0.6, 0.8] \\ 0.5 & 0.5 \end{bmatrix} \in \mathbb{IR}^{22}$$

then Π_k is varying in a convex polytope with vertices $\begin{bmatrix} 0.4 & 0.6 \\ 0.5 & 0.5 \end{bmatrix}$ and $\begin{bmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{bmatrix}$ as partly observed in figure 1, which just demonstrates the variation of the first row of Π_k.

Example 2: p.41 [10]
Let

$$\Pi_k = \begin{bmatrix} [0.35, 0.45] & [0.55, 0.65] \\ [0.25, 0.35] & [0.65, 0.75] \end{bmatrix} \in \mathbb{IR}^{22}$$

then Π_k is varying in a convex polytope with vertices $\begin{bmatrix} 0.35 & 0.65 \\ 0.25 & 0.75 \end{bmatrix}$, $\begin{bmatrix} 0.35 & 0.65 \\ 0.35 & 0.65 \end{bmatrix}$, $\begin{bmatrix} 0.45 & 0.55 \\ 0.25 & 0.75 \end{bmatrix}$ and $\begin{bmatrix} 0.45 & 0.55 \\ 0.35 & 0.65 \end{bmatrix}$.
One can observe that the number of vertices of convex polytope increases with increase in the dimension of matrix Π_k. Also, obtaining vertices of convex polytope from a stochastic interval matrix is a cumbersome procedure. Interested reader can find a detailed investigation of obtaining vertices of polytopes of interval stochastic matrices in [10]. In this article we confine ourselves to dealing with intervals $[P, Q]$ of Π_k as given in Definition 1.

Definitions of various second moment stability concepts for discrete-time MJLS are provided in the following [8]

Definition 2: For the system (1), the equilibrium point 0 is

- *Stochastically Stable* (SS), if for every initial state ($x_0 \in \mathbb{R}^n$) and for every initial probability distribution of σ_k
$$E\{\sum_{k=0}^{\infty} ||x_k(x_0, \omega)||^2\} < \infty \qquad (2)$$
where $x_k(x_0, \omega)$ is a sample solution of (1) initial from x_0.
- *Mean Square Stable* (MSS), if for every initial state ($x_0 \in \mathbb{R}^n$) and for every initial probability distribution of σ_k
$$\lim_{k\to\infty} E\{||x_k(x_0, \omega)||^2\} = 0 \qquad (3)$$
- *Exponentially Mean Square Stable* (EMSS), if for every initial state ($x_0 \in \mathbb{R}^n$) and for every initial probability distribution of σ_k , there exist constant $0 \leq \alpha \leq 1$ and $\beta > 0$ such that for all $k \geq 0$
$$E\{||x_k(x_0, \omega)||^2\} \leq \beta\alpha^k \, ||x_0||^2 , \qquad (4)$$
where α and β are independent of x_0.

III. Mathematical preliminaries

We begin this section with a brief survey of interval analysis [2], [16], which is necessary to understand the notions presented in the sequel.

We consider only real compact intervals, which are denoted by $[\underline{a}, \overline{a}]$, $\underline{a} \leq \overline{a}$, where $\underline{a}, \overline{a} \in \mathbb{R}$. We call $[\underline{a}, \overline{a}]$ a *non-degenerate interval* if $\underline{a} \neq \overline{a}$. Denote the set of all these intervals by $\mathbb{IR}$. Matrices with entries belonging to $\mathbb{IR}$ are denoted by $\mathscr{A}$, $\mathscr{B}$,..or by (A_{ij}), (B_{ij}),...; $\mathscr{A}$, $\mathscr{B}$,.. are called interval matrices; the set of all $n \times m$ interval matrices are denoted by $\mathbb{IR}^{nm}$. Denote $|[\underline{a}, \overline{a}]| \triangleq \max\{|\underline{a}|, |\overline{a}|\}$, and $|\mathscr{A}| \triangleq |A_{ij}|$ for each i, j. Denote $d([\underline{a}, \overline{a}]) \triangleq \overline{a} - \underline{a}$, and $d(\mathscr{A}) \triangleq d(A_{ij})$ for each i, j.

For $\mathscr{A} = (A_{ij})$, $\mathscr{B} = (B_{ij})$, the matrix operations +, − , . are defined as

$$\mathscr{A} \pm \mathscr{B} : (A_{ij} \pm B_{ij})$$
$$\mathscr{A}.\mathscr{B} : \left(\sum_s A_{is}B_{sj}\right)$$

For $a \in \mathbb{R}$, $a = [a, a] \in \mathbb{IR}$; for $A \in \mathbb{R}^{n\times n}$, $A = [A, A] \in \mathbb{IR}^{nn}$. These special intervals and matrices are called point intervals (or degenerate intervals) and point matrices. Point matrices are denoted by $\dot{\mathscr{A}}$, $\dot{\mathscr{B}}$, ..., using a dot above the character. It is obvious to note that the zero point matrix can be denoted by $\mathbb{O}$.

Definition 3: Let $\mathscr{A} \in \mathbb{IR}^{nn}$, then the powers of $\mathscr{A}^k = (A_{ij}^{(k)})$ of $\mathscr{A}$ are defined by

$$\mathscr{A}^0 := \mathbb{I}_n; \qquad \mathscr{A}^k := \mathscr{A}^{k-1}.\mathscr{A}, \quad k = 1, 2, ..$$

Definition 4: Let $\mathscr{A}^k = (A_{ij}^{(k)}) = ([\underline{a}_{ij}^{(k)}, \overline{a}_{ij}^{(k)}])$, we say that $\mathscr{A}$ is convergent (to zero) iff the sequence $\{\mathscr{A}^k\}$ converges to $\mathbb{O}$, i.e; $\lim_{k\to\infty} \underline{a}_{ij}^{(k)} = 0$, $\lim_{k\to\infty} \overline{a}_{ij}^{(k)} = 0$, $i, j = 1, 2, ..., n$. If $\mathscr{A}$ is not convergent, we call it divergent.

Definition 5: The directed graph $G(\mathscr{A})$ associated with $\mathscr{A} \in \mathbb{IR}^{nn}$ is the directed graph $G(|\mathscr{A}|)$ of a real matrix $|\mathscr{A}|$. Let a vertex set of $G(\mathscr{A})$ be $\{P_1, \cdots, P_n\}$. Any two vertices P_i, P_j are connected by a direct path $\Gamma_{ij} \triangleq \overrightarrow{P_i P_j}$ iff the corresponding interval matrix entry A_{ij} is not zero.

Definition 6: The jth column of $\mathscr{A}$ has property $(*)$ iff there exists a power $\mathscr{A}^m$ containing in the same jth column at least one interval not degenerated to a point interval.

An example is presented here to illustrate the $(*)$ property among the columns of $\mathscr{A}$.

Example 3: p.206 [15]

Let

$$\mathscr{A} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & [0, 0.5] & 0 \end{bmatrix}, \tag{5}$$

then

$$\mathscr{A}^k = 2^{k-4} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ [-1, 1] & [-1, 1] & 0 \end{bmatrix}, \; k \geq 3. \tag{6}$$

From (5), it is straightforward to say that the second column of $\mathscr{A}$ has $(*)$ property because it contains a non-degenerate interval $[0, 0.5]$. But the other columns of $\mathscr{A}$ may have $(*)$ property. To verify the $(*)$ property for other columns, consider (6). From (6), we can conclude that first and second columns of $\mathscr{A}$ has $(*)$ property because $\mathscr{A}^k, k \geq 3$ contains non-degenerate intervals $[-1, 1]$, $[-1, 1]$.

The procedure carried out in the above example is quite naive and cumbersome in finding the $(*)$ property among the columns of an interval matrix $\mathscr{A}$. To know whether the jth column of $\mathscr{A}$ has property $(*)$ and its further relevance, Lemma 1, Lemma 2, Lemma 3 are considered from [15].

Lemma 1: Each column of an irreducible interval matrix $\mathscr{A}$ has property $(*)$ provided $d(\mathscr{A}) \neq \mathbb{O}$.

Lemma 2: Let $1 \leq j \leq n$, then the jth column of $\mathscr{A}$ has property $(*)$ iff there exists a directed path $\Gamma_{ij} = \overrightarrow{P_i P_{i_1}}, \overrightarrow{P_{i_1} P_{i_2}}, \cdots \overrightarrow{P_{i_k} P_j}$ in $G(\mathscr{A})$ such that Γ_{ij} contains at least one non-degenerate interval.

Lemma 3: Let $\mathscr{A} \in \mathbb{IR}^{nn}$. Construct the real matrix $\dot{\mathscr{B}} = (b_{ij})$ by

$$b_{ij} = \begin{cases} |A_{ij}| & \text{if the j}^{\text{th}} \text{ column of } \mathscr{A} \text{ has property } (*) \\ A_{ij} & \text{otherwise} \end{cases}$$

Then $\mathscr{A}$ is convergent iff $\rho(\dot{\mathscr{B}}) < 1$.

The following lemma from graph theory will also be used in the sequel.

Lemma 4: [17] A square matrix A is reducible if for all permutation matrices P

$$P^T A P = \begin{bmatrix} X & Y \\ \mathbb{O} & Z \end{bmatrix}$$

where X, Z are square matrices of order at least 1. A matrix is said to be irreducible if it is not reducible. Equivalently, a square matrix A is irreducible if and only if its directed graph $G(A)$ is strongly connected, which in other words, A is irreducible if and only if for each pair of indices (i, j) there is a sequence of entries in A such that $a_{ik_1} a_{k_1 k_2} \cdots a_{k_t j} \neq 0$.

We say that the interval matrix $\mathscr{A}$ is irreducible iff either its directed graph $G(\mathscr{A})$ is strongly connected or $|\mathscr{A}|$ is irreducible.

IV. Main Results

In this section a sufficient condition for the MSS of the system (1) is established.

Let $\Pi_k = [P, Q]$, then

$$\Pi_k^T = [P^T, Q^T]$$

Since $P \geq_e 0$, $Q \geq_e 0$,

$$\Pi_k^T \otimes \mathbb{I}_{n^2} = [P^T \otimes \mathbb{I}_{n^2}, Q^T \otimes \mathbb{I}_{n^2}]$$

where $\otimes$ denotes a Kronecker product. Please see Appendix for a brief introduction about Kronecker products.

Consider

$$(\Pi_k^T \otimes \mathbb{I}_{n^2}) \text{diag}\{G(i)_N\} = [\underline{M}, \overline{M}] \triangleq \mathscr{A} \tag{7}$$

where

$$(\underline{M})_{ij} = \min \left\{ \sum_{l=1}^{N} (P^T \otimes \mathbb{I}_{n^2})_{il} (\text{diag}\{G(i)_N\})_{lj}, \right. \\ \left. \sum_{l=1}^{N} (Q^T \otimes \mathbb{I}_{n^2})_{il} (\text{diag}\{G(i)_N\})_{lj} \right\},$$

$$(\overline{M})_{ij} = \max \left\{ \sum_{l=1}^{N} (P^T \otimes \mathbb{I}_{n^2})_{il} (\text{diag}\{G(i)_N\})_{lj}, \right. \\ \left. \sum_{l=1}^{N} (Q^T \otimes \mathbb{I}_{n^2})_{il} (\text{diag}\{G(i)_N\})_{lj} \right\},$$

$$\text{diag}\{G(i)_N\} = \begin{bmatrix} G(1) & & & \\ & G(2) & & \\ & & \ddots & \\ & & & G(N) \end{bmatrix}, \tag{8}$$

and $G(i) = (H(i) \otimes H(i))$.

From (1),

$$\begin{aligned} x_k &= H(\sigma_{k-1}) x_{k-1} \\ &= H(\sigma_{k-1}) \cdots H(\sigma_0) x_0 \end{aligned}$$

Let

$$\Phi(k, 0) \triangleq \begin{cases} H(\sigma_{k-1}) \cdots H(\sigma_0) & \text{if } k \geq 1, \\ \mathbb{I}_n & \text{if } k = 0 \end{cases}$$

then

$$\begin{aligned} x_k &= \Phi(k, 0) x_0 \\ x_k x_k^T &= \Phi(k, 0) x_0 x_0^T \Phi^T(k, 0) \end{aligned}$$

From Lemma 7 of Appendix,

$$\begin{aligned} \text{vec}(x_k x_k^T) &= [\Phi(k, 0) \otimes \Phi(k, 0)] \text{vec}(x_0 x_0^T) \\ E\{\text{vec}(x_k x_k^T)\} &= E\{\Phi(k, 0) \otimes \Phi(k, 0)\} \text{vec}(x_0 x_0^T) \end{aligned} \tag{9}$$

Denote

$$\Phi_i(k) \triangleq Pr\{\sigma_k = i\}E\{\Phi(k,0)\otimes\Phi(k,0)/\sigma_k = i\}$$

then

$$\Phi_i(0) = \pi_i(0)\mathbb{I}_{n^2} \qquad \text{and}$$
$$\sum_{i=1}^{N}\Phi_i(k) = E\{\Phi(k,0)\otimes\Phi(k,0)\}. \tag{10}$$

Denote

$$V_\Phi(k) \triangleq \begin{bmatrix}\Phi_1(k)\\ \vdots \\ \Phi_N(k)\end{bmatrix}. \tag{11}$$

The following lemma will be used in the sequel.

Lemma 5: Suppose that $\{\sigma_k\}_{k\geq 0}$ is a finite state non-homogeneous Markov chain with TPM $\Pi_k = (\pi_{ij}(k))_{N\times N} \triangleq [P,Q] \in \mathbb{IR}^{NN}$, and let $\mathscr{A} = [\underline{M}, \overline{M}]$ (as defined in (7)), then $V_\Phi(k) = \mathscr{A}^k V_\Phi(0)$.

Proof: For the non-homogeneous case, by proceeding along the similar lines in Lemma 2.4 of [14],

$$V_\Phi(k) = A_k\ V_\Phi(k-1)$$

where

$$A_k = (\Pi_{k-1}^T \otimes \mathbb{I}_{n^2})\text{diag}\{G(i)_N\}$$

with $(\Pi_{k-1})_{ij} = \pi_{ij}(k-1)$. So

$$V_\Phi(k) = A_k \cdots A_1\ V_\Phi(0).$$

Since $\Pi_k = [P,Q]$, $A_k = \mathscr{A}$ for all k (from (7)). Thus

$$V_\Phi(k) = \mathscr{A}^k V_\Phi(0),$$

completing the proof. ■

To investigate further, let us consider two cases,

Case I: $\mathscr{A}$ *is irreducible and* $d(\mathscr{A}) \neq \mathbb{O}$

Theorem 1: Suppose that $\{\sigma_k\}_{k\geq 0}$ is a finite state non-homogeneous Markov chain with TPM $\Pi_k = (\pi_{ij}(k))_{N\times N} \triangleq [P,Q] \in \mathbb{IR}^{NN}$, and let $\mathscr{A} = [\underline{M}, \overline{M}]$ (as defined in (7)). Let $\mathscr{A}$ is irreducible and $d(\mathscr{A}) \neq \mathbb{O}$, then the system (1) is MSS if $\rho(|\mathscr{A}|) < 1$.

Proof: Since $\mathscr{A}$ is irreducible and $d(\mathscr{A}) \neq \mathbb{O}$, every column of $\mathscr{A}$ has $(*)$ property (Lemma 1). Thus from Lemma 3, $\dot{\mathscr{B}} = |\mathscr{A}|$, and the convergence of $\{\mathscr{A}^k\}$ to $\mathbb{O}$ is guaranteed if and only if $\rho(|\mathscr{A}|) < 1$. To arrive at the result, we prove in the sequel that the convergence of $\{\mathscr{A}^k\}$ to $\mathbb{O}$ implies MSS of (1) (as depicted with dashed lines in figure 2).

From (9), (10), (11) and using Lemma 5

$$\begin{aligned}E\{\text{vec}(x_k x_k^T)\} &= E\{\Phi(k,0)\otimes\Phi(k,0)\}\text{vec}(x_0x_0^T)\\ &= \sum_{i=1}^{N}\Phi_i(k)\ \text{vec}(x_0x_0^T)\\ &= [\mathbb{I}_{n^2}\cdots\mathbb{I}_{n^2}]V_\Phi(k)\ \text{vec}(x_0x_0^T)\\ &= [\mathbb{I}_{n^2}\cdots\mathbb{I}_{n^2}]\mathscr{A}^k V_\Phi(0)\ \text{vec}(x_0x_0^T)\end{aligned}$$

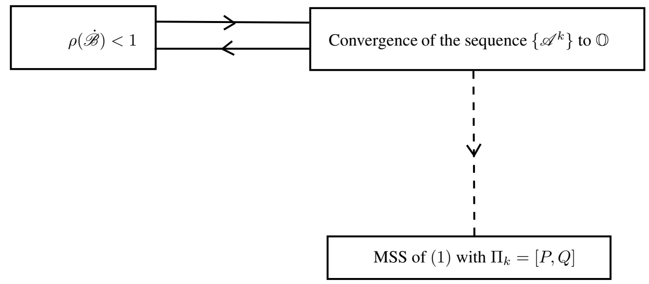

Fig. 2. Illustration of *Theorem 1*, where $\dot{\mathscr{B}} = |\mathscr{A}|$ if $\mathscr{A}$ is irreducible and $d(\mathscr{A}) \neq \mathbb{O}$

Since

$$E\{||x_k||^2\} = E\{x_k^Tx_k\} = tr(E\{x_kx_k^T\})$$

the convergence of $\{\mathscr{A}^k\}$ to $\mathbb{O} \Longrightarrow$ the system (1) is MSS. ■

If diag$\{G(i)_N\}$ is a non-negative matrix, then a simpler condition can be obtained for the MSS of (1). So, if diag$\{G(i)_N\} \geq_e 0$, then

$$\begin{aligned}\mathscr{A} &= (\Pi_k^T\otimes\mathbb{I}_{n^2})\text{diag}\{G(i)_N\}\\ &= [(P^T\otimes\mathbb{I}_{n^2})\text{diag}\{G(i)_N\}, (Q^T\otimes\mathbb{I}_{n^2})\text{diag}\{G(i)_N\}]\end{aligned}$$

So, $|\mathscr{A}| = (Q^T\otimes\mathbb{I}_{n^2})\text{diag}\{G(i)_N\}$, then Theorem 1 will be simplified as follows.

Corollary 1: Given the assumptions of Theorem 1, and if diag$\{G(i)_N\} \geq_e 0$, then the system (1) is MSS if $\rho((Q^T\otimes\mathbb{I}_{n^2})\text{diag}\{G(i)_N\}) < 1$.

Case II: $\mathscr{A}$ *is reducible*

Since $\mathscr{A}$ is reducible, we consider $\dot{\mathscr{B}}$ given in Lemma 3.

Theorem 2: Suppose that $\{\sigma_k\}_{k\geq 0}$ is a finite state non-homogeneous Markov chain with TPM $\Pi_k = (\pi_{ij}(k))_{N\times N} \triangleq [P,Q] \in \mathbb{IR}^{NN}$, and let $\mathscr{A} = [\underline{M}, \overline{M}]$ (as defined in (7)). Let $\mathscr{A}$ is reducible , then the system (1) is MSS if $\rho(\dot{\mathscr{B}}) < 1$.

Proof: The proof is similar to Theorem 1, hence omitted in the interests of brevity. ■

Also in this case, if diag$\{G(i)_N\} \geq_e 0$, then from Lemma 3,

$$\dot{\mathscr{B}} = |\mathscr{A}| = (Q^T\otimes\mathbb{I}_{n^2})\text{diag}\{G(i)_N\}$$

So, we can obtain a result analogous to Corollary 1.

Remark 2: When $\{\sigma_k\}_{k\geq 0}$ is a time-homogeneous Markov chain, i.e; $\Pi_k = \Pi$ for all k, then none of the columns of $\mathscr{A}$ has a non-degenerate interval. Hence, from Lemma 3,

$$\dot{\mathscr{B}} = \mathscr{A},$$

where

$$\mathscr{A} = (\Pi^T\otimes\mathbb{I}_{n^2})\text{diag}\{G(i)_N\}.$$

From Theorem 2, the system (1) is MSS if $\rho(\mathscr{A}) < 1$, which is the sufficient condition described in [4], [8], [14] for the MSS of time-homogeneous discrete-time MJLS.

V. Numerical Example

In this section, a numerical example is presented to illustrate the proposed results.

Consider

$$x_{k+1} = H(\sigma_k)x_k, \quad H(\sigma_k) \in \mathbb{R}^{2\times 2} \qquad (12)$$

with $\sigma_k = \{1, 2\}$.

Let $H(1) = \begin{bmatrix} 2 & -1 \\ 0 & 0.5 \end{bmatrix}$, $H(2) = \begin{bmatrix} 0.5 & 1 \\ 0 & 1 \end{bmatrix}$ and $\Pi_k = \begin{bmatrix} [0, 0.2] & [0.8, 1] \\ 0.15 & 0.85 \end{bmatrix}$. In this scenario if we write $\Pi_k = [P, Q]$, then $P = \begin{bmatrix} 0 & 0.8 \\ 0.15 & 0.85 \end{bmatrix}$ and $Q = \begin{bmatrix} 0.2 & 1 \\ 0.15 & 0.85 \end{bmatrix}$. Using (7), $\mathscr{A}$ can be obtained as given in (13).

To verify the irreducibility of the interval matrix $\mathscr{A}$, consider its direct graph in figure 3. Its vertices are denoted by the numbers $1, 2, \cdots 8$ for simplicity. If $\mathscr{A} = (A)_{ij}$ contains a non-degenerate and non-zero interval, then the directed edge of graph of $\mathscr{A}$ is denoted by a dashed arrow. From figure 3, it can be observed that the graph of $\mathscr{A}$ is not strongly connected, hence $\mathscr{A}$ is reducible.

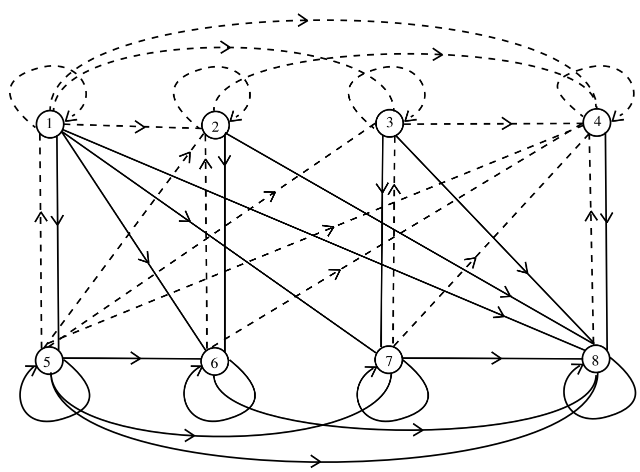

Fig. 3. Directed Graph $G(\mathscr{A})$ of the interval matrix $\mathscr{A}$ in (13)

So, we consider $\dot{\mathscr{B}}$. From figure 3, one can observe that for each vertex $i \in \{1, 2, \cdots 8\}$, there exists a directed path that ends at i and also contains a non-degenerate interval. So, Lemma 2 concludes that every column of the interval matrix $\mathscr{A}$ has $(*)$ property. Thus from Lemma 3, we obtain $\dot{\mathscr{B}} = |\mathscr{A}|$.

Here $\rho(|\mathscr{A}|) = 0.9923$, So the system (12) is MSS according to Theorem 2. We performed Monte Carlo simulations by considering realizations of the Markov chain Π_k. Figure 4 shows the mean square value of the system state obtained from the 10,000 realizations. One can observe that though one of the individual mode matrices $H(1)$ and $H(2)$ is unstable, the overall system is MSS in this case.

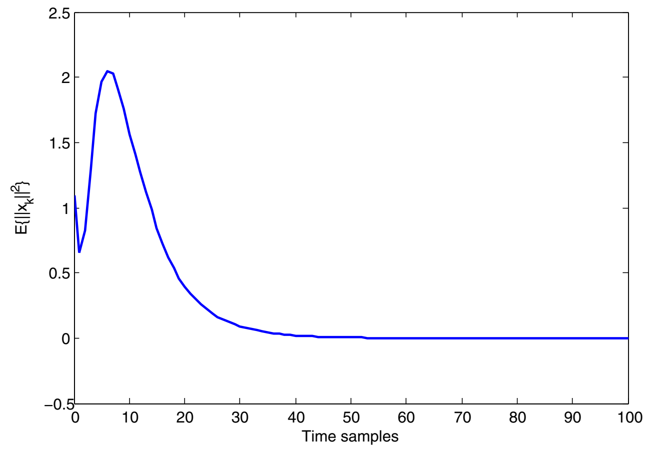

Fig. 4. Plot of $E\{||x_k||^2\}$

VI. Conclusions

We consider a discrete-time Markov jump linear system with a finite state non-homogeneous Markov chain whose transition probability matrix varying in an interval. A sufficient condition for the mean square stability of the system is obtained in terms of the spectral radius of a deterministic matrix using the results of Interval analysis and Kronecker products.

Appendix

Let $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{p\times q}$. Then the Kronecker product (or tensor product) of A and B is defined as the matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}_{mp\times nq}$$

For $A = (a_{ij})_{m\times n}$, the linear operator vec(A) is defined by

$$\text{vec}(A) = [a_{11} \ \cdots \ a_{m1} \ a_{12} \ \cdots \ a_{m2} \ a_{1n} \ \cdots \ a_{mn}]^T.$$

Lemma 6: [13]
a) $(AB \otimes CD) = (A \otimes C)(B \otimes D)$
b) $(A \otimes B)^T = (A^T \otimes B^T)$
c) $\text{vec}(A + B) = \text{vec}(A) + \text{vec}(B)$
d) $\text{vec}(ABC) = (C^T \otimes A)\text{vec}(B)$.

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$$
\mathscr{A} = \begin{bmatrix} [0,0.8] & [-0.4,0] & [-0.4,0] & [0,0.2] & 0.0375 & 0.075 & 0.075 & 0.15 \\ 0 & [0,0.2] & 0 & [-0.1,0] & 0 & 0.075 & 0 & 0.15 \\ 0 & 0 & [0,0.2] & [-0.1,0] & 0 & 0 & 0.075 & 0.15 \\ 0 & 0 & 0 & [0,0.05] & 0 & 0 & 0 & 0.15 \\ [3.2,4] & [-2,-1.6] & [-2,-1.6] & [0.8,1] & 0.2125 & 0.425 & 0.425 & 0.85 \\ 0 & [0.8,1] & 0 & [-0.5,-0.4] & 0 & 0.425 & 0 & 0.85 \\ 0 & 0 & [0.8,1] & [-0.5,-0.4] & 0 & 0 & 0.425 & 0.85 \\ 0 & 0 & 0 & [0.2,0.25] & 0 & 0 & 0 & 0.85 \end{bmatrix} \tag{13}
$$

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