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Fixed-order H_∞ and H_2 Controller Design for Continuous-time Polytopic Systems: An LMI-based Approach

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***Abstract*— In this paper, a new approach for fixed-order H_∞ and H_2 dynamic output-feedback controller design of continuous-time systems with state space polytopic uncertainty is proposed. This approach is based on the use of some instrumental matrices, which operate as a tool to decouple the controller parameters and the Lyapunov matrices. The stability of the closed-loop system as well as H_∞ and H_2 performance criteria are expressed by a set of linear matrix inequalities based upon linearly parameter-dependent Lyapunov matrices. Iterative procedure for updating the instrumental matrices using the previous controller can increase the performance of the approach. Simulation results demonstrate the effectiveness of the proposed method.**

I. INTRODUCTION

Fixed-order dynamic output-feedback control design is a challenging issue in the robust control theory and it has attracted considerable attention since the last four decades. The non-convexity of the set of all fixed-order stabilizing controllers for a given plant is the origin of difficulty in solving such problem [1]. Several approaches have been proposed to deal with the problem of fixed-order dynamic controller design. It has been shown that necessary and sufficient conditions for the existence of a fixed-order controller satisfying a certain stability or performance specification can be formulated in terms of linear matrix inequalities (LMIs) plus a non-convex rank constraint [2]. The existing methods for dealing with this problem include those based on alternating projections [3], augmented Lagrangian methods [4], [5], and nonlinear semidefinite programming (SDP) [6], [7].

Fixed-order controller design can also be formulated as a problem involving bilinear matrix inequalities (BMIs), which are non-convex. In [9], [10], several global and local methods for solving BMIs have been presented. In [11], a non-smooth non-convex optimization for fixed-order H_∞ synthesis has been developed. This approach led to the code *hinfstruct* in the MATLAB Robust Control Toolbox.

All these approaches propose a solution to the non-convex problem of fixed-order controller design; however, they do not consider systems subject to parametric uncertainty.

In [12] and [13], a non-smooth non-convex optimization method for fixed-order mixed H_∞ and H_2 controller design of systems with multi-model uncertainty has been presented. This method led to the software HIFOO (H_∞-H_2 Fixed Order Optimization) which is a public-domain MATLAB package. Although HIFOO simultaneously stabilizes and meet the performance conditions for the vertices of a polytopic system, it does not guarantee the stability conditions and H_∞ and H_2 performance for the whole polytope.

To overcome the non-convexity of fixed-order output-feedback controller design, some researchers have tried to convexify the problem and present sufficient conditions in terms of LMIs. They have proposed convex parameterizations of fixed-order stabilizing controllers with H_∞/H_2 performance for systems with polytopic uncertainty in the polynomial framework [14]–[17]. These approaches are based on the positivity of polynomials and Strictly Positive Realness (SPRness) of some transfer functions. The quality of these approaches depend on the choice of a so-called central polynomial. These approaches are limited to SISO polytopic systems and therefore they cannot consider state-space polytopic uncertainty which is more general than polynomial polytopic uncertainty.

In this paper, the problem of fixed-order H_∞ and H_2 output feedback controller design for MIMO continuous-time systems with polytopic uncertainty in their state space representations is considered. The solution is presented by a set of LMIs and using an iterative approach converges monotonically to a local optimum. The idea of this approach is based on a concept of SPR-pair matrices, which is used as a tool to convexify the originally non-convex problem of fixed-order controller design.

The paper is organized as follows. In the next section, the problem formulation and preliminaries are given. Section III presents a convex set of fixed-order stabilizing controllers for polytopic systems. The results are extended to fixed-order H_∞ and H_2 controller design in Sections IV and V, respectively. Section VI is devoted to simulation results. Finally, Section VII is devoted to concluding remarks.

The notation used throughout the paper is standard. In particular, I and $H^{-1}(s)$ are an identity matrix of an appropriate dimension and the inverse of a square transfer matrix $H(s)$, respectively. $P > 0$ and $P < 0$ means that the matrix P is positive-definite and negative-definite, respectively. The symbol $\star$ denotes symmetric blocks.

II. PROBLEM FORMULATION AND PRELIMINARIES

Consider a linear time-invariant polytopic system represented by the following state space realization:

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$$
\begin{aligned}
\dot{x}_g(t) &= A_g x_g(t) + B_g u(t) + B_w w(t) \\
z(t) &= C_z x_g(t) + D_{zu} u(t) + D_{zw} w(t) \\
y(t) &= C_g x_g(t) + D_w w(t)
\end{aligned} \tag{1}
$$

where $x_g \in \mathbb{R}^n$, $u \in \mathbb{R}^{n_i}$, $w \in \mathbb{R}^r$, $y \in \mathbb{R}^{n_o}$, and $z \in \mathbb{R}^s$ are the state, the control input, the exogenous input, the measured output, and a vector of output signals related to the performance of the control system, respectively. The real matrices A_g, B_g, B_w, C_z, C_g, D_{zu}, D_{zw}, and D_w are of appropriate dimensions. It is assumed that the matrices A_g and B_g have polytopic uncertainty as follows:

$$
A_g(\lambda) = \sum_{i=1}^{q} \lambda_i A_{g_i} \qquad B_g(\lambda) = \sum_{i=1}^{q} \lambda_i B_{g_i} \tag{2}
$$

where $\lambda = [\lambda_1 \cdots \lambda_q]^T \in \Lambda$,

$$
\Lambda = \left\{ \lambda \middle| \quad \sum_{i=1}^{q} \lambda_i = 1, \quad \lambda_i \geq 0; \quad i = 1, \dots, q \right\} \tag{3}
$$

$(A_{g_i}, B_{g_i}, C_g, 0)$ is the state space realization of i-th vertex of the polytope. It should be mentioned that if the matrix C_g has polytopic uncertainty and B_g is fixed, the same results are obtained. The case that both B_g and C_g have polytopic uncertainty is not treated in this contribution.

The objective is to present a convex set of fixed-order dynamic output feedback stabilizing controllers with H_∞ and H_2 performance for the polytopic system. The controller is represented by:

$$
\begin{aligned}
\dot{x}_c(t) &= A_c x_c(t) + B_c y(t) \\
u(t) &= C_c x_c(t) + D_c y(t)
\end{aligned} \tag{4}
$$

where $A_c \in \mathbb{R}^{m \times m}$ and B_c, C_c, and D_c are of appropriate dimensions. Then, the closed-loop system $H_{zw}(\lambda)$ has the following state space realization:

$$
\begin{aligned}
\dot{x}(t) &= A(\lambda)x(t) + B(\lambda)w(t) \\
z(t) &= Cx(t) + Dw(t)
\end{aligned} \tag{5}
$$

where,

$$
\begin{aligned}
A(\lambda) &= \begin{bmatrix} A_g(\lambda) + B_g(\lambda) D_c C_g & B_g(\lambda) C_c \\ B_c C_g & A_c \end{bmatrix} \\
B(\lambda) &= \begin{bmatrix} B_w + B_g(\lambda) D_c D_w \\ B_c D_w \end{bmatrix} \\
C &= [C_z + D_{zu} D_c C_g \quad D_{zu} C_c] \\
D &= D_{zw} + D_{zu} D_c D_w
\end{aligned} \tag{6}
$$

The basic idea of fixed-order controller design in the polynomial approaches [14], [18] is based on the strictly positive realness (SPRness) of some transfer function. This idea is presented as follows: Suppose that $c_i(s)$ for $i = 1, 2, \dots, q$ are the closed-loop characteristic polynomials at the i-th vertex, then the polytopic system is stable if the transfer function $c_i(s)/d(s)$ for $i = 1, 2, \dots, q$ is a strictly positive real (SPR) transfer function, where $d(s)$ is a given stable polynomial called the central polynomial. The choice of the central polynomial is very important since it affects the control performance as well as the conservatism of the approach.

In this paper, the idea of SPR transfer functions in the state space framework is used to propose a convex set of fixed-order stabilizing controllers with H_∞ and H_2 performance constraints for systems with polytopic uncertainty. The SPR condition can be given in the state space by Kalman-Yakubovic-Popov (KYP) Lemma.

Lemma 1: (KYP Lemma for continuous-time systems [19]) A transfer matrix $H(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is SPR if and only if there exists a symmetric matrix $P = P^T > 0$ such that:

$$
\begin{bmatrix} PA^T + AP & PC^T - B \\ CP - B^T & -D - D^T \end{bmatrix} < 0 \tag{7}
$$

Lemma 2: An SPR transfer matrix $H(s)$ and its inverse $H^{-1}(s)$ with the following realization are SPR with the same Lyapunov matrix.

$$
H^{-1}(s) = \left[\begin{array}{c|c} A - BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right] \tag{8}
$$

Proof: The proof is obtained easily by using KYP Lemma and then applying the Schur complement lemma [20] on it. ■

Remark: The matrices A and $A - BD^{-1}C$ are both stable with a common Lyapunov matrix.

III. Fixed-order Stabilizing Controller Design

A. LMI Representation

The problem addressed in this section is to provide a convex set of fixed-order stabilizing controllers for the systems subject to polytopic uncertainty in (1) and (2). For this purpose, the following definition is required to proceed.

Definition 1: Matrix $M \in \mathbb{R}^{n \times n}$ is an SPR-pair with matrix $A \in \mathbb{R}^{n \times n}$ if

$$
H(s) = \left[\begin{array}{c|c} M & M - A \\ \hline I & I \end{array}\right] \tag{9}
$$

is an SPR transfer matrix.

By applying Lemma 2, it is obvious that if $H(s)$ in (9) is SPR, $H^{-1}(s)$ with the following state space realization is also SPR.

$$
H^{-1}(s) = \left[\begin{array}{c|c} A & A - M \\ \hline I & I \end{array}\right] \tag{10}
$$

Therefore, if M and A are SPR-pair, then A and M are also SPR-pair and they are both stable with a common Lyapunov matrix. As a result, the following inequalities are equivalent:

$$
\begin{aligned}
&\begin{bmatrix} PM^T + MP & \star \\ P - M^T + A^T & -2I \end{bmatrix} < 0 \\
&\begin{bmatrix} PA^T + AP & \star \\ P + M^T - A^T & -2I \end{bmatrix} < 0
\end{aligned} \tag{11}
$$

The following theorem proposes a new convex parameterization of fixed-order stabilizing controllers for the polytopic system defined in (1) and (2).

Theorem 1: The fixed-order controller defined in (4) stabilizes the polytopic system in (1) and (2) if there exist a stable matrix M and a non-singular matrix T such that M makes an SPR-pair with $T^{-1}A_iT$ for $i = 1, \dots, q$, where A_i is the closed-loop state matrix of the i-th vertex defined by:

$$A_i = \begin{bmatrix} A_{g_i} + B_{g_i}D_cC_g & B_{g_i}C_c \\ B_cC_g & A_c \end{bmatrix} \tag{12}$$

Therefore, a convex set of stabilizing controllers can be given using KYP Lemma by the following set of LMIs:

$$\begin{bmatrix} P_iM^T + MP_i & \star \\ P_i - M^T + (T^{-1}A_iT)^T & -2I \end{bmatrix} < 0 \tag{13}$$

for $i = 1, \dots, q$.

The above inequalities are LMIs in terms of the controller parameters (A_c, B_c, C_c, D_c) and q symmetric matrices $P_i > 0$ for $i = 1, 2, \dots, q$.

Proof: Since M is an SPR-pair with $T^{-1}A_iT$ in (13), both are stable with the Lyapunov matrix P_i. By convex combination of (13) for all vertices, one can obtain:

$$\begin{bmatrix} P(\lambda)M^T + MP(\lambda) & \star \\ P(\lambda) - M^T + (T^{-1}A(\lambda)T)^T & -2I \end{bmatrix} < 0 \tag{14}$$

where $A(\lambda) = \sum_{i=1}^{q} \lambda_i A_i$, $P(\lambda) = \sum_{i=1}^{q} \lambda_i P_i$, and $\lambda_i \in \Lambda$ in (3). The above inequality shows that M and $T^{-1}A(\lambda)T$ are SPR-pair. Based on (11), the above inequality is equivalent to

$$\begin{bmatrix} P(\lambda)(T^{-1}A(\lambda)T)^T + (T^{-1}A(\lambda)T)P(\lambda) & \star \\ P(\lambda) + M^T - (T^{-1}A(\lambda)T)^T & -2I \end{bmatrix} < 0 \tag{15}$$

Now, multiply the above inequality on the right by $diag(T^T, T^T)$ and on the left by $diag(T, T)$:

$$\begin{bmatrix} (TP(\lambda)T^T)A^T(\lambda) + A(\lambda)(TP(\lambda)T^T) & \star \\ TP(\lambda)T^T + (TMT^T)^T - TT^TA^T(\lambda) & -2TT^T \end{bmatrix} < 0 \tag{16}$$

As a result, the closed-loop state matrix of the polytopic system $A(\lambda)$ is stable with a linearly parameter-dependent Lyapunov matrix $TP(\lambda)T^T$. ■

B. Choice of the central state matrix and the similarity transformation

The quality of this approach depends on the choice of M, the central state matrix, and T, the similarity transformation, which have to be selected in an appropriate way.

One method for choosing the matrices M and T is to use a set of initial stabilizing controllers designed for each vertex of the polytopic system. Since there are various methods for fixed-order controller design in the literature to deal with systems without polytopic uncertainty (e.g. [3], [11], [17], [21]–[23]), it is assumed that a fixed-order stabilizing controller for each vertex of the polytopic system is available.

Suppose that $\bar{A}_i$ is the closed-loop state matrix of the i-th vertex with its corresponding controller. Then, a suitable candidate for the central state matrix M will be a matrix which is SPR-pair with $T^{-1}\bar{A}_iT$ for $i = 1, 2, \dots, q$. This matrix and the non-singular matrix T are determined by the dual LMIs of (13), which are mentioned in the following remark:

Remark: Note that based on the results of Lemma 2, the inequality (13) is equivalent to the following inequality which means that $T^{-1}A_iT$ is also SPR-pair with M:

$$\begin{bmatrix} P_i(T^{-1}A_iT)^T + (T^{-1}A_iT)P_i & \star \\ P_i + M^T - (T^{-1}A_iT)^T & -2I \end{bmatrix} < 0 \tag{17}$$

for $i = 1, 2, \dots, q$.

Then, by multiplying this inequality on the right by $diag(T^T, T^T)$ and on the left by $diag(T, T)$, one obtains:

$$\begin{bmatrix} P_{T_i}A_i^T + A_iP_{T_i} & \star \\ P_{T_i} - XA_i^T + M_T^T & -2X \end{bmatrix} < 0 \tag{18}$$

where,

$$\begin{aligned} M_T &= TMT^T \\ P_{T_i} &= TP_iT^T \\ X &= TT^T \end{aligned} \tag{19}$$

for $i = 1, 2, \dots, q$. Therefore, the central state matrix M and the similarity transformation T can be chosen as follows:

$$\begin{aligned} T &= (chol(X))^T \\ M &= T^{-1}M_TT^{-T} \end{aligned} \tag{20}$$

where *chol* is Cholesky factorization and (M_T, X) are a feasible solution to the LMIs in (18) by replacing A_i with $\bar{A}_i$ for $i = 1, 2, \dots, q$.

IV. Fixed-order H_∞ Controller Design

In this section, the objective is to design a fixed-order stabilizing controller for a polytopic system which satisfies an infinity norm bound on some closed-loop transfer functions, i.e. $\|H_{zw}(\lambda)\|_\infty < \gamma$.

It is assumed that either D_{zu} or D_w in (1) are equal to zero. In what follows, $D_{zu} = 0$ is considered. Therefore, the controller matrices appear in the matrices $A(\lambda)$ and $B(\lambda)$.

The following Theorem presents a convex set of fixed-order H_∞ controllers.

Theorem 2: Suppose that a central stable matrix M and a nonsingular similarity transformation T are given. Then, the closed-loop system of the polytopic plant in (1) and (2) with the controller in (4) is stable and $\|H_{zw}(\lambda)\|_\infty < \gamma$ if there exist symmetric matrices $P_i > 0$ such that:

$$\begin{bmatrix} P_iM^T + MP_i & \star & \star & \star \\ P_i - M^T + (T^{-1}A_iT)^T & -2I & \star & \star \\ (T^{-1}B_i)^T & 0 & -\gamma I & \star \\ (CT)P_i & 0 & D & -\gamma I \end{bmatrix} < 0 \tag{21}$$

for $i = 1, 2, \dots, q$.

Proof: The proof is easily obtained by applying the Schur complement on (21). ■

The dual part of the LMIs in (21) is found by using the Schur complement lemma as follows:

$$\begin{bmatrix} P_{T_i}A_i^T + A_iP_{T_i} & \star & \star & \star \\ P_{T_i} + {M_T}^T - XA_i^T & -2X & \star & \star \\ B_i^T & 0 & -\gamma I & \star \\ CP_{T_i} & 0 & D & -\gamma I \end{bmatrix} < 0 \tag{22}$$

where M_T, P_{T_i}, and X have been defined in (19).

Now, consider a set of initial fixed-order H_∞ controllers independently designed for each vertex and compute $\bar{A}_i$, $\bar{B}_i$, $\bar{C}_i$, and $\bar{D}_i$ from (6) by replacing the initial controllers for A_c, B_c, C_c and D_c. Then, M_T and X can be obtained through an optimization problem that minimizes γ subject to the LMIs in (22) where (A_i, B_i, C_i, D_i) are replaced by ($\bar{A}_i$, $\bar{B}_i$, $\bar{C}_i$, $\bar{D}_i$).

Remarks:

1) The results can be further improved if the resulting controller is used as an initial controller in order to update the central state and similarity transformation matrices iteratively. This idea is applied to a simulation example in Section VI.
2) Theorem 2 can be developed for the case when $D_w = 0$ and $D_{zu} \neq 0$ as well.

V. FIXED-ORDER H_2 CONTROLLER DESIGN

The main objective of this section is to propose a convex set of fixed-order stabilizing controllers for polytopic systems which satisfies H_2 performance $\|H_{zw}(\lambda)\|_2^2 < \upsilon$.

It is assumed that $D_{zu} = 0$. In this way, the controller matrices appear in matrices $A(\lambda)$ and $B(\lambda)$.

Theorem 3: Suppose that a central stable matrix M and a nonsingular similarity transformation T are given. Then, the closed-loop system of the polytopic plant in (1) and (2) with the controller in (4) is stable and $\|H_{zw}(\lambda)\|_2^2 < \upsilon$ if there exist symmetric matrices $P_i > 0$ and $Q_i > 0$ such that:

$$\begin{gathered}\begin{bmatrix} P_i M^T + M P_i & \star & \star \\ P_i - M^T + (T^{-1}A_iT)^T & -2I & \star \\ (T^{-1}B_i)^T & 0 & -I \end{bmatrix} < 0 \\ \begin{bmatrix} P_i & \star \\ (CT)P_i & Q_i \end{bmatrix} > 0 \\ trace(Q_i) < \upsilon, \quad D = 0 \end{gathered} \tag{23}$$

for $i = 1, 2, \ldots, q$.

The quality of this approach depends on the choice of the state matrix M and the non-singular matrix T which can be acquired based upon the dual formulation of the above LMIs obtained by using the Schur Complement lemma as follows:

$$\begin{gathered}\begin{bmatrix} P_{T_i}A_i^T + A_iP_{T_i} & \star & \star \\ P_{T_i} + {M_T}^T - XA_i^T & -2X & \star \\ {B_i}^T & 0 & -I \end{bmatrix} < 0 \\ \begin{bmatrix} P_{T_i} & \star \\ CP_{T_i} & Q_i \end{bmatrix} > 0 \\ trace(Q_i) < \upsilon, \quad D = 0 \end{gathered} \tag{24}$$

where M_T, P_{T_i}, and X have been defined in (19).

Remarks:

1) Similar results can be obtained for the case when $D_w = 0$ and $D_{zu} \neq 0$.
2) The matrices M and T can be obtained by (20) where (M_T, X) is a feasible solution of an optimization problem which is minimizing υ subject to the LMI conditions in (24) (by simply replacing (A_i, B_i, C_i, D_i) with ($\bar{A}_i$, $\bar{B}_i$, $\bar{C}_i$, $\bar{D}_i$)).
3) Mixed H_∞ and H_2 control design can be addressed by gathering the LMIs in (21) and (23) in a single set of LMIs without using a common Lyapunov matrix and unique instrumental matrices (central state and similarity transform matrices) for both objectives.

VI. SIMULATION RESULTS

Consider the following fourth-order polytopic system. This example is borrowed from [26] and represents a two-mass-spring system.

$$\begin{aligned} A_g(k) &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & 0 & 0 \\ k & -k & 0 & 0 \end{bmatrix} \\ B_g &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}; \quad B_w = \begin{bmatrix} 0 \\ 0 \\ -0.75 \\ 0.75 \end{bmatrix} \\ C_g &= \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}; \quad C_z = \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix} \\ D_{zu} &= 0; \quad D_{zw} = 0; \quad D_w = 0 \end{aligned} \tag{25}$$

We consider that k is an uncertain parameter in the interval $[1.15, 2]$. Therefore, the polytopic system has two vertices.

The objective is to design a fixed-order H_∞ and H_2 dynamic output feedback controller in order to minimize $\|H_{zw}(\lambda)\|_\infty$ and $\|H_{zw}(\lambda)\|_2$ of the unstable system in (25). Note that since the state space realization of the system is not in the canonical form, the existing polynomial-based approaches in the literature for fixed-order controller design ([14], [17], [18]) cannot be used for the comparison purpose.

Case a: Fixed-order H_∞ controller design

First, two initial second-order H_∞ controllers are computed using the *hinfstruct* command in MATLAB (by setting the target gain to 1.4).

In this case, the infinity norm of two vertices are 1.2492 and 0.8957. Then, the central state matrix and the similarity transformation are computed using a feasible solution of (22) and (20) with $\gamma_{min} = 1.2557$. Finally, a second-order H_∞ controller is computed using the LMI in (21) with $\gamma_{min} = 1.0710$ as follows:

$$\begin{aligned} A_{c1} &= \begin{bmatrix} -2.769 & -2.284 \\ 2.02 & -0.0074 \end{bmatrix}; \quad B_{c1} = \begin{bmatrix} 4.362 \\ 0.005 \end{bmatrix} \\ C_{c1} &= \begin{bmatrix} -4.083 & -3.309 \end{bmatrix}; \quad D_{c1} = 6.295 \end{aligned} \tag{26}$$

The norm of $\|H_{zw_i}\|_\infty$ achieved at the two vertices are 0.9786 and 0.761, respectively.

Iterative approach for update on the instrumental matrices M and T based on the controller in the last iteration can improve the results. Figure 1 shows the values of γ after 20 iterations. The algorithm converges to $\gamma_{min} = 0.8645$ and it results in the following controller:

$$\begin{aligned} A_{c2} &= \begin{bmatrix} -2.759 & -2.245 \\ 1.950 & 0.0117 \end{bmatrix}; \quad B_{c2} = \begin{bmatrix} 4.52 \\ -0.044 \end{bmatrix} \\ C_{c2} &= \begin{bmatrix} -4.057 & -3.263 \end{bmatrix}; \quad D_{c2} = 6.561 \end{aligned} \tag{27}$$

with the infinity norms 0.8592 and 0.7809 at two vertices, respectively.

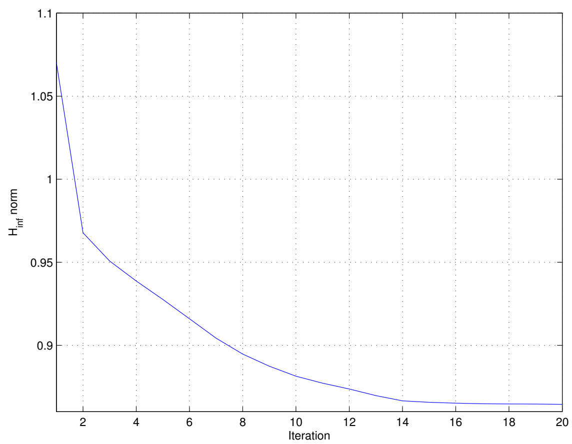

Fig. 1. Evolution of the infinity norm versus the iteration number

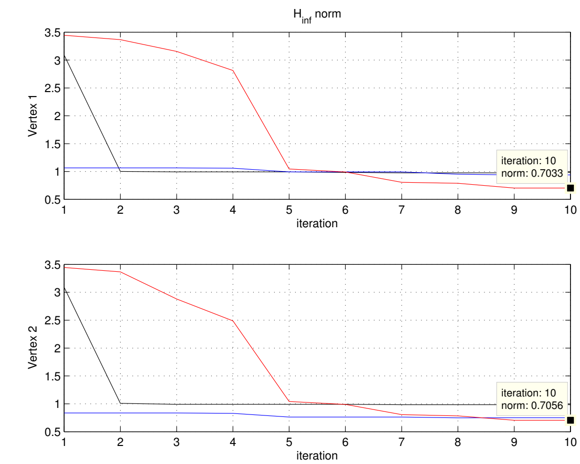

Fig. 2. H_∞ norm sequences optimized by HIFOO for the two vertices of the polytopic system

For the comparison purpose, HIFOO 3.5 (with HANSO 2.01) is used to design a fixed-order H_∞ controller for two vertices of the polytope. Since HIFOO uses random starting points, three sequences of optimized H_∞ norms with 10 iterations are generated. To improve the results with HIFOO, the designed controller in previous iteration is used as an initial guess. The results of H_∞ norms of two vertices are plotted in Figure 2. The minimum H_∞ norms of two vertices obtained by HIFOO after 10 iterations are 0.7033 and 0.7056, respectively.

HIFOO requires considerable computational time to find a controller. In this example, the average time, which has been computed using the *tic* and *toc* commands in MATLAB, for H_∞ controller design by the proposed approach (with 20 iterations) and HIFOO (with 10 iterations) are $32.36sec$ and $912.94sec$, respectively. If the option *options.cpumax* in HIFOO is set to reduce the running time, the performance will be reduced.

It should be mentioned that the controller obtained by HIFOO is valid just for the two vertices of the polytope and it does not guarantee the stability condition and the H_∞ performance constraints for any model within the polytope of these two vertices.

Case b: Fixed-order H_2 controller design

In this case, the same initial controllers (as the previous case) are employed. The central state matrix and the similarity transformation are computed using a feasible solution of (24) and (20). Finally, the results of Theorem 3 yield the value 0.7341 as an upper bound for the H_2 performance. The corresponding H_2 controller is:

$$\begin{aligned} A_{c1} &= \begin{bmatrix} -2.4705 & -2.561 \\ 1.929 & 0.0477 \end{bmatrix}; \quad B_{c1} = \begin{bmatrix} 4.99 \\ -0.0936 \end{bmatrix} \\ C_{c1} &= \begin{bmatrix} -3.656 & -3.7118 \end{bmatrix}; \quad D_{c1} = 7.2131 \end{aligned} \tag{28}$$

The results can be improved more by an iterative approach in which M and T are updated based on the controller in the last iteration as an initial controller. After just 5 iterations, the upper bound 0.6928 is obtained for the polytopic system

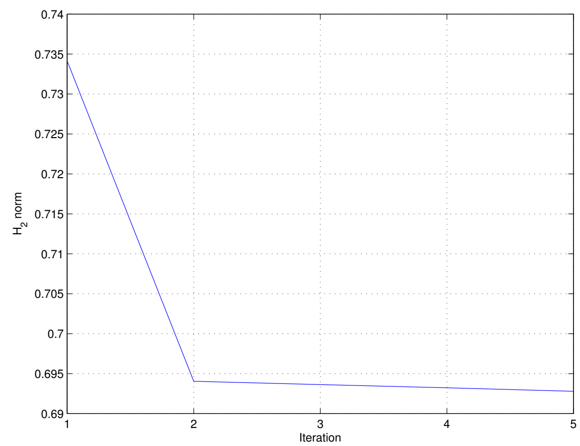

Fig. 3. Evolution of the two norm versus the iteration number

with the following controller:

$$\begin{aligned} A_{c2} &= \begin{bmatrix} -2.4685 & -2.56 \\ 1.9288 & 0.0469 \end{bmatrix}; \quad B_{c2} = \begin{bmatrix} 4.995 \\ -0.0956 \end{bmatrix} \\ C_{c2} &= \begin{bmatrix} -3.653 & -3.71 \end{bmatrix}; \quad D_{c2} = 7.223 \end{aligned} \tag{29}$$

Figure 3 shows the values $\sqrt{\upsilon}$ after 5 iterations. $\|H_{zw_i}\|_2$ of the two vertices are 0.6926 and 0.5335, respectively.

The results are compared with HIFOO 3.5 (with HANSO 2.01). Three sequences of optimized H_2 norms with 10 iterations are generated. At each iteration, the previous controller is used as an initial starting point. The results of H_2 norms of two vertices are plotted in Figure 4. The minimum H_2 norms of two vertices resulted by HIFOO after 10 iterations are 1.307 and 0.9171, respectively. The average time required to design the H_2 controller by the proposed approach (with 5 iterations) and HIFOO (with 10 iterations) are $12.90sec$ and $254.69sec$, respectively.

Remark: To solve the LMIs in MATLAB, YALMIP [27] as the interface and SDPT3 [28]/SeDuMi [29] as the solver have been used.

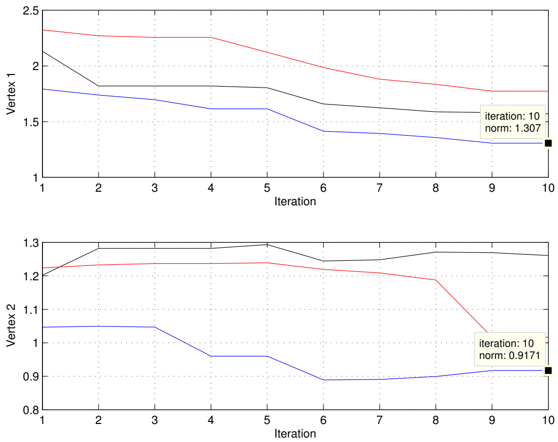

Fig. 4. H_2 norm sequences optimized by HIFOO for the two vertices of the polytopic system

VII. Conclusion

In this paper, a new approach for fixed-order H_∞ and H_2 output feedback control of continuous-time polytopic systems has been proposed. The approach is based upon the concept of SPR-pair matrices which operates as a tool to convexify the stability conditions, the H_∞ and H_2 performance constraints. The quality of this approach depends on the choice of the instrumental matrices which can be computed by means of a set of initial fixed-order controllers designed for each vertex of the polytopic system. An iterative approach for update on the instrumental matrices based on the previous controller can improve the H_∞ and H_2 performance. The simulation results demonstrated the effectiveness of the approach.

References

[1] M. Fu and Z.-Q. Luo, "Computational complexity of a problem arising in fixed order output feedback design," *Systems and Control Letters*, vol. 30, no. 5, pp. 209–215, 1997.

[2] P. Gahinet and P. Apkarian, "A linear matrix inequality approach to H_∞ control," *International Journal of Robust and Nonlinear Control*, vol. 4, pp. 421–448, 1994.

[3] K. M. Grigoriadis and R. E. Skelton, "Low order control design for LMI problems using alternating projection methods,," *Automatica*, vol. 32, no. 8, pp. 1117–1125, 1996.

[4] B. Fares, P. Apkarian, and D. Noll, "An augmented Lagrangian method for a class of LMI-constrained problems in robust control theory," *International Journal of Control*, vol. 74, no. 4, pp. 348–260, 2001.

[5] D. Ankelhed, A. Helmersson, and A. Hansson, "A partially augmented Lagrangian method for low order H-infinity controller synthesis using rational constraints," in *50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC)*, Orlando, FL, USA, 2011.

[6] C. W. J. Hol, C. W. Scherer, E. G. Van der Meché, and O. H. Bosgra, "A nonlinear SDP approach to fixed-order controller synthesis and comparison with two other methods applied to an active suspension system," *European Journal of Control*, vol. 9, no. 1, pp. 13–28, January 2003.

[7] P. Apkarian, D. Noll, J. B. Thevenet, and H. D. Tuan, "A spectral quadratic-SDP method with applications to fixed-order H_2 and H_∞ synthesis," *European Journal of Control*, vol. 10, no. 6, pp. 527–538, 2003.

[8] P. Apkarian, D. Noll, J.-B. Thevenet, and H. D. Tuan, "A spectral quadratic-SDP method with applications to fixed-order H_2 and H_∞ synthesis," in *Asian Control Conference*, Melbourne, Australia, 2004.

[9] T. Iwasaki, "The dual iteration for fixed-order control," *IEEE Transactions on Automatic Control*, vol. 44, no. 4, pp. 783–788, April 1999.

[10] E. Beran, "Methods for optimization-based fixed-order control design," Ph.D. dissertation, Technical University of Denmark, Denmark, 1997.

[11] P. Apkarian and D. Noll, "Nonsmooth H_∞ synthesis," *IEEE Transactions on Automatic Control*, vol. 51, no. 1, pp. 71–86, 2006.

[12] S. Gumussoy and M. L. Overton, "Fixed-order H-innity controller design via HIFOO, a specialized nonsmooth optimization package," in *IEEE American Control Conference*, Seattle, USA, 2008, pp. 2750–2754.

[13] D. Arzelier, G. Deaconu, S. Gumussoy, and D. Henrion, "H2 for HIFOO," in *International Conference on Control and Optimization with Industrial Applications*, Bilkent University, Ankara, Turkey, 2011.

[14] D. Henrion, M. Sebek, and V. Kucera, "Positive polynomials and robust stabilization with fixed-order controllers," *IEEE Transactions on Automatic Control*, vol. 48, no. 7, pp. 1178–1186, 2003.

[15] F. Yang, M. Gani, and D. Henrion, "Fixed-order robust H_∞ controller design with regional pole assignment," *IEEE Transactions on Automatic Control*, vol. 52, no. 10, pp. 1959–1963, 2007.

[16] A. Karimi, H. Khatibi, and R. Longchamp, "Robust control of polytopic systems by convex optimization," *Automatica*, vol. 43, no. 6, pp. 1395–1402, 2007.

[17] A. Sadeghzadeh, "Fixed-structure H_2 controller design: An LMI solution," in *The 7th IFAC Symposium on Robust Controller Design*, Aalborg, Denmark, 2012, pp. 325–330.

[18] H. Khatibi, A. Karimi, and R. Longchamp, "Fixed-order controller design for systems with polytopic uncertainty using LMIs," *IEEE Transactions on Automatic Control*, vol. 53, no. 1, pp. 428–434, 2008.

[19] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia: SIAM, 1994.

[20] A. Albert, "Conditions for positive and nonnegative definiteness in terms of pseudoinverses," *SIAM Journal on Applied Mathematics*, vol. 17, no. 2, pp. 434–440, 1969.

[21] J. V. Burke, D. Henrion, and A. S. L. M. L. Overton, "HIFOO : a MATLAB package for fixed-order controller design and H_∞ optimization," in *Fifth IFAC Symposium on Robust Control Design, Toulouse*, 2006.

[22] A. Karimi and G. Galdos, "Fixed-order H_∞ controller design for nonparametric models by convex optimization," *Automatica*, vol. 46, no. 8, pp. 1388–1394, 2010.

[23] G. Galdos, A. Karimi, and R. Longchamp, "H_∞ controller design for spectral MIMO models by convex optimization," *Journal of Process Control*, vol. 20, no. 10, pp. 1175 – 1182, 2010.

[24] U. Shaked, "An LPD approach to robust H_2 and H_∞ static output-feedback design," *IEEE Transactions on Automatic Control*, vol. 48, no. 5, pp. 866–872, 2003.

[25] C. Scherer, P. Gahinet, and M. Chilali, "Multiobjective output-feedback control via LMI optimization," *IEEE Transactions on Automatic Control*, vol. 42, no. 7, pp. 896–911, July 1997.

[26] P. Apkarian, "*hinfstruct*: Examples and applications," Available: http://pierre.apkarian.free.fr/Software.html.

[27] J. Löfberg, "YALMIP: A toolbox for modeling and optimization in MATLAB," in *CACSD Conference*, 2004. [Online]. Available: http://control.ee.ethz.ch/ joloef/yalmip.php

[28] K. C. Toh, M. J. Todd, and R. H. Tutuncu, "SDPT3: a MATLAB software package for semidefinite programming," *Optimization Methods and Software*, vol. 11, pp. 545–581, 1999.

[29] J. F. Sturm, "Using SeDuMi 1.02, a Matlab toolbox for optimization over symmetric cones," *Optimization Methods and Software*, vol. 11, pp. 625–653, 1999.